Midterm 1

Math 22A, Fall 2019

Name: Starkston Solutions

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You do not need to simplify numerical expressions for your final answers (e.g. you can write $3 - \frac{3}{4}$ instead of simplifying to $\frac{9}{4}$.)

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \cos\theta$$

Problem 1 (8pts): Calculate the vector given by the following linear combination

$$c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

Answer (2pts): $\begin{bmatrix} c+2d \\ 0 \\ -c-2d \end{bmatrix}$

All linear combinations of

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \text{ fill:}$$

A. A Line
B. A Plane
C. A Point
D. Three-dimensional space

Answer (3pts): A

All linear combinations of

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ fill:}$$

A. A Line
B. A Plane
C. A Point
D. Three-dimensional space

Answer (3pts): B

Problem 2 (10pts): Determine whether the following vectors are perpendicular:

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$3 \cdot 1 + 2(-2) + (-1)(-1)$
$= 3 - 4 + 1$
$= 0$

A. Perpendicular
B. Not perpendicular

Answer (2pts): A

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$1 \cdot 1 + 1 \cdot 3 + 1 \cdot 2 \neq 0$

A. Perpendicular
B. Not perpendicular

Answer (2pts): B

Calculate the length of the vector

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$\sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

Length (2pts): 3

Are the following matrix multiplications possible or impossible?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -2 & -1 \end{bmatrix}$$

A. Possible
B. Impossible

Answer (2pts): A

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

A. Possible
B. Impossible

Answer (2pts): B

Problem 3 (12pts): Find a matrix to fill in the blanks which performs the row operation which replaces row 3 by (row 3)+2(row 1) **(3pts)**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g+2a & h+2b & i+2c \end{bmatrix}$$

Write out the following system of equations in matrix form $Ax = b$ **(2pts)**:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -x_2 + 2x_3 &= 2 \\ -2x_1 + 4x_2 - 2x_3 &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 2 \\ -2 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Perform the row operation given by the matrix in the first part of this problem to the matrix equation $Ax = b$ in the second part of the problem to change the matrix A to one with a 0 in the position specified below **(4pts)**:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Does the system of equations have

A. One solution
B. Infinitely many solutions
C. No solutions

Answer (3pts): C

Problem 4 (14pts): State whether or not the following matrices have an inverse or not:

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

A. Has an inverse
B. No inverse

Answer (3pts): A

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

A. Has an inverse
B. No inverse

Answer (3pts): B

Calculate A^{-1} if A is the matrix

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 0 & -1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 \to R3 - R1} \left[\begin{array}{ccc|ccc} 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$R3 \to R3 + R2$

$$\left[\begin{array}{ccc|ccc} 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R2 \to -R2]{R1 \to \frac{1}{2}R1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$R1 \to R1 + 2R2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Answer (8pts): $\begin{bmatrix} 1/2 & -2 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

Problem 5 (12pts): Solve for the vector x:

Solve $Ax = b$ if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \text{and } A^{-1} = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$x = A^{-1}b = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Answer (4pts): $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Solve $Ax = b$ if $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ \text{and } b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$Ax = b \iff LUx = b \iff Ux = L^{-1}b$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad L^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$Ux = L^{-1}b \iff \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x + y + z &= 2 \rightarrow x = 2 - y - z = 2 - 1 - 1 = 0 \\ -y &= -1 \rightarrow y = 1 \\ 2z &= 2 \rightarrow z = 1 \end{aligned}$$

Answer (8pts): $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$