Subspaces

Is $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x^2+y^2=z^2 \right\}$ a subspace?

① Is it closed under scalar multiplication?

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in S $(x^2+y^2=z^2)$

then for any c in $\mathbb{R}$ $c\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ satisfies

$$\begin{aligned}(cx)^2+(cy)^2 &= c^2x^2+c^2y^2 \\ &= c^2(x^2+y^2) \\ &= c^2z^2 \\ &= (cz)^2\end{aligned}$$

so $(cx)^2+(cy)^2=(cz)^2$ so $\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$ is in S

Yes closed under scaler mult.

② Closed under addition?

No: For example $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are in S

$1^2+0^2=1^2$ ✓ $0^2+1^2=1^2$ ✓

but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is not in S because

$1^2+1^2 \neq 2^2$

Not closed under addition

$\Rightarrow$ Not a subspace

* Span$(v_1, v_2, \cdots, v_k)$ = all vectors which are linear combinations of $v_1, v_2, \cdots, v_k$

Is a subspace
Contains infinitely many vectors
dimension is less than or equal to k

Linear independence:

$$\left(\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right)$$ are linearly independent

<u>Proof</u>: If

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then
$$\begin{cases} 2c_1 - 3c_2 = 0 \\ -c_1 + 4c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$$

So the <u>only</u> linear combination giving $\vec{0}$ is when $c_1 = c_2 = 0$
Thus linearly independent.

On the other hand

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ <u>linearly dependent</u> because

If $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} c_1 + 2c_2 = 0 \\ 3c_2 + c_3 = 0 \end{cases} \Rightarrow \begin{matrix} c_1 = -2c_2 \\ c_3 = -3c_2 \end{matrix} \quad c_2 \text{ free}$$

Say $c_2 = 1$, then $c_1 = -2$, $c_3 = -3$ $\quad (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Combination with some nonzero coefficients

~~Row space (...)~~
~~S_3 possible options~~
~~determ...~~

Basis ~~is~~ spans + is linearly independent. dim = # vectors in basis

If $(v_1, \dots, v_k)$ span U

can ~~also~~ remove 0 or more vectors to get a basis

$\Rightarrow \dim U \leq k$

If $(v_1, \dots, v_\ell)$ are linearly independent vectors in ~~$\mathbb{R}$~~ V

can add 0 or more vectors to get a basis.

$\Rightarrow \dim V \geq \ell$.

Linear independent list	Basis	Spanning list
Could be too small	Just right # vectors is dimension	Could be too big

~~Any subspace If U is a subspace of V~~
~~$\dim(U) \leq \dim(V)$~~

~~If let~~ $\dim \mathbb{R}^n = n$

If have more than n vectors in $\mathbb{R}^n$ can't be linearly independent (too long list)

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$ linearly ind, do not span $\mathbb{R}^3$

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$ span $\mathbb{R}^3$, not linearly independent

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ do not span $\mathbb{R}^3$, not linearly independent

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right)$ do not span $\mathbb{R}^3$, not linearly independent

span + linearly indep $\Rightarrow$ basis $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ span + lin. ind $\Rightarrow$ basis

[1 2 ...] ↑ ... c ... b

Matrix A : Find N(A), C(A) basis for each

nullity   rank

Reduced row echelon form

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 3 & -1 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

linearly independent columns for basis of C(A)   pivots   ~~free variables~~   for N(A)

$x_1 \quad -2x_3 + 3x_4 = 0$

$x_2 + x_3 - 4x_4 = 0$

$$N(A) = \begin{bmatrix} 2x_3 - 3x_4 \\ -x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

C(A) basis: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ rank = 2

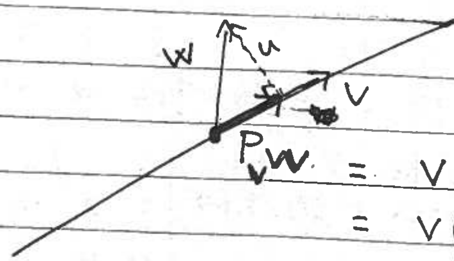
N(A) basis $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$ nullity = 2

Orthogonality

If v, w are column vectors $\quad v \cdot w = v^T w$

NOT vw^T

Projection



$$P_v w = v(v^T v)^{-1} v^T w$$
$$= v(v \cdot v)^{-1}(v \cdot w)$$
$$= v\left(\frac{v \cdot w}{v \cdot v}\right)$$

(If $\|v\| = 1$, $v \cdot v = 1^2 = 1$ so then $P_v w = v(v \cdot w)$)

For an orthogonal vector to v: $u = w - P_v w$

(goes into Gram Schmidt procedure)

To project to Span$(v_1, v_2, \ldots, v_k)$

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T$$

To project w to Span$(v_1, v_2, \ldots, v_k)$

$$Pw = A(A^TA)^{-1}A^Tw$$

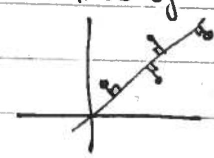
When $Ax = b$ has no solution, the best approximate solution is obtained from solving

$$A^TAx = A^Tb$$

(because then $Ax = A(A^TA)^{-1}A^Tb = Pb$)

Application: least squares line of best fit

$$y = C + Dx$$



Determinants: Use row operations to make calculations easier

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 12 & 12 & 12 & 12 \end{bmatrix}$$

columns are linearly dependent $\rightarrow \det = 0$

subtract R1 from each of R2, R3, R4

$$= \det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

cofactor expansion using row 3 or 4 $= 0$