Midterm 2

Math 22A, Fall 2019

Name: Solutions

Student ID: ____________________

You do not need to simplify numerical expressions for your final answers (e.g. you can write $3 - \frac{3}{4}$ instead of simplifying to $\frac{9}{4}$.)

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \, ||\mathbf{w}||} = \cos\theta$$

The formula for the projection matrix is

$$P = A(A^T A)^{-1} A^T$$

$$Pv = A(A^T A)^{-1} A^T v$$

When the projection is onto a line spanned by a single vector a, the projection matrix is

$$P_a = \frac{aa^T}{a^T a}$$

$$P_a v = \frac{aa^T}{a^T a} v = a\left(\frac{a^T v}{a^T a}\right)$$

Problem 1 (8 pts): Consider the subset S of $\mathbb{R}^2$ of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $xy^2 = 0$.

(a) (3pts) Is S closed under scalar multiplication? Prove it is or give an example of a vector v in S and a scalar c such that cv is not in S.

If $\begin{bmatrix} x \\ y \end{bmatrix}$ is in S $xy^2 = 0$

For any c in $\mathbb{R}$, $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ satisfies $(cx)(cy)^2 = c^3xy^2 = c^3(0) = 0$

So $c\begin{bmatrix} x \\ y \end{bmatrix}$ is in S.

Therefore S is closed under scalar multiplication

(b) (3pts) Is S closed under addition? Prove it is or give an example of vectors v_1, v_2 in S such that $v_1 + v_2$ is not in S.

S is not closed under addition

For example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in S because

$1 \cdot 0^2 = 0$ and $0 \cdot 1^2 = 0$

but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in S because

$1 \cdot 1^2 \neq 0$

(c) (2pts) Is S a subspace of $\mathbb{R}^2$? **Subspace** **Not Subspace**

Problem 2 (8 pts):

$$U = \begin{bmatrix} -1 & 8 & 1 & 4 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (3pts) Write down 3 columns of U which are linearly independent.

$$\underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{V_1}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{V_3} \qquad \left(\text{OR } \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

(b) (5pts) Give a proof that these 3 columns are linearly independent.

If $c_1 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

then $\begin{bmatrix} -c_1 + c_2 + 4c_3 \\ 2c_2 \\ c_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ so $-c_1 + c_2 + 4c_3 = 0$, $2c_2 = 0$ and $c_3 = 0$

$\Rightarrow$ $c_1 = c_2 + 4c_3 = 0$

$c_2 = 0$

$c_3 = 0$

Therefore the only linear combination of v_1, v_2, v_3 giving $\vec{0}$ is when $c_1 = c_2 = c_3 = 0$

Thus (v_1, v_2, v_3) are linearly independent.

Problem 3 (10 pts): Let

$$A = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix}$$

Find bases for the null space $N(A)$ and the column space $C(A)$. What is the nullity, $\dim(N(A))$? What is the rank $\dim(C(A))$?

$$\begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

pivot free

$-x_1 + 4x_2 = 0$

$x_1 = 4x_2$

$$N(A) = \left\{ \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$$

Basis for $N(A) = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$

1 dimensional

$$C(A) = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -8 \end{bmatrix} \right) = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} \right)$$

linearly dependent

Basis for $C(A) = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

1 dimensional

Basis for $N(A)$ (4pts): $\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$

Basis for $C(A)$ (4pts): $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

Nullity (1pt): 1

Rank (1pt): 1

Problem 4 (8 pts): In each sentence, *circle the best choice* among the choices in parentheses.

1. Suppose $v_1, v_2, \ldots, v_6$ are vectors in $\mathbb{R}^4$.

   (a) Those six vectors **(do)** / **(do not)** / **(might not)** span $\mathbb{R}^4$.

   (b) Those six vectors **(are)** / **(are not)** / **(might not be)** linearly independent.

   (c) Those six vectors **(are)** / **(are not)** / **(might not be)** a basis for $\mathbb{R}^4$.

2. Choose true or false for the following statements.

   (a) Every subspace contains the zero vector.

   (True) / **(False)**

   (b) A basis is a spanning set that is as large as possible.

   (True) / **(False)**

   (c) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

   (True) / **(False)**

   (d) A basis is a linearly independent set that is as large as possible.

   (True) / **(False)**

   (e) A single nonzero vector v_1 by itself is linearly independent.

   (True) / **(False)**

Problem 5 (8 pts):

1. (4 pts) If A is a 3×3 matrix with three column vectors c_1, c_2, c_3 which are orthogonal to each other such that $||c_1|| = 1$, $||c_2|| = 2$, and $||c_3|| = 3$, what is $A^T A$?

$$\begin{bmatrix} - & c_1^T & - \\ - & c_2^T & - \\ - & c_3^T & - \end{bmatrix}\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix}1&0&0\\0&4&0\\0&0&9\end{bmatrix}}$$

$$c_1 \cdot c_1 = ||c_1||^2 = 1$$
$$c_2 \cdot c_2 = ||c_2||^2 = 4$$
$$c_3 \cdot c_3 = ||c_3||^2 = 9$$

2. (4 pts) Calculate the projection $P_v w$ of the vector w onto the line $L = \text{Span}(v)$ where

$$v = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 2 \end{bmatrix}$$

$$P_v w = \frac{\overset{v}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}} \overset{v^T}{\begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}} \overset{w}{\begin{bmatrix} 3 \\ 2 \\ -2 \\ 2 \end{bmatrix}}}{\underset{v^T}{\begin{bmatrix} 1 & -1 & 0 & -1 \end{bmatrix}} \underset{v}{\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \frac{[-1]}{[3]}$$

$$= -\frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -1/3 \\ +1/3 \\ 0 \\ +1/3 \end{bmatrix}}$$

Problem 6 (8 pts): Find an orthonormal basis for the subspace

$$U = \text{Span}\left(\underset{v_1}{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \right)$$

$$\|v_1\| = \sqrt{2^2+0^2+(-2)^2+0^2+(1)^2} = \sqrt{9} = 3$$

$$e_1 = \tfrac{1}{3} v_1 = \begin{bmatrix} 2/3 \\ 0 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$u_2 = v_2 - P_{e_1} v_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \underbrace{\underbrace{(e_1 \cdot v_2)}_{2+\frac{2}{3}+\frac{1}{3}}}_{3} \begin{bmatrix} 2/3 \\ 0 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\|u_2\| = \sqrt{1^2+1^2+1^2+(-1)^2+0^2} = \sqrt{4} = 2$$

$$e_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\text{ON Basis: } \left\{ \begin{bmatrix} 2/3 \\ 0 \\ -2/3 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \\ 0 \end{bmatrix} \right\}$$

Problem 7 (8 pts):

(a) (4pts) Calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\det A = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= -1 \cdot 1 \cdot (-1) \cdot 4$$

$$= 4$$

(b) (4pts) Determine a value of k which ensures the following matrix has $\det(A) = 0$:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 5 & 3 & k & 3 \end{bmatrix}$$

If $K = 5$ the first + third columns are the same

So the columns are linearly ~~[illegible]~~ dependent So

$\det(A) = 0$.

Or by ~~[illegible]~~ row operations

$$\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 5 & 3 & k & 3 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & k-5 & 0 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & k-5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= -(K-5) \cdot 5$$

So $-5(K-5) = 0$ when $\boxed{K = 5}$