1. (a) $2\begin{bmatrix}1\\2\\1\end{bmatrix}+3\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\4\\2\end{bmatrix}+\begin{bmatrix}-3\\3\\0\end{bmatrix}=\begin{bmatrix}-1\\7\\2\end{bmatrix}$

(b) $\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix}\begin{bmatrix}-2\\14\\4\end{bmatrix}$: ~~linearly independent~~

$\begin{bmatrix}-2\\14\\4\end{bmatrix}=4\begin{bmatrix}1\\2\\1\end{bmatrix}+6\begin{bmatrix}-1\\1\\0\end{bmatrix}$ So the linear combination forms a plane : $\boxed{B}$

(c) $\begin{bmatrix}1\\2\\1\end{bmatrix}=-1\begin{bmatrix}-1\\-2\\-1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}3\\6\\3\end{bmatrix}$ So the linear combination forms a line : $\boxed{A}$

2. (a) $\begin{bmatrix}3\\0\\-1\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-6\\0\\2\end{bmatrix}$ So they are parallel $\boxed{B}$

(b) $\begin{bmatrix}1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\\1\end{bmatrix}=0$ So they are perpendicular $\boxed{A}$

(c) $\cos\theta=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}=\frac{1\cdot1+2\cdot(-2)+0\cdot2}{\sqrt{1^2+2^2+0^2}\sqrt{1^2+(-2)^2+2^2}}=-\frac{1}{\sqrt{5}}$

(d) $\begin{bmatrix}-1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}4\\1\\-2\end{bmatrix}=\begin{bmatrix}-1\times4+1\times1+0\times2\\0\times4+1\times1+1\times(-2)\end{bmatrix}=\begin{bmatrix}-3\\-1\end{bmatrix}$

(e) $\begin{bmatrix}1&3\\2&1\\0&-1\end{bmatrix}\begin{bmatrix}3\\1\\-2\end{bmatrix}$ impossible since the length of row & column doesn't match

3. (a) $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}=\begin{bmatrix}a&b&c\\g&h&i\\d&e&f\end{bmatrix}$ swap row 2 & 3

(b) $\begin{bmatrix}1&-2&1\\1&-2&2\\0&4&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

Contiue of 3

(c) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

(e) The system has a unique solution $\boxed{A}$

with nonzero diagonal

4. (a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has an inverse : upper triangular matrix with no zero on diagonal always has inverse $\boxed{A}$

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ has no inverse : not full-rank is upper triangular and one diagonal entry is 0 $\boxed{B}$

(c) $A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} -1 & 2 \\ 2 & 1 \end{matrix}\right]$ $\left[\begin{matrix} 1 & 0 \\ 2 & 1 \end{matrix} \middle| \begin{matrix} -1 & 2 \\ 0 & 5 \end{matrix}\right]$ $\left[\begin{matrix} 1 & 0 \\ \frac{2}{5} & \frac{1}{5} \end{matrix} \middle| \begin{matrix} -1 & 2 \\ 0 & 1 \end{matrix}\right] \to \left[\begin{matrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{matrix} \middle| \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}\right]$

$\to \left[\begin{matrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right]$

So the inverse $A^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. (a) $b = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ $Ux = L^{-1}b$ $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$x + y + z = -1$
$y + z = 3$
$0 = 0$

$L^{-1}b = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

It has Infinitely many solutions $\boxed{B}$

(b) $b = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

The solution is

$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $LUx = b \Leftrightarrow Ux = L^{-1}b$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

$x + y + z = 2$ $\quad x = 2 - y - z \to x = 2 + 2 - 1 = 3$
$-y = 2 \to y = -2$
$3z = 3 \to z = 1$

$\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$