Practice Midterm 2

Math 22A, Fall 2019

Name: Solutions

Student ID: ______________________________

You do not need to simplify numerical expressions for your final answers (e.g. you can write $3 - \frac{3}{4}$ instead of simplifying to $\frac{9}{4}$.)

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||\,||\mathbf{w}||} = \cos\theta$$

The formula for the projection matrix is

$$P = A(A^TA)^{-1}A^T$$

Problem 1 (pts): Determine whether each of the following sets is a subspace of P_2 the polynomials of degree 2. If it is a subspace, prove it is closed under scalar multiplication and addition. If it is not a subspace give an example showing it is not closed under one of the two operations.

(a) The set of polynomials of the form $p(t) = at^2$ where a is in $\mathbb{R}$.

Closed under scalar multiplication:

If $p(t) = at^2$ then $cp(t) = c(at^2) = (c\cdot a)t^2$

$c\cdot a$ is in $\mathbb{R}$ so $cp(t)$ has the right form.

Closed under addition:

If $p_1(t) = a_1t^2$ and $p_2(t) = a_2t^2$

then $p_1(t) + p_2(t) = a_1t^2 + a_2t^2 = (a_1+a_2)t^2$

Since $a_1 + a_2$ is in $\mathbb{R}$ $p_1(t) + p_2(t)$ has the right form.

Thus this set is a subspace

(b) The set of polynomials of the form $p(t) = t^2 + a$ where a is in $\mathbb{R}$.

This set is not a subspace because it is not closed under ~~addition~~ Scalar multiplication

For example $p(t) = t^2 + 1$ has the required form

but $2\cdot p(t) = 2t^2 + 2$ does not have the required form

should be a 1

Problem 2 (pts):

(a) Find the value of k for which the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -1 & 2 \\ 1 & 1 & k \end{bmatrix}$$

has rank 2.

$$4\begin{bmatrix}1\\0\\1\end{bmatrix} - 2\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\2\end{bmatrix}$$

So if $\boxed{K=2}$ the 3 columns will be linearly dependent

but $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ are linearly independent

since $c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \iff \begin{matrix} c_1 = 0 \\ -c_2 = 0 \\ c_1 + c_2 = 0 \end{matrix} \iff c_1 = c_2 = 0.$

(b) Give an example of a matrix whose column space contains $(1, 2, 5)$ and $(0, 4, 1)$ and whose null space contains $(1, -1, 2)$.

in column space, in null space

$$\begin{bmatrix} 1 & 0 & a \\ 2 & 4 & b \\ 5 & 1 & c \end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 1+2a \\ -2+2b \\ 4+2c \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \begin{matrix} a = -1/2 \\ b = 1 \\ c = -2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 2 & 4 & 1 \\ 5 & 1 & -2 \end{bmatrix}$$

Problem 3 (pts): State whether the following vectors are linearly independent or dependent. If they are linearly independent, prove it. If they are linearly dependent, give coefficients c_1, c_2, c_3, c_4 such that
$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

These vectors are linearly dependent.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} c_1 + c_2 + c_4 = 0 \\ c_1 + c_3 + 2c_4 = 0 \\ c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 = -c_4 \\ c_3 = -c_1 - 2c_4 = -c_4 \\ c_2 = 0 \end{cases}$$

c_4 is a free choice

Pick $c_4 = 1$ then $c_3 = -1$ $c_2 = 0$ $c_1 = -1$

$$-1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 4 (pts): Find bases for the null space $N(A)$ and the column space $C(A)$.

pivots

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 1 & 3 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

free

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{aligned} x_1 + 3x_2 &= 0 \\ x_3 &= 0 \\ 4x_4 &= 0 \end{aligned}$$

$$\Leftrightarrow \begin{aligned} x_1 &= -3x_2 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$N(A) = \left\{ \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

basis for $N(A)$: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$C(A) = \text{Span}\left(\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}}_{\text{linearly dependent}} \right)$$

linearly dependent b/c $(-3)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} + (0)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

remove

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right)$$

Since there are 3 pivots a basis for $C(A)$ should have 3 vectors

Basis for $C(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Problem 5 (pts):

1. Amongst the following subspaces, specify all pairs which are orthogonal to each other.

$$U_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right), \quad U_2 = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right), \quad U_3 = \text{Span}\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}\right),$$

$$U_4 = \text{Span}\left(\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}\right), \quad U_5 = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\right), \quad U_6 = \text{Span}\left(\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}\right)$$

U_1 and U_2 , U_1 and U_3 , U_1 and U_5 , U_1 and U_6

U_3 and U_6 , U_4 and U_6

2. Calculate the projection matrix which projects vectors onto the following subspace

$$U = \text{Span}\left(\begin{bmatrix} 2/3 \\ 1/3 \\ 0 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \\ 0 \end{bmatrix}\right)$$

These vectors are orthonormal.

$\Downarrow$

$$A = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 2/3 \\ 0 & 2/3 \\ 2/3 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{So} \quad (A^TA)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T = AA^T = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 2/3 \\ 0 & 2/3 \\ 2/3 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 & 0 & 2/3 \\ -1/3 & 2/3 & 2/3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 & 0 & -2/9 & 4/9 \\ 0 & 5/9 & 4/9 & 2/9 \\ -2/9 & 4/9 & 4/9 & 0 \\ 4/9 & 2/9 & 0 & 4/9 \end{bmatrix}$$

Problem 6 (pts): Let

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 2 & 6 \\ 4 & -1 & -3 \end{bmatrix}$$

Find an *orthonormal basis* for the column space of A.

Basis for $C(A)$: $\left\{ \underbrace{\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{v_2} \right\}$ because $\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ so linearly dependent

linearly independent

$$\|v_1\| = \sqrt{3^2+0^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$e_1 = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$$

$$u_2 = v_2 - P_{e_1} v_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \underbrace{(e_1 \cdot v_2)}_{\frac{3}{5}-\frac{4}{5} = -1/5} \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 3/25 \\ 0 \\ 4/25 \end{bmatrix} = \begin{bmatrix} 28/25 \\ 2 \\ -21/25 \end{bmatrix}$$

$$\|u_2\| = \sqrt{\frac{28^2}{25^2}+2^2+\frac{21^2}{25^2}}$$

$$e_2 = \begin{bmatrix} \dfrac{28}{25\sqrt{\frac{28^2}{25^2}+2^2+\frac{21^2}{25^2}}} \\[2ex] \dfrac{2}{\sqrt{28^2/25^2+2^2+21^2/25^2}} \\[2ex] \dfrac{-21}{25\sqrt{\frac{28^2}{25^2}+2^2+\frac{21^2}{25^2}}} \end{bmatrix}$$

Problem 7 (pts):

1. Answer whether each of the following statements is true or false:

   (a) The determinant of $I + A$ is $1 + \det(A)$. (True) / (False)

   (b) The determinant of ABC is $\det(A)\det(B)\det(C)$. (True) / (False)

   (c) The determinant of $4A$ is $4\det(A)$. (True) / (False)

2. Determine the value of k which ensures the following matrix has $\det(A) = 5$:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k-2 \end{bmatrix}$$

$$\det(A) = -\det\left(\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & k-2 \end{bmatrix}\right) = -(-1)(k-2) = k-2$$

So $\det(A) = 5$ when $k-2=5$

So $\boxed{k=7}$