1. (a) $C(A) = \text{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$ by choosing 1st and 3rd column $\dim(C(A)) = 2$

$N(A) = \text{span}\left\{ \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}5\\-2\\0\\1\end{pmatrix} \right\}$ by choosing 3rd and 4th as free variable $\dim(N(A)) = 2$

(b) take any $\vec{b}$ such that $b_3 - b_1 \neq 2(b_2 - b_1)$ e.g. $\vec{b} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

(c) Special solution $\begin{pmatrix}2\\0\\0\\1\end{pmatrix}$ by choosing 2nd & 3rd as free variable

General solution $\begin{pmatrix}2\\0\\0\\1\end{pmatrix} + t_1 \begin{pmatrix}1\\-1\\1\\0\end{pmatrix} + t_2 \begin{pmatrix}5\\-2\\0\\1\end{pmatrix}$

2. (a) $C(A) = \text{span}\left\{ \begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\3\\1\\-1\end{pmatrix} \right\}$ by choosing 1st 2nd 4th column $\dim(C(A)) = 3$

$N(A) = \text{span}\left\{ \begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix} \right\}$ by taking 3rd 5th as free variable $\dim(N(A)) = 2$
(2nd 4th must be 0)

(b) take any $\vec{b}$ s.t. $b_4 \neq b_1$ e.g. $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

(c) Special solution $\begin{pmatrix}3\\3\\0\\-1\\0\end{pmatrix}$ by taking 3rd & 5th as free variable

General solution $\begin{pmatrix}3\\3\\0\\-1\\0\end{pmatrix} + t_1 \begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix} + t_2 \begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}$

3. (a)
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & 2 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ -2 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ -2 & 1 & 0 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 3 \end{array}\right]$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) $A^{-1} = \begin{bmatrix} \frac{13}{6} & \frac{7}{6} & -\frac{1}{6} \\ -2 & -1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$ (from form of U we know A is invertible)

(c) $\vec{x} = A^{-1}b = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$

4. (a) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 3 & 3 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 3 & 1 \\ -3 & 0 & 1 & 0 & 3 & -1 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 3 & 1 \\ -4 & -1 & 1 & 0 & 0 & -2 \end{array}\right]$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} -3 & -1 & 1 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 2 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

(c) $x = A^{-1}b = \begin{bmatrix} -7 \\ -\frac{5}{6} \\ \frac{9}{2} \end{bmatrix}$

5. (a) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -4 & -2 & -1 \\ 0 & 0 & 1 & 4 & 2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 & 0 & 4 \end{array}\right]$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

(b) A^{-1} does not exist since U is not full-rank

(c) Special solution: $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

All solution : $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

6. (a) dim=2 plane (b) dim=2 plane (c) dim=1 line

(d) dim = 1 line (e) dim = 2 plane

(f) dim = 3 all of $\mathbb{R}^3$ (g) dim=0 point