Chapter 6 Practice Problems:

#1. $A - \lambda I = \begin{bmatrix} 4-\lambda & 5 \\ -1 & -2-\lambda \end{bmatrix}$, $\det(A-\lambda I) = (4-\lambda)(-2-\lambda) + 5$

$= \lambda^2 - 2\lambda - 3$

Set $\det(A-\lambda I) = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0 \Rightarrow \lambda = 3$ or -1

$\lambda = 3$: $A - 3I = \begin{bmatrix} 1 & 5 \\ -1 & -5 \end{bmatrix} \Rightarrow$ eigenvector $= \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

$\lambda = -1$ $A + I = \begin{bmatrix} 5 & 5 \\ -1 & -1 \end{bmatrix} \Rightarrow$ eigenvector $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

#2. $\det(A-\lambda I) = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2.$

$A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow$ eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Not diagonalizable.

Eigenvector is calculating $\vec{x}$ such that $(A-\lambda I)\vec{x} = \vec{0}$

#3. $\det(A-\lambda I) = \det\begin{bmatrix} -3-\lambda & 0 & 0 \\ -8 & 1-\lambda & 0 \\ 16 & -8 & -3-\lambda \end{bmatrix} = (-3-\lambda)^2(1-\lambda) = 0$

$\Rightarrow \lambda = -3, \quad \lambda = 1.$

$\lambda = -3. \qquad A+3I = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 4 & 0 \\ 16 & -8 & 0 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1 \qquad A - I = \begin{bmatrix} -4 & 0 & 0 \\ -8 & 0 & 0 \\ 16 & -8 & -4 \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3 & & \\ & -3 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1}$$

#4. $\det(A-\lambda I) = \lambda^2 - 6\lambda + 10. = 0 \Rightarrow \lambda = 3 \pm i$

$\lambda = 3+i \qquad A-(3+i)I = \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$

$\lambda = 3-i \qquad A-(3-i)I = \begin{bmatrix} -1+i & -1 \\ 2 & 1+i \end{bmatrix} \Rightarrow \text{eigenvector} = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 1 \\ 1+i & -1+i \end{bmatrix} \begin{bmatrix} 3+i & \\ & 3-i \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1+i & -1+i \end{bmatrix}^{-1}$$

#5. $\det(A-\lambda I) = \lambda^2 - 3\lambda - 10 = 0 \Rightarrow \lambda = 5$ or -2

$\lambda = 5.$ $A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\lambda = -2$ $A + 2I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 5 & \\ & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}^{-1}$$

#6. $\det(A-\lambda I) = \lambda^2 - 2\lambda + 5 = (\lambda-1)^2 + 4 = 0 \Rightarrow \lambda = 1 \pm 2i$

$\lambda = 1+2i$ $A - (1+2i)I = \begin{bmatrix} 2-2i & 4 \\ -2 & -2-2i \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 1 \\ -\frac{1}{2}+\frac{1}{2}i \end{bmatrix}$

$\lambda = 1-2i$ $A - (1-2i)I = \begin{bmatrix} 2+2i & 4 \\ -2 & -2+2i \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1 \\ -\frac{1}{2}-\frac{1}{2}i \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2}+\frac{1}{2}i & -\frac{1}{2}-\frac{1}{2}i \end{bmatrix} \begin{bmatrix} 1+2i & \\ & 1-2i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\frac{1}{2}+\frac{1}{2}i & -\frac{1}{2}-\frac{1}{2}i \end{bmatrix}^{-1}$$

#7. $\det(A-\lambda I) = \det\begin{bmatrix} -5-\lambda & 0 & -4 \\ 20 & 2-\lambda & 11 \\ 6 & 0 & 5-\lambda \end{bmatrix} = (2-\lambda)(-1)^{2+2}\begin{vmatrix} -5-\lambda & -4 \\ 6 & 5-\lambda \end{vmatrix}$

$= (2-\lambda)\left[(-5-\lambda)(5-\lambda)+24\right] = (2-\lambda)(\lambda^2-1) = 0$

$= (2-\lambda)(\lambda+1)(\lambda-1) = 0$

$\Rightarrow \lambda = 2.\ \lambda = 1,\ \lambda = -1$

$\lambda = 2 \quad A-2I = \begin{bmatrix} -7 & 0 & -4 \\ 20 & 0 & 11 \\ 6 & 0 & 3 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1 \quad A-I = \begin{bmatrix} -6 & 0 & -4 \\ 20 & 1 & 11 \\ 6 & 0 & 4 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}$

~~Not diagonalizable~~. $\lambda = -1 \quad A+I = \begin{bmatrix} -4 & 0 & -4 \\ 20 & 3 & 11 \\ 6 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} -4 & 0 & 4 \\ 0 & 3 & -9 \\ 0 & 0 & 0 \end{bmatrix} \begin{cases} -4x+4z=0 \\ 3y-9z=0 \end{cases} \to x = z,\ y = 3z$

$V_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

#8. $\det.(A-\lambda I) = -(\lambda+2)(\lambda-1)^2 = 0 \Rightarrow \lambda = -2, 1$

$\lambda = -2 \quad A+2I = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ -1 & -14 & 0 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1 \quad A-I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & -14 & -3 \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}$

Not diagonalizable.

#9. $\det(A-\lambda I) = (3-\lambda)(\lambda^2+1) = 0 \Rightarrow \lambda = 3, \pm i$

$\lambda = 3.$ $A - 3I = \begin{bmatrix} -3 & 0 & -1 \\ 9 & 0 & 3 \\ 1 & 0 & -3 \end{bmatrix} \Rightarrow V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = i$ $A - iI = \begin{bmatrix} -i & 0 & -1 \\ 9 & 3-i & 3 \\ 1 & 0 & -i \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} i \\ -3i \\ 1 \end{bmatrix}$

$\lambda = -i$ $A + iI = \begin{bmatrix} i & 0 & -1 \\ 9 & 3+i & 3 \\ 1 & 0 & i \end{bmatrix} \Rightarrow V_3 = \begin{bmatrix} -i \\ 3i \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & i & -i \\ 1 & -3i & 3i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & & \\ & i & \\ & & -i \end{bmatrix} \begin{bmatrix} 0 & i & -i \\ 1 & -3i & 3i \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

#10. (a). $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, F.

(b). $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, F.

(c). T.

(d). $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ F.

(e). $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ F.

(f). $\begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}$ F.

(g). $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ F.

(h). T

(i) T

(j) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ F

(k) T

(l) T

(m) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ F

(n) T

(o) T

(p). $\begin{bmatrix} 4 & 5 \\ -1 & -2 \end{bmatrix}$ F.

#11. $\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & -2 \\ c & 1-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 5 + 2c$

one real eigenvalue $\Rightarrow (-6)^2 - 4(5+2c) = 0 \Rightarrow c = 2.$

$\Rightarrow \det(A-\lambda I) = \lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = 3.$

$A - 3I = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Not diagonalizable.

#12. $\det(A-\lambda I) = \det\begin{bmatrix} 3-\lambda & 3 \\ c & 5-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 15 - 3c.$

Two distinct eigenvalue : $(-8)^2 - 4(15-3c) > 0 \Rightarrow c > -\frac{1}{3}.$

one real eigenvalue : $(-8)^2 - 4(15-3c) = 0 \Rightarrow c = -\frac{1}{3}$

two complex eigenvalue : $(-8)^2 - 4(15-3c) < 0 \Rightarrow c < -\frac{1}{3}$

$c = 0. \Rightarrow \det(A-\lambda I) = \lambda^2 - 8\lambda + 15 = 0 \Rightarrow \lambda = 3$ or 5

$\lambda = 3.$ $A - 3I = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Diagonalizable.

$\lambda = 5.$ $A - 5I = \begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$c = -2. \Rightarrow \det(A-\lambda I) = \lambda^2 - 8\lambda + 21 \Rightarrow \lambda = 4 \pm \sqrt{5}i$

$\lambda = 4+\sqrt{5}i$ $A - (4+\sqrt{5}i)I = \begin{bmatrix} -1-\sqrt{5}i & 3 \\ -2 & 1-\sqrt{5}i \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} \frac{1-\sqrt{5}i}{2} \\ 1 \end{bmatrix}$

$\lambda = 4-\sqrt{5}i$ $A - (4-\sqrt{5}i)I = \begin{bmatrix} -1+\sqrt{5}i & 3 \\ -2 & +1+\sqrt{5}i \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} \frac{1+\sqrt{5}i}{2} \\ 1 \end{bmatrix}$ Diagonalizable.