Drinfeld-Sokolov Flows and the Local Geometric Langlands Correspondence

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Abstract

We discuss and compare various strategies to give meaning to the flows of the Drinfeld-Sokolov integrable hierarchies within the context of the local geometric Langlands correspondence.

1 Introduction

The notion of oper plays an important role in two distinct parts of mathematics. First, in the Lie theoretic generalizations of the KdV integrable hierarchy by Drinfeld and Sokolov in [8] and second in the geometric Langlands correspondence, see for example [3], [4]. Within the integrable system context these opers flow along the time variables of the hierarchy. In the present work we describe an interpretation of this dynamical aspect of opers within the geometric Langlands program.

Let $\mathfrak{g}^\circ$ be a simple complex Lie algebra and fix a description of the formal punctured disc as $D^\times = \text{Spec } \mathbb{C}((x))$ for an indeterminate x. The local geometric Langlands parameters associated to the loop group of $\mathfrak{g}^\circ$ are given by the space $\text{Conn}_{\mathfrak{g}^\circ}(D^\times)$ of $\mathfrak{g}^\circ$ local systems on $D^\times$. A candidate for the local geometric Langlands correspondence is constructed by Frenkel and Gaitsgory in [11]. This construction aims to attach to each Langlands parameter a categorical loop group representation by exploiting the Feigin-Frenkel isomorphism [9]. Consequently, one does not work directly with $\text{Conn}_{\mathfrak{g}^\circ}(D^\times)$ but rather with the closely related space $\text{Op}_{\mathfrak{g}^\circ}(D^\times)$ of $\mathfrak{g}^\circ$-opers on the disc $D^\times$. There is a forgetful map $\text{Op}_{\mathfrak{g}^\circ}(D^\times) \to \text{Conn}_{\mathfrak{g}^\circ}(D^\times)$ from opers to Langlands parameters and the strategy in [11] is to attach loop group representations to each oper in a way that hopefully descends to a true Langlands correspondence for the $\mathfrak{g}^\circ$ local systems themselves. Though it is not yet known if all opers with isomorphic underlying connections indeed yield isomorphic categorical loop group representations, it is at least known by work of Frenkel and Zhu [14], see also a global-to-local proof by Arinkin [2], that the forgetful map is surjective and hence gives a genuine candidate construction for each Langlands parameter. In this manner opers on a disc play a crucial role in the local geometric Langlands correspondence.

The way such opers can be used to describe integrable hierarchies of generalized KdV type involves a second indeterminate, say z, in addition to the coordinate x on the disc $D^\times$. One can define the space $\text{Aff-Op}_{\mathfrak{g}^\circ}(D^\times)$ of affine opers on the disc that recovers $\text{Op}_{\mathfrak{g}^\circ}(D^\times)$ when setting the spectral parameter z to 0. The opers in $\text{Aff-Op}_{\mathfrak{g}^\circ}(D^\times)$ are the Lax operators of the Drinfeld-Sokolov hierarchy and hence flow along the time variables. A natural way to look for Langlands theoretic meaning of Drinfeld-Sokolov flows is via the composition of the two previously described maps:

$$\text{Aff-Op}_{\mathfrak{g}^\circ}(D^\times) \to \text{Op}_{\mathfrak{g}^\circ}(D^\times) \to \text{Conn}_{\mathfrak{g}^\circ}(D^\times)$$

However, as described by Frenkel and Gaitsgory in [11] (Section 4) the flows do not change the isomorphism class of the underlying connection. Hence, a different approach to incorporate the flows into the Langlands correspondence in an interesting manner is needed.

Our strategy in the present work has two aspects. We first switch from a description of the Drinfeld-Sokolov phase space in terms of opers to a description in terms of the infinite-dimensional Sato Grassmannian. It turns out that in this manner one obtains a clear Langlands theoretic interpretation of the flows in terms of the ramification of the Langlands parameter. The second aspect of our strategy is that the above Grassmannian notion of flows on $\text{Conn}_{\mathfrak{g}^\circ}(D^\times)$ *can* be related to the usual flows on opers, but not directly via the forgetful map $\mathrm{Op}_{\mathfrak{g}^\circ}(D^\times) \to \mathrm{Conn}_{\mathfrak{g}^\circ}(D^\times)$. Rather, there is an intermediate step that is a central aspect of the construction of the integrable hierarchy. Each oper can be conjugated to a differential operator with values in the principal Heisenberg algebra of the affine Lie algebra of $\mathfrak{g}^\circ$. Out of these Heisenberg algebra elements one can define a very natural map

$$\text{Aff-Op}_{\mathfrak{g}^\circ}(D^\times) \to \mathrm{Conn}_{\mathfrak{g}^\circ}(D^\times)$$

Letting this construction flow along the Drinfeld-Sokolov times gives another notion of flow on $\mathrm{Conn}_{\mathfrak{g}^\circ}(D^\times)$ in addition to the Grassmannian perspective. In Theorem 1 we show that the two notions correspond to each other in a precise sense whenever a comparison makes sense.

Having defined meaningful Drinfeld-Sokolov flows on $\mathrm{Conn}_{\mathfrak{g}^\circ}(D^\times)$ we then investigate their Langlands theoretic meaning. We show that a nice answer exists when restricting to what we call the locus of Heisenberg connections (for Lie algebras of type A one can use classical results of Levelt and Turrittin to show that this locus is large). Turning on more and more Drinfeld-Sokolov times corresponds on this locus exactly to increasing the ramification of the Langlands parameter.

2 Flows on connections

As in the introduction let $\mathfrak{g}^\circ$ denote a simple complex Lie algebra and we denote by $\mathfrak{g}$ the corresponding untwisted affine Lie algebra. The space of local geometric Langlands parameters on a formal punctured disc $D_t^\times = \mathrm{Spec}\ \mathbb{C}(\!(t)\!)$ is the space of connections of the form

$$\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t^\times) = \left\{ \partial_t + A \,\middle|\, A \in \mathfrak{g}^\circ(\!(t)\!) \right\}$$

Our aim is to obtain for each element of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$ a notion of letting the connection flow along the flows of the integrable hierarchy associated to $\mathfrak{g}$ by Drinfeld and Sokolov in [8]. We start by recalling some basic features of this hierarchy that are important for our subsequent constructions.

A crucial role is played by the principal Heisenberg subalgebras of $\mathfrak{g}$ and we now recall their basic properties and their role in defining Lax equations on opers. Let l be the rank of $\mathfrak{g}^\circ$ and fix Chevalley generators e_i, f_i $(1 \le i \le l)$ of $\mathfrak{g}^\circ$. Let z be an indeterminate and fix an isomorphism $\mathfrak{g} \cong \mathfrak{g}^\circ[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}\mathfrak{d}$ where c is a non-zero central term and $\mathfrak{d}$ a scaling element. Let $\mathrm{E}(\mathfrak{g})$ denote the multi-set of exponents of $\mathfrak{g}$ and let $\mathrm{E}(\mathfrak{g})^{>0}$ and $\mathrm{E}(\mathfrak{g})^{<0}$ be the multi-sets of positive and negative exponents, respectively. Let E_0 be a non-zero vector in the lowest root space of $\mathfrak{g}^\circ$ and let $e_0 = z \cdot E_0$. The cyclic element in the sense of Kostant [20] associated to our Lie theoretic choices is the element of $\mathfrak{g}$ given by

$$\Lambda_z = \sum_{i=0}^{l} e_i$$

The centralizer of Λ_z in $\mathfrak{g}$ is the principal Heisenberg algebra associated to our choice of cyclic element. For our purposes it is often useful to work with the centralizer within the loop part of $\mathfrak{g}$ and so we define

$$\mathfrak{h}_{\mathrm{pri}} := \mathrm{Cent}_{\mathfrak{g}^\circ[z,z^{-1}]}(\Lambda_z)$$

It is known that $\mathfrak{h}_{\mathrm{pri}}$ has basis elements Λ_i where i ranges through the multi-set of exponents of $\mathfrak{g}$ and Λ_i is of degree i with respect to the principal gradation on $\mathfrak{g}$. This gradation is obtained by putting all the positive Chevalley generators e_i of $\mathfrak{g}^\circ$ in degree 1 and the f_i's in degree -1. Multiplication by z raises the degree by the Coxeter number h. So in particular Λ_z is homogeneous of degree 1 since E_0 is of degree $1-h$ and one can always assume $\Lambda_1 = \Lambda_z$.

The Drinfeld-Sokolov hierarchy associated to $\mathfrak{g}$ can be formulated in terms of letting certain type of differential operators flow via Lax equations corresponding to elements of $\mathfrak{h}_{\mathrm{pri}}$. We start by describing the differential operators. To do so fix yet another indeterminate (in addition to the variables z and t that we already introduced), let us denote it by x. Let D_x be the formal disc with coordinate x. The space $\mathrm{Op}_{\mathfrak{g}^\circ}(D_x)$ of opers on D_x consists of operators of the form

$$\partial_x + \sum_{i=1}^{l} e_i + q(x)$$

where $q(x)$ is an element of $\mathfrak{g}^\circ$ that is non-positively graded with respect to the principal gradation and the x dependency is of $\mathbb{C}[\![x]\!]$ type. The gauge transformations are given by $\exp(\mathrm{ad}\ N)$ for N an element of $\mathfrak{g}^{\circ,<0}[\![x]\!]$ where the superscript indicates negatively graded elements with respect to the principal gradation. The variant $\text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ of $\text{Op}_{\mathfrak{g}^\circ}(D_x)$ that occurs for Drinfeld-Sokolov hierarchies involves a second parameter z, the spectral parameter. The elements of $\text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ are operators of the form

$$\partial_x + \Lambda_z + q(x)$$

where $q(x)$ is as before. On this space of opers Drinfeld and Sokolov define flows via a collection of Lax equations. Underlying this construction is the fact that one can conjugate an affine oper into the Heisenberg algebra in a suitable sense. Namely, for every element L of $\text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ there exists α in $\mathfrak{g}$, negatively graded with respect to the principal gradation and depending on x, such that

$$\exp(\mathrm{ad}\ \alpha)\ \mathrm{L} = \partial_x + \Lambda_z + \sum_{i \in \mathrm{E}(\mathfrak{g})^{<0}} H_i =: \partial_x + \mathrm{Heis}(\mathrm{L}) \tag{1}$$

with H_i in the i'th graded piece of the principal Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$. See [8], [17] for proofs. Note that while the initial definition of affine opers has very mild dependency on z (through the term $e_0 = zE_0$), the transform of L described by Equation (1) really employs the affine algebra $\mathfrak{g}$ in a very non-trivial manner. Fix H in $\mathfrak{h}_{\mathrm{pri}}$ that is positively graded with respect to the principal gradation. One can then define a flow (with time variable denoted by t, say) on $\text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ via

$$\partial_t\ \mathrm{L} = \left[(\tilde{H})_+, \mathrm{L}\right] \tag{2}$$

where $\tilde{H}$ is a suitable dressed version of H whose definition involves the α from Equation 1 and the $+$ subscript denotes the non-negative part with respect to the principal gradation. See for example [17] (Proposition 3.1). The standard choice for these flows is to let H be of the form Λ_i where i is a positive exponent and note that it is known that all these flows commute. The Drinfeld-Sokolov flows are then parameterized by the multi-set $\mathrm{E}(\mathfrak{g})^{>0}$ of positive exponents of $\mathfrak{g}$ and we denote by

$$\mathbf{t} = \{t_1, \cdots\} = \{t_i \mid i \in \mathrm{E}(\mathfrak{g})^{>0}\}$$

the corresponding collection of Drinfeld-Sokolov time variables.

A natural attempt to use the constructions of Drinfeld and Sokolov is to identify the disc D_x of the integrable system with the disc D_t of the Langlands correspondence and letting $z = 0$ pass to the underlying element of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t^\times)$. However, as indicated in the introduction, this does not yield an interesting theory. We therefore switch now to a description of the Drinfeld-Sokolov phase space in terms of points of an infinite-dimensional Grassmannian.

2.1 Grassmannian perspective

The definition of the Sato Grassmannian involves (at least implicitly) a punctured disc. As an example, the algebro-gemetric locus of the Grassmanian is obtained via the Krichever map from a complex algebraic curve with extra data when restricted to a disc on the curve, see for example the survey [22]. For our constructions it is most useful to identify this Grassmannian disc with the disc $D_t^\times$ of the local Langlands correspondence. To follow standard coordinate conventions for the Sato Grassmannian, we denote the coordinate on $D_t^\times$ by $1/z$ for some indeterminate z. Let $\mathcal{H} = \mathbb{C}(\!(1/z)\!)$ and $\mathcal{H}^+ = \mathbb{C}[z]$. For a positive integer n, the (index zero part of the big cell of the) vector Sato Grassmannian $Gr(n)$ is defined to be the set of complex subspaces of $\mathbb{C}(\!(1/z)\!)^n$ whose projection to $\mathbb{C}[z]^n$ is an isomorphism. To each point of $Gr(n)$ Cafasso and Wu associate in [5] a function in $\mathbb{C}[\![\mathbf{t}]\!]$ that gives a candidate of a Drinfeld-Sokolov tau function. More precisely, let $\mathfrak{g}$ be of type A, B, C, D, or G and realize $\mathfrak{g}^\circ$ via its first fundamental representation. Let L be an affine oper and let α be as in Equation (1). Cafasso and Wu show in [5] (Theorem 4.4) that (up to multiplication by an explicit non-zero scalar that is independent of L) the function associated to a point

$$V := \exp(\alpha)^{-1} \cdot \mathcal{H}^{+,n}$$

in $Gr(n)$ agrees with the tau function associated to L. This allows to describe the phase space of the Drinfeld-Sokolov hierarchy in terms of points in the Grassmannian instead of in terms of opers. We can now describe the action of the Drinfeld-Sokolov flows on points of the Sato Grassmannian, at least in the cases considered in [5], namely $\mathfrak{g}$ is of type A, B, C, D, or G and $\mathfrak{g}^\circ$ is realized via its first fundamental representation. Cafasso and Wu show that infinitesimally the flows act on a point V of $Gr(n)$ via

$$V(\mathbf{t}) = \exp\left(\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i\right) V \tag{3}$$

To account for the fact that this describes the flows infinitesimally one can treat the time variables t_i as formal nilpotent parameters (the degree of nilpotency will not be relevant for our subsequent considerations). Equation (3) yields a natural strategy to define of Drinfeld-Sokolov flows on the space $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t^\times)$ of connections.

Definition 1. For a connection ∇ in $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$ define

$$\begin{aligned} \nabla(\mathbf{t}) \quad &:= \quad \exp\left(\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i\right) \nabla \exp\left(-\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i\right) \\ &= \quad \exp(\ \mathrm{ad} \sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i)\ \nabla = \nabla + [\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i, \nabla] + \frac{[\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i, [\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} t_i \Lambda_i, \nabla]]}{2!} + \cdots \end{aligned}$$

where we identify the coordinate t on the disc $D^\times$ with $1/z$.

In Definition 3 we introduce the sub-locus of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t^\times)$ consisting of connections associated to elements in the Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$. It will follow from the known commutativity of the (loop part of the) Heisenberg algebra, see [8] (Proposition 5.16), that for such connections Definition 1 yields well defined flows even if the time variables are not assumed to be nilpotent but rather just usual complex parameters. In Theorem 2 we show that the so obtained flows have a very nice description and have clear Langlands theoretic meaning through their relation with the slope of the underlying connections (a rational number describing the ramification of the connection).

2.2 Heisenberg connections

As indicated above, we aim to attach to each element H of the principal Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$ a connection on the disc $D_t^\times$. A natural procedure to do so would be to associate to H the connection $\partial_z + H$. For us it is slightly more convenient to work with a variant of ∂_z. The derivation ∂_z has an analogue for all the different realizations (in the sense of [18]) of the affine Lie algebra $\mathfrak{g}$ and ∂_z corresponds to the standard loop or "homogeneous" realization. The Heisenberg algebra is intimately related to the principal realization and it will be more convenient to replace ∂_z by its analogue for this principal realization. We recall its definition now. Let $s = (s_0, \cdots, s_l) \in (\mathbb{Z}^{\geq 0})^{l+1}$ be a gradation vector, meaning not all s_i's are zero. Let $m_s = \sum_{i=0}^{l} a_i s_i$ where the a_i's are the Kac labels. We denote by $\mathfrak{g}(s)$ the realization of $\mathfrak{g}$ of type s as defined by Kac in [18]. The grading on $\mathfrak{g}$ associated to the gradation vector s satisfies for Chevalley generators e_i, f_i that $\deg e_i = s_i = -\deg f_i$ for i between 0 and l. For each i in $\mathbb{Z}$ one considers the derivation d_i^s of the derived algebra of $\mathfrak{g}(s)$ that satisfies for each X_j in the j'th graded piece with respect to the s gradation that $\mathrm{d}_i^s(X_j) = j \cdot X_{j+im_s}$ and $\left[\mathrm{d}_i^s, \mathrm{d}_j^s\right] = m_s \cdot (i-j) \cdot \mathrm{d}_{i+j}^s$ Then, see for example [23], one has

$$\mathrm{d}_i^s = \partial_z + \left[g_{i,s}, -\right]$$

for an element $g_{i,s}$ of $\mathfrak{g}(s)$. The homogeneous gradation corresponds to $s = (1, 0, \cdots, 0)$ and in this case $g_{i,s} = 0$. As indicated before we switch instead to the principal gradation and this corresponds to $s = (1, \cdots, 1)$. In this case $g_{i,s} = \rho^\vee/(hz)$ where $\rho^\vee$ in $\mathfrak{g}^\circ$ is half the sum of positive co-roots and h is the Coxeter number of $\mathfrak{g}$.

Note, see [18] for details, that the dimension of the subspace of $\mathfrak{h}_{\mathrm{pri}}$ of principal degree i (with i an exponent) is usually 1, the only exception is the case where $\mathfrak{g} \cong \mathfrak{so}_{2n}^{(1)}$ and $i \equiv n-1 \mod 2n-2$ in which case the dimension is 2. We deal with this multiplicity issue in the following way: Consider the standard $2n$-dimensional representation of $\mathfrak{so}_{2n}$ and let Λ_1 be a cyclic element. It is then known that apart from the case $i \equiv n-1 \mod 2n-2$ the degree i part of the Heisenberg algebra is spanned by $\Lambda_i = \Lambda_1^i$.

Definition 2. We call the span of the Λ_1^i's the standard part of $\mathfrak{h}_{\mathrm{pri}}$ and denote it by $\mathfrak{h}_{\mathrm{pri}}^{\mathrm{st}}$ (hence this equals $\mathfrak{h}_{\mathrm{pri}}$ except for Lie algebras of type D).

Definition 3. The space of Heisenberg connections on the formal punctured disc $D^\times = \mathrm{Spec}\ \mathbb{C}((t)) = \mathrm{Spec}\ \mathbb{C}((1/z))$ is given by

$$\mathrm{Conn}_{\mathfrak{g}^\circ}^{\mathrm{Heis}}(D^\times) := \left\{ (\mathbb{C}((1/z))^n, \mathrm{d}_{-1}^{(1,\cdots,1)} + H) \;\middle|\; H \text{ in } \mathfrak{h}_{\mathrm{pri}}^{\mathrm{st}} \right\}$$

We call $\mathbf{0} = \mathrm{d}_{-1}^{(1,\cdots,1)}$ the origin of the space of Heisenberg connections.

We obtain in Theorem 2 a good notion of Drinfeld-Sokolov flows on Heisenberg connections. While not strictly speaking necessary for this aim, it is useful to understand how restrictive a condition it is to restrict to the Heisenberg locus. It turns out that this locus of Heisenberg connections is relatively large within the space $\mathrm{Conn}_{\mathfrak{g}^\circ}(D^\times)$ of Langlands parameters. We now describe this for Lie algebras of type A using the classical results of Levelt and Turrittin. Let $n > 0$ and let f be an element of $\mathbb{C}((t^{1/n}))$ and suppose that f is not an element of $\mathbb{C}((t^{1/m}))$ for $1 \le m < n$. Decompose $f = f^{\mathrm{hol}} + f^{\mathrm{reg}} + f^{\mathrm{irreg}}$ where $f^{\mathrm{hol}} = \sum_{i>0} a_i t^{i/n}$, $f^{\mathrm{reg}} = a_0$, and $f^{\mathrm{irreg}} = \sum_{i<0} a_i t^{i/n}$. Define a connection on $D^\times$ by

$$\mathcal{E}(f,n) := (\mathbb{C}((t^{1/n})), \partial_t + f(t)/t)$$

with $f^{\mathrm{hol}} = 0$. It is known that $\mathcal{E}(f,n)$ is irreducible and f^{irreg} is uniquely determined by the isomorphism class of the connection, up to a coordinate change $t^{1/n} \mapsto \zeta_n t^{1/n}$ for an n'th root of unity ζ_n and f^{reg} is determined up to adding an element in $\mathbb{Z} \cdot (1/n)$. Define now a second type of connection on $D^\times$ by

$$\mathrm{Nil}_N := (\mathbb{C}((t))^N, \partial_t + (\sum_{i=1}^{N-1} e_{i,i+1})/t)$$

where $e_{i,i+1}$ is the matrix with zeros everywhere except a 1 at the $(i, i+1)$ entry. The classification result of Levelt and Turrittin is as follows. For every connection ∇ on $D^\times$ there exists $r \ge 1$ and f_i's, n_i's, and N_i's with

$$\nabla \cong \bigoplus_{i=1}^{r} (\mathcal{E}(f_i, n_i) \otimes \mathrm{Nil}_{N_i})$$

where the n_i's, and N_i's are unique up to permutation and the f_i's are unique up to adding elements in $\mathbb{Z} \cdot (1/n_i)$ and up to a substitution $t^{1/n_i} \mapsto \zeta_{n_i} \cdot t^{1/n_i}$ for a n_i'th root of unity ζ_{n_i}. We now relate this to Heisenberg connections for $\mathfrak{sl}_n^{(1)}$ (in the standard defining representation of $\mathfrak{sl}_n$). Let $\Lambda = \sum_{i=1}^{n-1} e_{i+1,i} + z \cdot e_{1,n}$ be a standard choice of cyclic element of $\mathfrak{sl}_n^{(1)}$. Suppose that the constant term of $f = \sum a_i t^{i/n}$ is given by $(1-n)/2$. Then

$$\mathcal{E}(f,n) \cong (\mathbb{C}((t))^n, \mathrm{d}_{-1}^{\mathrm{pri}} + \sum a_i \Lambda^i)$$

Hence, for suitably fixed regular singular term, all the basic building blocks $\mathcal{E}(f,n)$ of the Levelt-Turrittin normal form are given by Heisenberg connections.

2.3 Heisenberg form of opers

In the previous section we have given a notion of letting an element of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$ flow along the Drinfeld-Sokolov times $\mathbf{t}$ that was motivated by the description of the integrable hierarchy in terms of points in the Sato Grassmannian. We are of course free to define whatever we like to, but a natural question is to give some relation of these constructions with the flow description in terms of opers. In the present section we do so, by taking an intermediate step: Pass from an oper L in $\text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ to $\partial_x + \mathrm{Heis(L)}$ as in Equation (1). The latter differential operator has two variables, x and z (recall that $t = 1/z$ is the disc of the local Langlands correspondence). Our approach is drastic, but we justify it through our theorems: Instead of working with the differential operator above that has the "mixed" variables x and z, we consider (see Definition 4 below) the connection on $D_t^\times$ defined by Heis(L). Note that Heis(L) is an element of $\mathfrak{g}^\circ((1/z))$ but not in general of $\mathfrak{g}^\circ((z))$ (since it can have elements of arbitrary negative principal degree) and hence to get a meromorphic connection we should indeed work on the disc $D_t^\times$ rather than $D_z^\times$. We now describe all this in detail.

First note that the element Heis(L) of the Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$ occurring in Equation (1) is not unique in general. Nonetheless, Wu has shown in [24] how to gauge fix the degrees of freedom to arrive at a unique Heis(L) in such a manner that it can serve as the defining information for a good theory of tau functions (note that the tau function perspective on integrable hierarchies of generalized KdV type was not considered in the orginal work of Drinfeld and Sokolov [8]). Consider as before the decomposition $\mathfrak{g} \cong \mathfrak{g}^\circ[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}\mathfrak{d}$ for a central element c and let $\mathfrak{c}$ denote the projection from $\mathfrak{g}$ to $\mathbb{C}c$. The gauge fixing condition of [24] is that the element α occurring in Equation (1) has to satisfy an additional set of equations: For all positive exponents i of $\mathfrak{g}$ one requires

$$\mathfrak{c}(\exp(\mathrm{ad}\ \alpha)\Lambda_i) = 0 \tag{4}$$

That this condition is important might be surprising from the point of view of the original development of the Drinfeld-Sokolov hierarchy: In [8] Drinfeld and Sokolov work purely with the loop part $\mathfrak{g}^\circ[z, z^{-1}]$ of $\mathfrak{g}$. However, in [24] Wu reformulates the hierarchy in terms of the full affine algebra $\mathfrak{g}$ and shows that in this case Equation (4) is a good choice of fixing the element Heis(L). We assume from now on that Heis(L) denotes this specific Heisenberg description of the Lax operator and note that by [24] (Lemma 3.3) it only depends on the gauge equivalence class of the oper L. So one has a well defined process

$$\text{affine oper L} \quad \rightsquigarrow \quad \mathrm{Heis(L)} \in \mathfrak{h}_{\mathrm{pri}}$$

We call the resulting element of the Heisenberg algebra the Heisenberg form of L. As L flows along the Drinfeld-Sokolov times so does its Heisenberg form.

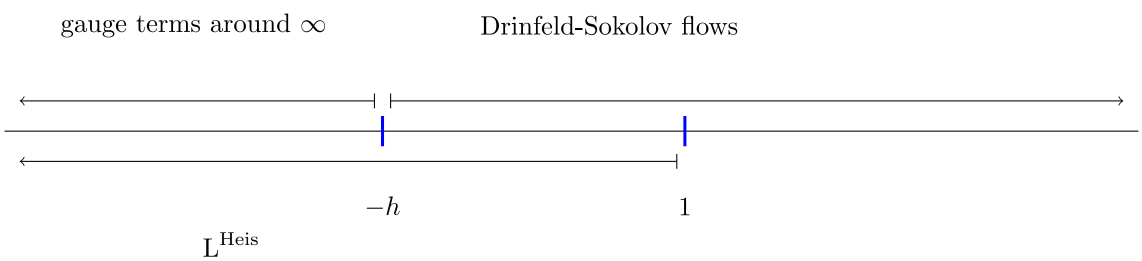
This leads to a second approach to define Drinfeld-Sokolov flows on elements of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$:

Definition 4. For an affine oper $\mathrm{L} \in \text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ define the element $\mathrm{L}^{\mathrm{Heis}}(\mathbf{t})$ of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$

$$\mathrm{L}^{\mathrm{Heis}}(\mathbf{t}) \quad := \quad \Big(\mathbb{C}((1/z))^n, \mathrm{d}_{-1}^{(1,\cdots,1)} + \mathrm{Heis(L)}\Big)$$

Note that $\mathrm{L}^{\mathrm{Heis}}(\mathbf{t})$ is a Heisenberg connection except that the dependency on $\mathbf{t}$ might necessitate to allow coefficients to be formal power series, at least for some choices of L. In the case relevant for Theorem 2 this will however not be an issue: We show that all coefficients depend polynomially on the time variables $\mathbf{t}$.

We have arrived at the following picture: Firstly, for every element of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$ we have a notion of letting it flow along $\mathbf{t}$ that is inspired by the Grassmannian approach to the Drinfeld-Sokolov hierarchy (though we have not discussed in what way this is well defined on gauge equivalence classes yet). Secondly, via the description of the integrable hierarchy in terms of opers we have another notion of flows for those Heisenberg connections of the form $\mathrm{L}^{\mathrm{Heis}}$ as in Definition 4. We show in Theorem 1 that these two notions can be related. As a first observation in this direction note that from Equation 1 one sees that Heis(L) has possibly non-zero terms only in principal degrees at most 1. In comparison, we show in Theorem 2 that in general for Heisenberg connections the terms in principal degree less than $-h$ can be gauged away and the terms in degree bigger than $-h$ correspond to the Drinfeld-Sokolov flows. Hence, to understand the relation of the flows $\mathrm{L}^{\mathrm{Heis}}(\mathbf{t})$ to the flows given in Definition 1 there is a finite range of principal degrees in which the two theories overlap. Namely, in the "critical range" of degrees i with $-h < i < 1$.



We show in Theorem 1 that in this critical range the flow orbit of the vacuum $\mathbf{0}$ of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t)$ can be described as the flow of a Heisenberg form of an oper (and this oper happens to play an important role in quantum field theory).

Theorem 1. *Let $\mathfrak{g}^\circ$ be a simple complex Lie algebra with Coxeter number h. There exists an affine oper L in $\text{Aff-Op}_{\mathfrak{g}^\circ}(D_x)$ such that there is an isomorphism*

$$\mathbf{0}(t_1, \cdots, t_{h-1}) \cong \mathrm{L}^{\mathrm{Heis}}(t_1, \cdots, t_{h-1})$$

of elements of $\mathrm{Conn}_{\mathfrak{g}^\circ}(D_t^\times)$.

In fact, Theorem 1 was obtained in full generality in [21]. We present here an alternative proof for Lie algebras of type A that shows its relation to a duality of conformal field theories and should be of independent interest.

The construction of the oper L in Theorem 1 involves the notion of the corresponding tau function which we now briefly recall. For each oper there exists a dressing operator Θ, see [24] for the precise definition, such that

$$\mathrm{L} = \Theta(\partial_x + \Lambda_z)\Theta^{-1} + \omega \cdot c$$

for a scalar function ω and a choice c of central element of the affine algebra $\mathfrak{g}$. The tau function $\tau(\mathbf{t})$ associated to L can be defined, up to multiplication by a non-zero scalar, via the collection of differential equations for each $i \in \mathrm{E}(\mathfrak{g})^{>0}$

$$\Theta\Lambda_i\Theta^{-1} = \tilde{\Lambda}_i - (\partial_{t_i} \log \tau) \cdot c$$

where $\tilde{\Lambda}_i$ has no central term. Wu has shown in [24] that this tau function is related to the Heisenberg form

$$\mathrm{Heis}(\mathrm{L}) = \Lambda + \sum_{i \in \mathrm{E}(\mathfrak{g})^{<0}} H_i$$

by

$$\partial_x \partial_{t_{|i|}} \log \tau(\mathbf{t}) = |i| \cdot H_i \tag{5}$$

Define now a differential operator $\mathcal{L}_{\mathfrak{g}}$ by

$$\mathcal{L}_{\mathfrak{g}} = \sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} \left(\frac{i+h}{h} t_{i+h} - \delta_{i,1} \right) \partial_{t_i} + \frac{1}{2h} \sum_{i,j \in \mathrm{E}(\mathfrak{g})^{>0}, i+j=h} ij t_i t_j \tag{6}$$

It is known, see [5], that there is a unique affine oper L_{WK} whose associated τ function satisfies

$$\mathcal{L}_{\mathfrak{g}}\ \tau(\mathbf{t}) = 0 \tag{7}$$

For example for $\mathfrak{g} = \mathfrak{sl}_2^{(1)}$ this equation defines the suitably normalized Witten-Kontsevich tau function of topological gravity. As an example, consider the unique tau function of the $\mathfrak{sl}_4^{(1)}$ Drinfeld-Sokolov hierarchy that satisfies Equation

(7). If all times except t_1, t_2, t_3 are turned off then it follows from [1] (Lemma 3.7) that

$$\log \tau(t_1, t_2, t_3) = -\frac{3}{8} \cdot t_1^2 t_3 - \frac{1}{2} \cdot t_1 t_2^2 - \frac{9}{16} \cdot t_2^2 t_3^2 - \frac{81}{1230} \cdot t_3^3$$

From Equation (5) one then obtains $H_{-1} = -\frac{3}{4}t_3$, $H_{-2} = -\frac{2}{4}t_2$, $H_{-3} = -\frac{1}{4}t_1$ and hence a clear pattern for H_i emerges. Using the $p-q$ duality [15] from quantum field theory we now prove the corresponding result for $\mathfrak{g}^\circ = \mathfrak{sl}_n$.

So from now on we assume that L is an element of $\text{Aff-Op}_{\mathfrak{sl}_n}(D_x)$. Drinfeld and Sokolov show in [8] (Proposition 3.20) a nice correspondence between a variant of the Heisenberg form Heis(L) and the *scalar* Lax operator

$$\mathrm{L}_{\mathrm{sc}} = \partial_x + \sum_{i<0} c_i(\mathbf{t})\partial_x^i$$

associated to L. We cannot directly use their result since, as indicated before, in [8] the integrable hierarchy is formulated purely in terms of the loop algebra part of $\mathfrak{g}$ whereas the definition of Heis(L) employs a reformulation of the hierarchy in terms of the central extension $\mathfrak{g}$. Nonetheless, as shown by Wu in [24] (Section 5) one obtains that for each negative exponent i

$$H_i = \frac{1}{i} \cdot \mathrm{Res}\ (\mathrm{L}_{\mathrm{sc}}^{|i|}) \cdot \Lambda^i \tag{8}$$

where the residue Res(−) of a pseudo-differential operator is defined to be the coefficient of ∂_x^{-1} and as before $\Lambda = \sum_{i=1}^{n-1} e_{i+1,i} + z \cdot e_{1,n}$ is a standard choice of cyclic element of $\mathfrak{sl}_n^{(1)}$.

We show how to compute the residues in Equation (8) by using the $p-q$ duality of two-dimensional quantum gravity. This is a duality of conformal field theories described from a mathematical point of view by Fukuma, Kawai, Nakayama in [15]. One can write

$$\mathrm{L}_{\mathrm{sc}}^n = x + \sum_{i=0}^{n} (n+1-i)\hat{t}_{n+1-i}\partial_x^{n-i}$$

for some functions $\hat{t}_i$ depending on $\mathbf{t}$. The notation $\hat{t}_i$ for the above coefficients indicates that under the duality described in [15] they can be interpreted to be Drinfeld-Sokolov times of a different integrable hierarchy. Relating these times with the original flows $\mathbf{t} = \{t_1, \cdots\}$ is a key result of [15]. Their relation is described in a purely algebraic manner. We assume for simplicity that $t_n = 0$ and $t_{n+1} = -n/(n+1)$ and hence, see [15] (Equation 4.6), one also has $\hat{t}_{n+1} = 1/(n+1)$. It is shown in [15] (Equation (4.13)) that for an indeterminate z the following two formal Laurent series with respect to $1/z$ are compositional inverses

$$\nu(z) := \sum_{i=0}^{\infty} \frac{(n+1-i)t_{n+1-i}}{-n} \cdot z^{1-i} \quad , \quad \xi(z) := \left(\sum_{i=0}^{\infty} (n+1-i)\hat{t}_{n+1-i} z^{n-i} \right)^{1/n}$$

Here the n'th root involved in the definition of $\xi(z)$ is chosen to give the asymptotic $\xi(z) = z + \text{lower order terms}$. Note that up to the substitution $z \rightsquigarrow \partial_x$ one sees that $\xi(z)^n$ almost equals $\mathrm{L}_{\mathrm{sc}}^n$. Since $\nu(z)$ and $\xi(z)$ are compositional inverses one can use Lagrange inversion to express the residues of powers of $\xi(z)$ in terms of the time variables t_i. A version of Lagrange inversion for two formal Laurent series can be described as follows, see for example [16] (Section 4). Suppose f and g are two Laurent series in $\mathbb{C}(\!(1/z)\!)$ such that the composition $f \circ g$ is well defined and such that $g = a_k z^k + a_{k-1}z^{k-1} + \cdots$ with $a_k \neq 0$. Then the formal residue satisfies

$$(-k) \cdot \mathrm{Res}\ f(z) = \mathrm{Res}\ ((f \circ g)(z) g'(z))$$

where the residue of an element of $\mathbb{C}(\!(1/z)\!)$ is simply the coefficient of z. Since ξ and ν are compositional inverses one can apply this result with $g = \nu$ and $f = \xi^i$ to obtain

$$\mathrm{Res}\ \xi(z)^i = \frac{i(n-i)t_{(n-i)}}{n}$$

One has (even for negative j) $\partial_x^j x = x\partial_x^j + \mathcal{O}(\partial_x^{j-1})$. Hence, since only the constant term of $\mathrm{L}_{\mathrm{sc}}^n$ depends on x and since that constant term equals $\hat{t}_1 + x$ one sees that for each $1 \le i \le n-1$ one has $\mathrm{Res}\, \mathrm{L}_{\mathrm{sc}}^i = \mathrm{Res}\, \xi(z)^i$ where the residue on the left-hand side is the coefficient of ∂_x^{-1}. By Equation (8) it follows that

$$\mathrm{L}^{\mathrm{Heis}} = \partial_z + \frac{\rho^\vee}{h} \cdot \frac{1}{z} + \Lambda_z + \sum_{i=1}^{n-1} \frac{i-n}{n} \cdot t_{n-i} \cdot \Lambda_{-i}$$

as desired.

2.4 Letting the vacuum flow

In Theorem 2 we describe how Heisenberg connections flow along the Drinfeld-Sokolov flows via Definition 1. The Langlands theoretic meaning turns out to be related to the change of slope of the connection.

This slope of a connection ∇ is a rational number that is 0 if ∇ is isomorphic to an at most regular singular connection in the following sense. Suppose

$$\nabla \cong \partial_t + A_{-1}t^{-1} + A_0 + A_1 t + \cdots$$

If there is such a member of the gauge equivalence class of ∇ with $A_{-1} = 0$ then the connection is called regular and otherwise it is called regular singular. Suppose now that ∇ is a connection that is neither regular nor regular singular. In this case its slope is the rational number a/b where the pull-back of ∇ to the field extension $\mathbb{C}(\!(t^{1/b})\!)$ of $\mathbb{C}(\!(t)\!)$ has leading order singular term $t^{(1+a)/b}$ and the coefficient is a non-nilpotent element of $\mathfrak{g}^\circ$. It is known that the slope is well defined, see for example [7] (Lemma 3) for a proof.

Recall that it follows from our definitions that when the "vacuum connection" $\mathbf{0}$ (see Definition 3) flows as in Definition 1 one obtains a family of Heisenberg connections. The following result describes some of the finer points of this relation. The theorem yields in particular a clear Langlands theoretic meaning of the flows: The ramification of a connection (which is partially measured by the slope) should be preserved under a Langlands correspondence (see [12] and [19] for relevant results in this direction for the local geometric Langland correspondence) and the theorem implies that turning on more and more Drinfeld-Sokolov times corresponds precisely to increasing the slope of the Langlands parameters.

Theorem 2. *Let $\mathfrak{g}^\circ$ be a simple complex Lie algebra with Coxeter number h. All regular Heisenberg connections are isomorphic to $\mathbf{0}$ and the Drinfeld-Sokolov flows are well defined on this isomorphism class. Every Heisenberg connection ∇^{Heis} can be written as*

$$\nabla^{\mathrm{Heis}} \cong \mathbf{0}(\mathbf{t})$$

for suitable values $\mathbf{t}$ of the time variables and for any $c \ge 1$ one has

$$\text{slope } \mathbf{0}(\mathbf{t}) \le -1 + 2c + c/h \quad \rightsquigarrow \quad t_i = 0 \text{ for } i > c + h$$

Proof. We start the proof of Theorem 2 with a slope calculation that is very similar to the discussion by Frenkel and Gross in [13] (Section 5). Consider a Heisenberg connection $\nabla = \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i \le c} a_i \Lambda_i$ with $a_c \neq 0$. Let v be such that $v^{2h} = t$ and consider the pull back of ∇ to $\mathbb{C}(\!(v)\!)$. It is given by

$$\partial_v - \frac{2\rho^\vee}{v} - 2hv^{2h-1} \sum_{i \le c} a_i \left(\sum_{j=1}^{r} e_j \frac{1}{v^{4h}} + e_0 \frac{1}{v^{6h}} \right)^i$$

Using the same gauge transformation as in [13] one obtains

$$\partial_v - 2v^{2h-1} \sum_{i \le c} a_i \left(\sum_{j=0}^{r} e_j \frac{1}{v^{4h+2}} \right)^i$$

Since by results of Kostant $\sum_{j=0}^{r} e_j$ is not nilpotent one obtains that the pole is of order $c(4h+2)-2h+1$ and hence

$$\mathrm{slope}(\nabla) = -1 + 2c + c/h \tag{9}$$

The slope only depends on the isomorphism class of a connection and hence Equation (9) implies that a Heisenberg connection is regular (at ∞) precisely if it is of the form $\nabla = \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i<-h} H_i$. We now show that this implies that all regular Heisenberg connections are isomorphic. Consider the gauge transformation $\exp(\mathrm{ad}\ \phi)$ with $\phi = -\sum_{j<0} H_j/j$. Then

$$\begin{aligned}
\exp(\mathrm{ad}\ \phi)\,\nabla &= \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i\ll\infty} H_i - \sum_{j<0}\left[\mathrm{d}_{-1}^{(1,\cdots,1)}, \frac{H_j}{j}\right] - \sum_{\substack{i\ll\infty \\ j<0}}\left[H_i, \frac{H_j}{j}\right] + \text{ higher brackets} \\
&= \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i\ll\infty} H_i - \sum_{j\leq 0} H_{j-h} + \text{ higher brackets}
\end{aligned}$$

where

$$\text{higher brackets } = \frac{1}{2!}\left[\phi, \left[\phi, \mathrm{d}_{-1}^{(1,\cdots,1)}\right]\right] + \frac{1}{2!}\left[\phi, \left[\phi, \sum H_i\right]\right] + \cdots$$

We claim that all higher brackets in fact vanish. By [5] (Equation 3.26) that

$$\left[\mathrm{d}_{-1}^{(1,\cdots,1)}, H_i\right] = (i/h)\cdot H_{i-h} \tag{10}$$

Furthermore, the Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$ is commutative by [8] (Prop. 5.16) (recall that we only work with the loop part, not the central term contribution). Hence, since all the higher brackets involve at least two elements of the Heisenberg algebra this implies that all higher brackets vanish. It follows that

$$\left(\mathbb{C}(\!(1/z)\!)^n, \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i\ll\infty} H_i\right) \cong \left(\mathbb{C}(\!(1/z)\!)^n, \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{1-h\leq i\ll\infty} H_i\right) \tag{11}$$

and in particular all regular Heisenberg connections are isomorphic to $\mathbf{0}$.

We now describe how this connection flows under the Drinfeld-Sokolov action given by Definition 1. First let $\nabla = \mathrm{d}_{-1}^{(1,\cdots,1)} + H$ be a Heisenberg connection. Then $\nabla(\mathbf{t})$ is given by

$$\begin{aligned}
\exp(\mathrm{ad}\sum t_i\Lambda_i)\ \mathrm{d}_{-1}^{(1,\cdots,1)} = \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i\ll\infty} H_i \quad &- \quad \left[\mathrm{d}_{-1}^{(1,\cdots,1)}, \sum_i t_i\Lambda_i\right] \\
-\sum_{i,j}[H_i, t_j\Lambda_j] \quad &+ \quad \frac{1}{2!}\left[\sum_i t_i\Lambda_i, \left[\sum_i t_i\Lambda_i, \mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i\ll\infty} H_i\right]\right] + \cdots
\end{aligned}$$

This equals

$$\mathrm{d}_{-1}^{(1,\cdots,1)} + \sum_{i\ll\infty} H_i - \left[\mathrm{d}_{-1}^{(1,\cdots,1)}, \sum_i t_i\Lambda_i\right]$$

since the higher brackets vanish since they include at least two elements of the Heisenberg algebra and hence vanish up to a central term and one can formulate the Drinfeld-Sokolov hierarchy via the loop algebra of $\mathfrak{g}^\circ$ and hence can assume the central term to vanish. and hence

$$\nabla(\mathbf{t}) \cong \nabla - \sum_{i\in \mathrm{E}(\mathfrak{g})^{>0}} \frac{it_i}{h}\cdot\Lambda_{i-h} \tag{12}$$

One can see that in general it is not true that if $\nabla_1 \cong \nabla_2$ then $\nabla_1(\mathbf{t}) \cong \nabla_2(\mathbf{t})$. However, our earlier arguments show that on the isomorphism class of the vacuum **0** everything works out fine and its orbit reaches every isomorphism class of Heisenberg connections as claimed in Theorem 2. □

The upshot of our results is that, loosely speaking, the Drinfeld-Sokolov flows on local geometric Langlands parameters correspond to the following deformation of local arithmetic Langlands parameters. Consider the local field $K = \mathbb{F}_q(\!(z)\!)$. Typically, a local consider Langlands parameter will be of the form

$$\phi = \mathrm{Ind}_L^K(\chi)$$

where K/L is a finite extension and χ is a character. The geometric analogue of these induced representations are push-forwards of one-dimensional connections. Suppose now that the Lie algebra $\mathfrak{g}$ is of type A. In this case such push-forwards along a field extension $\mathbb{C}(\!(z^{1/n})\!)/\mathbb{C}(\!(z)\!)$ are described in terms of powers of the cyclic element $\Lambda = \sum_{i=1}^{n-1} e_{i+1,i} + z \cdot e_{1,n}$ of $\mathfrak{sl}_n^{(1)}$. One therefore obtains (Heisenberg) connections associated to Lie algebra elements $\sum a_i \Lambda^i$. We have shown in the present work that, when correctly interpreted, letting the Drinfeld-Sokolov times flow corresponds to varying the coefficients a_i. While on the arithmetic side the variation of these a_i's corresponds to varying the character χ in the induced representation $\phi = \mathrm{Ind}_L^K(\chi)$, on the geometric side we have shown that this type of variation is genuinely an incarnation of an integrable system.

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