Feigin-Frenkel image of Witten-Kontsevich points

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Abstract

The Witten-Kontsevich KdV tau function of topological gravity has a generalization to an arbitrary Drinfeld-Sokolov hierarchy associated to a simple complex Lie algebra. Using the Feigin-Frenkel isomorphism we describe the affine opers describing such generalized Witten-Kontsevich functions in terms of Segal-Sugawara operators associated to the Langlands dual Lie algebra. In the case where the Lie algebra is simply laced there is a second role these Segal-Sugawara operators play: Their action, in the basic representation of the affine algebra associated to the Lie algebra, singles out the Witten-Kontsevich tau function within the phase space. We show that these two Langlands dual roles of Segal-Sugawara operators correspond to a duality between the first and last operator for a complete set of Segal-Sugawara operators.

1 Introduction

In the early 1990's Witten conjectured in [16] that the partition function of topological gravity gives rise to a very specific KdV tau function. This was shown to be the case by Kontsevich in [13]. Since then, many generalizations of this set-up have been studied. In particular, if $\mathfrak{g}$ is a simple finite-dimensional complex Lie algebra, there is a certain point in the phase space $\Phi_{\mathfrak{g}}$ of the Drinfeld-Sokolov hierarchy associated to $\mathfrak{g}$ whose tau function generalizes the above mentioned KdV tau function (which corresponds to the choice $\mathfrak{g} = \mathfrak{sl}_2$). We call this point the Witten-Kontsevich point $P_{\mathfrak{g}}$.

The Drinfeld-Sokolov phase space $\Phi_{\mathfrak{g}}$ has various descriptions. In particular, when it is described not in terms of tau functions but in terms of affine opers, then one can apply, after passing to the underlying non-affine oper, the Feigin-Frenkel isomorphism to each point of $\Phi_{\mathfrak{g}}$ (it is these non-affine opers that are part of the local geometric Langlands correspondence as formulated by Frenkel and Gaitsgory in [7]). This process associates to each point in phase space a function on the center $\mathfrak{z}({}^{L}\mathfrak{g})$ of the critical level vertex algebra associated to the Langlands dual algebra ${}^{L}\mathfrak{g}$. Our Theorem 1 describes the function that one obtains when this process is applied to the Witten-Kontsevich point $P_{\mathfrak{g}}$.

Whether or not $\mathfrak{g}$ is simply laced, the center $\mathfrak{z}({}^{L}\mathfrak{g})$ has a description in terms of ${}^{L}\mathfrak{g}$ Segal-Sugawara operators. Something interesting happens for the special point $P_{\mathfrak{g}}$ in phase space if indeed $\mathfrak{g}$ is simply laced. In this case, $P_{\mathfrak{g}}$ can be defined in terms of the action of Segal-Sugawara operators on a highest weight representation (of non-criticial level) of the affine algebra $\mathfrak{g}^{(1)}$ associated to $\mathfrak{g}$. Via Theorem 1, the Witten-Kontsevich point therefore gives rise to two different occurrences of Segal-Sugawara operators. To understand their relation (which in a sense, via the results of Feigin and Frenkel, comes from a Langlands duality for simple Lie algebras) is the main motivation of the present work. In Theorem 2 we give the answer: Loosely speaking, it corresponds to a duality between the first and last Segal-Sugawara operator.

The considerations of the present work are restricted to the Witten-Kontsevich points in phase space, but one can ask for a much more general relation between Segal-Sugawara operator action on tau functions and the associated functions on Segal-Sugawara operators that are obtained via the Feigin-Frenkel isomorphism.

1.1 General case

We start by stating our main result, Theorem 1. Let $\mathfrak{g}$ be a simple complex Lie algebra with Langlands dual algebra ${}^{L}\mathfrak{g}$ and associated untwisted affine algebra $\mathfrak{g}^{(1)}$. Let $\mathrm{E}(\mathfrak{g})$ denote the set of exponents of $\mathfrak{g}^{(1)}$: They are the translates by integer multiples of the Coxeter number h of $\mathfrak{g}$ of the finite set of exponents of $\mathfrak{g}$. Let $\mathrm{E}(\mathfrak{g})^{>0}$ denote the subset of positive exponents of $\mathfrak{g}^{(1)}$. Let

$$\mathbf{t} = \{t_1, \cdots, t_i, \cdots \mid i \in \mathrm{E}(\mathfrak{g})^{>0}\} \tag{1}$$

These are the time variable of the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy. The phase space of this integrable hierarchy can be described in terms of the space of tau functions $\mathrm{Tau}_{\mathfrak{g}}$ which is a subset of the space of formal power series $\mathbb{C}[\![\mathbf{t}]\!]$.

Definition 1. The Witten-Kontsevich point $P_{\mathfrak{g}}$ (also called the topological string point) of the Drinfeld-Sokolov hierarchy of $\mathfrak{g}$ is defined in terms of its tau function via

$$\left(\sum_{i \in \mathrm{E}(\mathfrak{g})^{>0}} \frac{i+h}{h} \cdot t_{i+h} \partial_{t_i} + \frac{1}{2h} \cdot \sum_{\substack{i,j \in \mathrm{E}(\mathfrak{g})^{>0} \\ i+j=h}} ij \cdot t_i t_j \right) \tau(\mathbf{t}) = \partial_{t_1} \tau(\mathbf{t}) \tag{2}$$

Cafasso and Wu show in [2] that there is a unique $\mathfrak{g}$ Drinfeld-Sokolov tau function $\tau_{\text{string}}(\mathbf{t})$ satisfying Equation (2). To apply the results of Feigin and Frenkel to this situation a different description of the Drinfeld-Sokolov phase space is needed. Rather than in terms of tau functions, this space can be described in terms of the space $\mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}}$ of affine opers on a formal disc $D = \mathrm{Spec}\ \mathbb{C}[\![x]\!]$, where x is some indeterminate (we recall the definition in Section 2.1). Indeed, this is the formalism in which Drinfeld and Sokolov first developed the theory in [5].

Remark 1. It is known that the indeterminate x can be identified with the first flow variable t_1 of the Drinfeld-Sokolov hierarchy, see for example [2] (Section 3.2) for a detailed discussion, and we will freely do so throughout this work.

Let us denote the affine oper corresponding to $\tau_{\text{string}}(\mathbf{t})$ by $\mathrm{L}_{\mathfrak{g}}^{\text{string}}$. The space of affine opers has a non-affine variant $\mathrm{Op}_{\mathfrak{g}}(D)$ and there is a map

$$\mathrm{Aff} : \mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}} \longrightarrow \mathrm{Op}_{\mathfrak{g}}(D) \tag{3}$$

which passes to the underlying non-affine oper, see Section 2 for details. In particular, one obtains a non-affine oper $\mathrm{Aff}(\mathrm{L}_{\mathfrak{g}}^{\text{string}})$ associated to the Witten-Kontsevich point and this is now finally an object to which the results of Feigin and Frenkel can be applied.

Let $\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$ denote the center of the critical level (meaning negative of the dual Coxeter number) vertex algebra $\mathrm{V}_{\text{crit}}({}^{\mathrm{L}}\mathfrak{g})$ associated to $({}^{\mathrm{L}}\mathfrak{g})^{(1)}$. Due to the work of Feigin and Frenkel [6] the space of (isomorphism classes of) opers gives a description of the functions on this center. Namely, there is an isomorphism

$$\mathrm{FF} : \mathrm{Op}_{\mathfrak{g}}(D)_{/\sim} \cong \mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})^{\vee} \tag{4}$$

One can then ask to what function on the center $\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$ the (non-affine) oper $\mathrm{Aff}(\mathrm{L}_{\mathfrak{g}}^{\text{string}})$ of the Witten-Kontsevich point corresponds. We answer this in Theorem 1. To formulate this result it is useful to have a more explicit description of the center. This can be done in terms of the state-field correspondence of the vertex algebra $\mathrm{V}_{\text{crit}}({}^{\mathrm{L}}\mathfrak{g})$:

Let w be an indeterminate and let $Y(-, w)$ denote the state-field correspondence of $\mathrm{V}_{\text{crit}}({}^{\mathrm{L}}\mathfrak{g})$. Let r denote the rank of $\mathfrak{g}$ and let $\mathfrak{S}^{(1)}_{{}^{\mathrm{L}}\mathfrak{g}}, \cdots, \mathfrak{S}^{(r)}_{{}^{\mathrm{L}}\mathfrak{g}}$ be a complete set of Segal-Sugawara operators for ${}^{\mathrm{L}}\mathfrak{g}$ in the sense of [14], see Section 3.1 for the definition. This means in particular that they are elements of the center $\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$ and their Fourier coefficients $\mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g},k}$ given by

$$Y\left(\mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}}, w\right) = \sum_k \mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g},k}\ w^{-k-1}$$

generate the center, see Section 2. Hence, describing a function on the center of $\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$ corresponds to describing how it acts on these Fourier coefficients. It is in this manner that we describe the function corresponding to the

Witten-Kontsevich point. Define the "characteristic functions"

$$\mathbf{1}_{\mathfrak{S}^{(j)}_{{}^{\mathrm{L}}\mathfrak{g},k}} \in \mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})^{\vee} \quad \text{via} \quad \mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g},s} \mapsto \begin{cases} 1 \text{ if } (i,s)=(j,k) \\ \\ 0 \text{ otherwise} \end{cases}$$

It is useful to align certain scaling choices that are part of the construction of the Drinfeld-Sokolov hierarchy as well as the Segal-Sugawara operators. We make this precise in Definition 4, where we introduce the notion of alignment between ${}^{\mathrm{L}}\mathfrak{g}$ Segal-Sugawara operators and the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy.

Theorem 1. *Let $\mathfrak{g}$ be a simple complex Lie algebra of rank r and suppose $\mathfrak{S}^{(1)}_{{}^{\mathrm{L}}\mathfrak{g}}, \cdots, \mathfrak{S}^{(r)}_{{}^{\mathrm{L}}\mathfrak{g}}$ is a complete set of ${}^{\mathrm{L}}\mathfrak{g}$ Segal-Sugawara operators aligned with the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy. Then*

$$\mathrm{FF}\left(\mathrm{Aff}\left(\,\mathrm{L}^{\mathrm{string}}_{\mathfrak{g}}\right)\right) = \mathbf{1}_{\mathfrak{S}^{(r)}_{{}^{\mathrm{L}}\mathfrak{g},-2}}$$

The theorem attains a more symmetric form in the case where $\mathfrak{g}$ is simply laced. In this case, $\mathrm{L}^{\mathrm{string}}_{\mathfrak{g}}$ itself can be defined in terms of the action of a suitable Segal-Sugawara operator and Theorem 1 gives rise to a non-trivial duality between the first and last Segal-Sugawara operator. We state this in a precise manner in the next section.

1.2 Simply laced case

Assume now that $\mathfrak{g}$ is simply laced, in particular one has $\mathfrak{g} = {}^{\mathrm{L}}\mathfrak{g}$. As in the previous section let $\mathfrak{S}^{(1)}_{\mathfrak{g}}, \cdots, \mathfrak{S}^{(r)}_{\mathfrak{g}}$ be a complete set of Segal-Sugawara operators. Each Fourier coefficient $\mathfrak{S}^{(i)}_{\mathfrak{g},k}$ is an element of the center $\mathfrak{z}(\mathfrak{g})$ and hence gives rise, via the Feigin-Frenkel isomorphism FF and the map Aff to a function on the space of affine opers. Switching to the tau function description of this Drinfeld-Sokolov phase space we obtain that each $\mathfrak{S}^{(i)}_{\mathfrak{g},k}$ gives rise to a function

$$^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k} : \mathrm{Tau}_{\mathfrak{g}} \longrightarrow \mathbb{C} \tag{5}$$

Since $\mathfrak{g}$ is simply laced, Kac-Kazhdan-Lepowsky-Wilson have shown in [11] that the basic representation of the affine algebra $\mathfrak{g}^{(1)}$ associated to $\mathfrak{g}$ has a concrete realization acting on the space of formal power series $\mathbb{C}[\![\mathbf{t}]\!]$. It follows from the construction of the vertex algebra $\mathrm{V}_{\mathrm{crit}}(\mathfrak{g})$ that each of the Fourier coefficients $\mathfrak{S}^{(i)}_{\mathfrak{g},k}$ can be viewed as an element of the universal enveloping algebra of $\mathfrak{g}^{(1)}$. Hence, via the basic representation one obtains functions

$$^{(2)}\mathfrak{S}^{(i)}_{\mathfrak{g},k} : \mathbb{C}[\![\mathbf{t}]\!] \longrightarrow \mathbb{C}[\![\mathbf{t}]\!] \tag{6}$$

Since $\mathrm{Tau}_{\mathfrak{g}}$ is contained in $\mathbb{C}[\![\mathbf{t}]\!]$, the functions in (6) give in particular rise to functions on $\mathrm{Tau}_{\mathfrak{g}}$. It is an interesting task to relate them to the functions on $\mathrm{Tau}_{\mathfrak{g}}$ described in Equation (5). Our calculations are a step in this direction. Namely, Theorem 1 (for $\mathfrak{g}$ simply laced) yields the following (we refer to Definition 5 for the meaning of normalized set of Segal-Sugawara operators).

Theorem 2. *Let $\mathfrak{g}$ be a complex simple and simply laced Lie algebra of rank r. Suppose $\mathfrak{S}^{(1)}_{\mathfrak{g}}, \cdots, \mathfrak{S}^{(r)}_{\mathfrak{g}}$ is a complete set of normalized Segal-Sugawara operators aligned with the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy. If τ in $\mathrm{Tau}_{\mathfrak{g}}$ satisfies*

$$\left(-\partial_{t_1} + {}^{(2)}\mathfrak{S}^{(1)}_{\mathfrak{g},0}\right)(\tau) = 0$$

then it follows that

$$\left(-\partial_{t_1}{}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k}\right)(\tau) = \begin{cases} 1 \quad \text{if } i=r \text{ and } k=-1 \\ \\ 0 \text{ otherwise} \end{cases}$$

The collections of functions ${}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k}$ and ${}^{(2)}\mathfrak{S}^{(i)}_{\mathfrak{g},k}$ are both associated to $\mathfrak{g} \cong {}^{\mathrm{L}}\mathfrak{g}$. Nonetheless, the first set of functions is constructed via the Feigin-Frenkel isomorphism and hence is naturally associated to ${}^{\mathrm{L}}\mathfrak{g}$, whereas the second set of functions is constructed via the basic representation and is naturally associated to $\mathfrak{g}$. Theorem 2 shows that this hidden Langlands duality for finite-dimensional Lie algebras is reflected in a duality for Segal-Sugawara operators. To obtain a symmetrical description of this duality one should be careful with the indexing of the Fourier coefficients of the vertex algebra:

It turns out that for any choice of complete set of Segal-Sugawara operators, the first operator $\mathfrak{S}^{(1)}_{\mathfrak{g}}$ is always a non-zero scalar multiple of the conformal vector of the vertex algebra. Hence its Fourier coefficients satisfy the Virasoro algebra relations, but there is a change by 1 of the indices: In any vertex operator algebra with state-field correspondence denoted by $Y(-,w)$, the conformal vector v is such that for

$$Y(v,w) = \sum v_k w^{-k-1}$$

the elements $L_k := v_{k-1}$ satisfy the Virasoro algebra relations. Note the re-indexing by 1. Hence, if for the first Segal-Sugawara operator we re-index the coefficients by 1, then Theorem 2 can be phrased, up to the $T := -\partial_{t_1}$ action as a correspondence

$$\text{kernel of (re-indexed) } \mathfrak{S}^{(1)}_{\mathfrak{g},-1} \quad \Longleftrightarrow \quad \text{characteristic function of } \mathfrak{S}^{(r)}_{\mathfrak{g},-1}$$

This gives an answer to the question of how the two roles of Segal-Sugawara operators for Witten-Kontsevich points are related: They correspond to an interchange of first and last Segal-Sugawara operator.

2 Witten-Kontsevich points

After recalling the definition of affine opers in Section 2.1 we calculate in the present section the oper corresponding to the Witten-Kontsevich point.

2.1 Opers

As in the introduction, let $\mathfrak{g}$ be a simple complex Lie algebra with associated untwisted affine Lie algebra $\mathfrak{g}^{(1)}$. We work with the loop realization (also called homogeneous realization) of this affine Lie algebra and we denote the loop variable by z. Let r denote the rank of $\mathfrak{g}$ and let e_i, f_i with $1 \leq i \leq r$ be Chevalley generators of $\mathfrak{g}$. Let θ_0 denote the lowest root of $\mathfrak{g}$ and choose a generator E_0 of its root space. Let

$$\Lambda_1 = \sum_{i=1}^{r} e_i + z \cdot E_0 \tag{7}$$

and define the corresponding principal Heisenberg algebra

$$\mathfrak{h}_{\mathrm{pri}} = \mathrm{Cent}_{\mathfrak{g}^{(1)}}(\Lambda_1)$$

Recall, see for example [17] (Section 2.1), that the Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$ has (up to central terms) a $\mathbb{C}$-basis of elements Λ_j where j is an exponent of $\mathfrak{g}^{(1)}$ such that

$$[\Lambda_i, \Lambda_j] = i \cdot \delta_{i,-j} \cdot c$$

where c is the canonical central element (see [10]) of $\mathfrak{g}^{(1)}$.

The phase space of the Drinfeld-Sokolov hierarchy can be defined as the space of (isomorphism classes of) affine opers on the disc $D = \mathrm{Spec}\ \mathbb{C}[\![x]\!]$. This uses the principal gradation on $\mathfrak{g}^{(1)}$, which in particular satisfies

$$\deg\,(e_i) = 1 = -\deg\,(f_i) \quad (1 \leq i \leq r)$$

$$\deg\,(z\cdot E_0)=1$$

We refer to [10] for more details. Let in particular $\mathfrak{g}^{<0}$ and $\mathfrak{g}^{\leq 0}$ denote the subspaces of $\mathfrak{g}$ corresponding to negatively and non-positively graded elements, respectively. The space of affine opers is given by

$$\mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}}=\left\{\partial_x+\Lambda_1+q\;\middle|\;q\in\mathfrak{g}^{\leq 0}[\![x]\!]\right\} \tag{8}$$

The gauge transformations $\sim$ are of the form $\exp(\mathrm{ad}\,N)$ with N in $\mathfrak{g}^{<0}[\![x]\!]$. Note that the affine opers are the Lax operators of the Drinfeld-Sokolov hierarchy.

It will be convenient to make the following normalization assumption regarding the choice of basis $\{\Lambda_j\}$ of the principal Heisenberg algebra. Let as before denote h the Coxeter number of $\mathfrak{g}$ and note that $1-h$ is an exponent of $\mathfrak{g}^{(1)}$. Consider the standard loop realization of $\mathfrak{g}^{(1)}$ with loop variable z. Then $\Lambda_1\otimes z^{-1}$ is an element in $\mathfrak{h}_{\mathrm{pri}}$ of principal degree $1-h$. Furthermore, every other element in $\mathfrak{h}_{\mathrm{pri}}$ of degree $1-h$ is a scalar multiple of it since the space is one-dimensional: The only situation with exponents of multiplicity greater than 1 is $\mathfrak{so}_{2n}^{(1)}$ $(n\geq 4)$ where the integers congruent to $n-1$ modulo $h=2n-2$ are exponents of multiplicity two. We therefore can scale the basis elements of $\mathfrak{h}_{\mathrm{pri}}$ such that

$$\Lambda_{1-h}=\Lambda_1\otimes z^{-1} \tag{9}$$

We fix from now on such a basis.

2.2 Heisenberg form of Witten-Kontsevich points

Cafasso and Wu show in [1] (Theorem 3.10) that each Drinfeld-Sokolov hierarchy has a unique Witten-Kontsevich point. To prove Theorem 1 we switch from the tau function definition of this point given in Equation (2) to the explicit description of the corresponding (affine) oper which we denote by $\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}}$. To do so we first recall necessary aspects of tau functions of Drinfeld-Sokolov hierarchies. It is known, see for example [17], that every L in $\mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}}$ can be gauge transformed into the principal Heisenberg algebra $\mathfrak{h}_{\mathrm{pri}}$ in the following sense. There is α in the affine Lie algebra $\mathfrak{g}^{(1)}$ such that

$$\exp(\mathrm{ad}\,\alpha)\,\mathrm{L}=\exp(\mathrm{ad}\,\alpha)\,(\partial_x+\Lambda_1+q)=\partial_x+\Lambda_1+H \tag{10}$$

for H in the principal Heisenberg subalgebra $\mathfrak{h}_{\mathrm{pri}}$ of $\mathfrak{g}^{(1)}$. Here the exponentiated adjoint action is given by

$$\begin{aligned}\exp(\mathrm{ad}\,\alpha)\,(\partial_x+\Lambda_1+q)=\partial_x+\Lambda_1+q\quad&+\quad[\alpha,\Lambda_1+q]+\frac{1}{2!}[\alpha,[\alpha,\Lambda_1+q]]+\cdots\\&-\quad\partial_x\alpha+\frac{1}{2!}[\alpha,-\partial_x\alpha]+\cdots\end{aligned}$$

The special case of $\mathfrak{g}=\mathfrak{sl}_n$ suggests that the tau function of the Lax operator L can be defined in terms of the so obtained element H of the Heisenberg algebra. Namely, let L_{sc} denote the scalar Lax operator associated to L as constructed in [5] (Section 3.3). Working purely with the loop algebra quotient of $\mathfrak{sl}_n^{(1)}$, Drinfeld and Sokolov show in [5] (Proposition 3.20) that one has (up to total derivatives)

$$H_i=-\frac{1}{|i|}\cdot(\mathrm{L}_{\mathrm{sc}}^{|i|})_{-1}\cdot\Lambda_i$$

where the subscript -1 indicates the coefficient of ∂_x^{-1} and H_i denotes the i'th graded piece of H with respect to the principal gradation. In this manner one can see the tau functions enter the picture: It is known, see for example [8] (Appendix A), that

$$\partial_{t_1}\partial_{t_i}\log\tau(\mathbf{t})=(\mathrm{L}_{\mathrm{sc}}^{i})_{-1}$$

The idea of Wu [17] to define tau functions for all Drinfeld-Sokolov hierarchies is that even though H in Equation

(10) is in general not unique, one can gauge fix it in such a manner to arrive at a Heisenberg element that allows to define a tau function mimicking the above described $\mathfrak{sl}_n^{(1)}$ case. The gauge fixing condition imposed in [17] is that for all j in $\mathrm{E}(\mathfrak{g})^{>0}$ one has

$$(\exp(\mathrm{ad}\ \alpha)\ \Lambda_j)_c = 0 \tag{11}$$

where $(...)_c$ denotes the coefficient of the canonical central element c. Wu shows by induction on the degree with respect to the principal gradation that indeed there is α satisfying Equation (10) as well as Equation (11) simultaneously.

Definition 2. For L in $\mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}}$ and α satisfying Equation (10) and Equation (11) we call

$$\mathrm{Heis}(\mathrm{L}) := \exp(\mathrm{ad}\ \alpha)\ \mathrm{L} = \partial_x + \Lambda_1 + \sum_{i<0, i\in \mathrm{E}(\mathfrak{g})} H_i$$

the Heisenberg description of L.

We now calculate this expression in the case where L is the oper $\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}}$ of the Witten-Kontsevich point.

Lemma 2.3. *Let $\mathfrak{g}$ be a simple complex Lie algebra with Coxeter number h. The Heisenberg form of the Witten-Kontsevich point is given by*

$$\mathrm{Heis}(\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}}) = \partial_x + \Lambda_1 - \frac{x}{h}\cdot \Lambda_{1-h} + \text{ lower order terms}$$

where the lower order terms are sums of elements H_i in the Heisenberg algebra of principal degree less than $-h$.

Proof. The starting point is to differentiate Equation (2) with respect to t_k where k in $\mathrm{E}(\mathfrak{g})^{>0}$ satisfies $1 \leq k < h$. One obtains

$$\left(\frac{1+h}{h} t_{1+h} - 1\right)\partial_1\partial_k \log\tau(\mathbf{t}) + \sum_{\substack{i\in \mathrm{E}(\mathfrak{g})^{>0}\\ i>1}} \frac{i+h}{h} t_{i+h}\partial_i\partial_k \log\tau(\mathbf{t}) + \frac{k(h-k)}{h} t_{h-k} = 0$$

If one sets $t_i = 0$ for all $i \geq 1+h$ one therefore obtains

$$\partial_1\partial_k \log\tau(\mathbf{t}) = \frac{k(h-k)}{h} t_{h-k}$$

See also [2] (Lemma 3.6) for this type of calculation. Furthermore, Wu has shown in [17] (Section 3.2) that for all Drinfeld-Sokolov hierarchies one has for each k in $\mathrm{E}(\mathfrak{g})^{>0}$

$$\partial_1\partial_k \log\tau(\mathbf{t}) = -k\cdot \frac{(\Lambda_k, H)}{(\Lambda_k, \Lambda_{-k})} \tag{12}$$

where $(-,-)$ denotes an arbitrary non-degenerate symmetric invariant bilinear form. It follows that

$$\mathrm{Heis}\left(\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}}(t_1, \cdots, t_{h-1})\right) = \partial_x + \Lambda_1 - \frac{1}{h}\cdot \sum_{\substack{i\in \mathrm{E}(\mathfrak{g})\\ 1-h\leq i\leq -1}} (h+i)t_{h+i}\cdot \Lambda_i + \sum_{\substack{i\in \mathrm{E}(\mathfrak{g})\\ i\leq -h}} H_i$$

In particular, when all times except $t_1 = x$ (see Remark 1) are turned off, one obtains the lemma. □

2.4 Oper description of Witten-Kontsevich points

We now use the Heisenberg description of the Witten-Kontsevich point to calculate the Lax operator $\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}}$ itself.

Lemma 2.5. *The affine oper of the Witten-Kontsevich point is given by*

$$\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}} = \partial_x + \Lambda_1 - x\cdot E_0$$

Proof. Consider an element U of $\mathfrak{g}^{(1)}$ that can be written as $U = \sum_{i<0} U_i$ with U_i in the i'th principal grade and $U_i = 0$ for $i > -h$. We solve inductively with respect to principal degree the equation

$$\exp(\text{ad } U)\left(\partial_x + \Lambda_1 - \frac{x}{h}\cdot\Lambda_{1-h} + \text{ lower order terms }\right) = \partial_x + \Lambda_1 + *\cdot E_0 \tag{13}$$

for some yet to be determined scalar $*$. In degree i with $2-h \le i \le 1$ the equation plainly holds. To solve Equation (13) in degree $1-h$ note that E_0 is of degree $1-h$: Recall that E_0 is a generator of the lowest root space. In terms of the simple roots α_i and the Kac labels a_i the lowest root is given by $\theta_0 = -\sum_{i=1}^r a_i\alpha_i$. The sum $\sum_{i=1}^r a_i$ is known to equal $h-1$ and hence the height of θ_0 is $1-h$ as is the principal degree of E_0. It now follows that Equation (13) in degree $1-h$ yields

$$[U_{-h}, \Lambda_1] - \frac{x}{h}\cdot\Lambda_{1-h} = *\cdot E_0 \tag{14}$$

Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$ corresponding to our choice of simple roots α_i. We work with the standard loop (or homogeneous) realization of $\mathfrak{g}^{(1)}$ and we let

$$U_{-h} = g\otimes z^{-1} \quad \text{with } g\in\mathfrak{t}$$

Since by Equation (9) one has $\Lambda_{1-h} = \Lambda_1\otimes z^{-1}$, it follows that Equation (14) corresponds to

$$\frac{1}{z}\cdot\sum_{i=1}^r [g, e_i] + [g, E_0] - \frac{x}{h\cdot z}\cdot\sum_{i=1}^r e_i - \frac{x}{h}\cdot E_0 = *\cdot E_0$$

The left hand side is given by

$$\frac{1}{z}\cdot\sum_{i=1}^r \alpha_i(g)e_i + \theta_0(g)E_0 - \frac{x}{h\cdot z}\sum_{i=1}^r e_i - \frac{x}{h}\cdot E_0$$

Hence this equation can be solved if g satisfies

$$\alpha_i(g) = \frac{x}{h} \quad \text{for each} \quad 1\le i\le r \tag{15}$$

$$\theta_0(g) = \frac{x}{h} + * \tag{16}$$

One then obtains, where the a_i's are the Kac labels, that

$$* = \theta_0(g) - \frac{x}{h} = -\sum_{i=1}^r a_i\alpha_i(g) - \frac{x}{h} = -\frac{x}{h}\cdot\left(1 + \sum_{i=1}^r a_i\right) = -\frac{x}{h}\cdot\sum_{i=0}^r a_i = -x$$

since the sum $\sum_{i=0}^r a_i$ of all Kac labels equals the Coxeter number. Furthermore one gets

$$g = \frac{x}{h}\cdot\rho^\vee$$

where $\rho^\vee$ is half the sum of positive co-roots. Now observe that the condition given by Equation (11) automatically holds, since $U_i = 0$ for $i > -h$ and since there is nothing to check for U_{-h} since $-h$ is not an exponent. Note that $\partial_x + \Lambda_1 - x\cdot E_0$ is an operator of the form to which Proposition 3.1 of [17] can be applied. Due to the inductive nature, with respect to principal gradation degree, of the proof of the proposition in loc. cit. we have hence shown that indeed

$$\partial_x + \Lambda_1 - x\cdot E_0 = \exp(\text{ad } U)\left(\partial_x + \Lambda_1 - \frac{x}{h}\cdot\Lambda_{1-h} + \text{ lower order terms }\right)$$

for suitable U. It follows that

$$\mathrm{L}_{\mathfrak{g}}^{\text{string}} = \partial_x + \Lambda_1 - x \cdot E_0$$

as desired. □

Remark 2. The relation between $\mathrm{Heis}(\mathrm{L}_{\mathfrak{g}}^{\text{string}})$ and $\mathrm{L}_{\mathfrak{g}}^{\text{string}}$ obtained by Lemma 2.3 and Lemma 2.5 can in the case of $\mathfrak{g} = \mathfrak{sl}_2$ quickly be checked via the results in [17] (Section 5.1): Realizing $\mathfrak{sl}_2$ as traceless 2×2 matrices, one can take

$$E_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \Lambda_1 = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$$

and the principal Heisenberg algebra is spanned (modulo the center) by the odd powers of Λ_1. It is shown by Wu that if

$$\exp(\mathrm{ad}\, U)\left(\partial_x + \Lambda_1 + \begin{pmatrix} 0 & v(\mathbf{t}) \\ 0 & 0 \end{pmatrix}\right) = \partial_x + \Lambda_1 + \sum_{i>0,\text{odd}} h_i(\mathbf{t})\Lambda_1^{-i}$$

for U such that the gauge fixing condition Equation (11) holds, then

$$h_1(\mathbf{t}) = \frac{v(\mathbf{t})}{2}$$

$$h_3(\mathbf{t}) = -\frac{1}{8}\left(v(\mathbf{t})^2 + \frac{v(\mathbf{t})_{xx}}{3}\right)$$

$$\vdots$$

In particular, one obtains the desired relation between $\mathrm{Heis}(\mathrm{L}_{\mathfrak{g}}^{\text{string}})$ and $\mathrm{L}_{\mathfrak{g}}^{\text{string}}$.

Remark 3. Note that the occurrence of $-xE_0$ rather than say xE_0 in the various oper descriptions of the Witten-Kontsevich point is due to normalization choices. Consider for example the case of the Drinfeld-Sokolov hierarchy of $\mathfrak{g}^{(1)} = \mathfrak{sl}_n^{(1)}$ in the scalar formalism: Let

$$\mathrm{L}_{\text{sc}}^{\text{string}}(\mathbf{t}) = \partial_x + \sum_{i \leq 0} a_i(\mathbf{t})\partial_x^i$$

be the scalar Lax operator of the the $\mathfrak{sl}_n^{(1)}$ Drinfeld-Sokolov hierarchy associated to $\mathrm{L}_{\mathfrak{sl}_n}^{\text{string}}$. It is described via a string equation

$$\left[\left(\mathrm{L}_{\text{sc}}^{\text{string}}(\mathbf{t})\right)^n, Q(\mathbf{t})\right] = 1$$

It is known, see for example [4] (Section 3.1), that

$$Q(\mathbf{t}) = -\sum_{i \geq 1, n \nmid i} \frac{(i+n)t_{i+n}}{n}\left(\mathrm{L}_{\text{sc}}^{\text{string}}(\mathbf{t})\right)_+^i$$

where the $+$ subscript corresponds to the part of a pseudo-differential operator with non-negative powers of ∂_x. Note in particular when all times t_j with $j > 1+n$ are turned off then $Q(\mathbf{t}) = \mu \cdot \partial_x$ for some constant μ and changing the sign of t_{1+n} changes the sign of μ. Since the string equation with all times turned off except $t_1 = x$ and t_{1+n} is essentially of the form

$$[\partial_x^n - \mu^{-1} \cdot x, \mu \cdot \partial_x] = 1$$

one can see the effect of normalization choice of t_{1+n} on the Lax operator of the Witten-Kontsevich point. Note that this kind of normalization ambiguity is also the reason for the minus sign in Definition 4 later on.

As indicated before, the space of opers involved in the Feigin-Frenkel isomorphism is not $\mathrm{Op}_{\mathfrak{g}}(D)^{\text{aff}}$ itself but rather its non-affine variant $\mathrm{Op}_{\mathfrak{g}}(D)$. It is defined in a very similar manner as Equation (8):

$$\mathrm{Op}_{\mathfrak{g}}(D) = \left\{ \partial_x + \sum_{i=1}^{r} e_i + q \right\} \tag{17}$$

with q in $\mathfrak{g}^{\leq 0}[\![x]\!]$ and the gauge transformations are of the the same form as for affine opers. Put differently, these non-affine opers correspond to setting the variable z equal to 0. We hence define the map

$$\mathrm{Aff} : \mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}} \longrightarrow \mathrm{Op}_{\mathfrak{g}}(D)$$

coming from $z \mapsto 0$:

$$\partial_x + \Lambda_1 + q \mapsto \partial_x + \sum_{i=1}^{r} e_i + q$$

One obtains from Lemma 2.5:

Corollary 2.6. *The non-affine oper of the Witten-Kontsevich point is given by*

$$\mathrm{Aff}\left(\mathrm{L}_{\mathfrak{g}}^{\mathrm{string}}\right) = \partial_x + \sum_{i=1}^{r} e_i - x \cdot E_0 \tag{18}$$

3 Proof of the theorems

3.1 Proof of Theorem 1

Using the explicit description in Corollary 2.6 of the oper of the Witten-Kontsevich point we complete the proof of Theorem 1 in the present section. To do so, we first recall the description of functions on the space $\mathrm{Op}_{\mathfrak{g}}(D)$ in terms of Segal-Sugawara operators of the Langlands dual algebra ${}^{\mathrm{L}}\mathfrak{g}$. A key tool is a theory of normal forms for opers.

Let as before r denote the rank of $\mathfrak{g}$ and let $d_1, \cdots, d_r$ denote the exponents of $\mathfrak{g}$, ordered in non-decreasing order. Note that the smallest exponent of $\mathfrak{g}$ is 1 and the largest exponent is $h-1$ for the Coxeter number h of $\mathfrak{g}$. Then for each $1 \leq j \leq r$ choose a subspace V_j of $\mathfrak{g}_{-d_j}$ (the degree $-d_j$ part of $\mathfrak{g}$ with respect to principal gradation) such that

$$\mathfrak{g}_{-d_j} = \left[\sum_{i=1}^{r} e_i, \mathfrak{g}_{-d_j-1} \right] \oplus V_j$$

Define now

$$V^{\mathrm{can}} := \bigoplus_{j=1}^{r} V_j$$

Then it is known, see for example [9], that every oper can be gauge transformed to a unique element of the form

$$\partial_x + \sum_{i=1}^{r} e_i + v \quad \text{with} \quad v \in V^{\mathrm{can}}[\![x]\!] \tag{19}$$

Hence the space of opers is parametrized by $V^{\mathrm{can}}[\![x]\!]$ and the functions on the space of opers can be described in the following manner:

Choose a basis $\{v_j\}$ for each space V_j. For each oper L in $\mathrm{Op}_{\mathfrak{g}}(D)$ let $v(\mathrm{L})$ be the corresponding element of $V^{\mathrm{can}}[\![x]\!]$ and write

$$v(\mathrm{L}) = \sum_j \left(\sum_{k<0} v_{j,k} \cdot x^{-k-1} \right) \cdot v_j$$

for scalars $v_{j,k}$. Define the functions

$$w_{j,k} : \mathrm{Op}_{\mathfrak{g}}(D) \longrightarrow \mathbb{C}$$

by

$$w_{j,k}(\mathrm{L}) = v_{j,k}$$

A special role is played by the functions $w_{j,-1}$ that pick up the various constant terms: Via the results of Feigin and Frenkel they can be viewed as a generating set of the center of the vertex algebra $\mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g})$.

To describe the relevant details of this we first recall some aspects of the the vertex algebra $\mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g})$. For any u in $\mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g})$ denote by $Y(u,w) = \sum_n u_n w^{-n-1}$ the corresponding field. The center $\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$ of $\mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g})$ is defined to be

$$\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g}) = \left\{ v \in \mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g}) \;\middle|\; u_n(v) = 0 \ \text{ if } n \geq 0 \text{ and } u \in \mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g}) \right\}$$

The analogous space for non-critical level is known to always be isomorphic to $\mathbb{C}$. However, at the critical level the center is much larger and can be described as a polynomial algebra in infinitely many variables. In fact, there is a finite set of central elements that generate it and the important notion of a complete set of Segal-Sugawara operators is such a choice of finite generating set. Let us make this more precise, see [14] (Section 6.3) for more details:

Consider the enveloping algebra $\mathrm{U}_-({}^{\mathrm{L}}\mathfrak{g}) := \mathrm{U}(z^{-1}\, {}^{\mathrm{L}}\mathfrak{g}[z^{-1}])$ and let $|0\rangle$ denote the vacuum vector of $\mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g})$. Then as a vector space, this vertex algebra is isomorphic to $\mathrm{U}_-({}^{\mathrm{L}}\mathfrak{g})|0\rangle$ and this gives rise to an isomorphism of vector spaces

$$\xi : \mathrm{V}_{\mathrm{crit}}({}^{\mathrm{L}}\mathfrak{g}) \to \mathrm{U}_-({}^{\mathrm{L}}\mathfrak{g}) \tag{20}$$

Via ξ define the translation operator

$$T : \mathfrak{z}({}^{\mathrm{L}}\mathfrak{g}) \to \mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$$

to be the map coming from the map $-\partial_z$, meaning $g \otimes z^i \mapsto -ig \otimes z^{i-1}$ for g in ${}^{\mathrm{L}}\mathfrak{g}$ and $i < 0$. The following definition is given by Molev in [14] (Section 6.3).

Definition 3. Let r denote the rank of ${}^{\mathrm{L}}\mathfrak{g}$ and let $d_1 \leq \cdots \leq d_r$ denote the exponents of ${}^{\mathrm{L}}\mathfrak{g}$ (these are also the exponents of $\mathfrak{g}$). A complete set of Segal-Sugawara operators for ${}^{\mathrm{L}}\mathfrak{g}$ is a set

$$\left\{ \mathfrak{S}^{(1)}_{{}^{\mathrm{L}}\mathfrak{g}}, \cdots, \mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}}, \cdots, \mathfrak{S}^{(r)}_{{}^{\mathrm{L}}\mathfrak{g}} \right\}$$

of elements of the center $\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g})$ such that:

(i) $\deg \xi \left(\mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}} \right) = -(d_i + 1)$ for all i

(ii) the elements $T^j \mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}}$ with i as above and $j \geq 0$ are algebraically independent and generate the center:

$$\mathfrak{z}({}^{\mathrm{L}}\mathfrak{g}) = \mathbb{C}\left[T^j \mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}} \right]_{i,j}$$

It follows from the work of Feigin and Frenkel that such a complete set of Segal-Sugawara operators always exists. Namely, fix a canonical oper description as in Equation (19) and recall the Feigin-Frenkel isomorphism FF given in Equation (4). For each i let $\mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}}$ be the central element such that

$$\mathrm{FF}\left(\mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}} \right) = w_{i,-1} \tag{21}$$

The collection of the elements $\mathfrak{S}^{(i)}_{{}^{\mathrm{L}}\mathfrak{g}}$ then forms a complete set of Segal-Sugawara operators, see for example [9] (Section 4.3) and [14] (Theorem 6.3.1) for details.

Note that various choices are made in the definition of the Drinfeld-Sokolov hierarchy associated to $\mathfrak{g}$. In particular, the definition of the element Λ_1 in Equation (7) involves the choice of a generator E_0 of the lowest root space. It is useful for our considerations to compare this to the freedom in choosing a complete set of Segal-Sugawara operators. We formalize this as follows:

Definition 4. A complete set $\mathfrak{S}^{(1)}_{{}^L\mathfrak{g}}, \cdots, \mathfrak{S}^{(r)}_{{}^L\mathfrak{g}}$ of Segal-Sugawara operators for ${}^L\mathfrak{g}$ is aligned with the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy if the operators satisfy Equation (21) for a choice of canonical oper description for $\mathrm{Op}_{\mathfrak{g}}(D)$ that satisfies $v_r = -E_0$.

Remark 4. Note that v_r spans the space V_r and this space consists of elements of degree $1-h$ with respect to the principal gradation, where h is the Coxeter number. This is precisely the space spanned by E_0 and hence v_r and E_0 are non-zero multiples of each other.

Assume now that a complete set of Segal-Sugawara operators for ${}^L\mathfrak{g}$ is chosen which is aligned with the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy. In order to prove Theorem 1 note that the properties of the Feigin-Frenkel isomorphism imply that Equation 21 in fact contains further information. Let $|0\rangle$ be the vacuum vector in the vertex algebra $\mathrm{V}_{\mathrm{crit}}({}^L\mathfrak{g})$ and let as before $Y(-,w)$ denote the state-field correspondence. Recall that the Fourier coefficients $\mathfrak{S}^{(i)}_{{}^L\mathfrak{g},k}$ of the Segal-Sugawara operators are defined via

$$Y\left(\mathfrak{S}^{(i)}_{{}^L\mathfrak{g}}, w\right) = \sum_{k\in\mathbb{Z}} \mathfrak{S}^{(i)}_{{}^L\mathfrak{g},k}\, w^{-k-1}$$

Equation (21) then implies (see [9], Theorem 4.3.2, for details) that

$$\mathrm{FF}\left(\mathfrak{S}^{(i)}_{{}^L\mathfrak{g},k}\right) = w_{i,k}$$

for all $k<0$, not just for $k=-1$. Since

$$\mathrm{Aff}\left(\mathrm{L}^{\mathrm{string}}_{\mathfrak{g}}\right) = \partial_x + \sum_{i=1}^{r} e_i - x\cdot E_0$$

and since by Definition 4 one has $v_r = -E_0$ it follows that

$$\mathrm{FF}\left(\mathfrak{S}^{(i)}_{{}^L\mathfrak{g},k}\right)\left(\mathrm{Aff}\left(\mathrm{L}^{\mathrm{string}}_{\mathfrak{g}}\right)\right) = \begin{cases} 1 & \text{if } (i,k) = (r,-2) \\ \\ 0 & \text{otherwise} \end{cases}$$

This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

In the present section we deduce Theorem 2 from Theorem 1. Assume therefore from now on that $\mathfrak{g}$ is simply laced and hence in particular $\mathfrak{g} = {}^L\mathfrak{g}$. The Witten-Kontsevich point is a special point in $\mathrm{Op}_{\mathfrak{g}}(D)^{\mathrm{aff}}$ whose definition in terms of a differential equation for the tau function is given in Equation (2). As indicated in the introduction, if $\mathfrak{g}$ is simply laced the defining differential equation can be expressed in terms of Segal-Sugawara operators and we now recall the details.

Let as before $\mathrm{V}_{\mathrm{crit}}(\mathfrak{g})$ denote the critical level vertex algebra associated to $\mathfrak{g}$ with state-field correspondence $Y(-,w)$ and vacuum vector $|0\rangle$. Let $\{J_a\}$ be a basis of $\mathfrak{g}$ and let $\{J^a\}$ be the dual basis with respect to a choice of non-degenerate invariant bi-linear form $(-,-)$ on $\mathfrak{g}$. Whenever g is an element of $\mathfrak{g}$ we denote by $g[i]$ the element $g\otimes z^i$ in $\mathfrak{g}[z,z^{-1}]$. Let

$$J_a(w) = \sum_i J_a[i] w^{-i-1}$$

$$J^a(w) = \sum_i J^a[i] w^{-i-1}$$

Then, see for example [9] (Section 3.1), in the homogeneous description of the vertex algebra $\mathrm{V}_{\mathrm{crit}}(\mathfrak{g})$ one has

$$Y\left(\sum_{a=1}^{\dim\,\mathfrak{g}} J_a[-1]J^a[-1]\,|0\rangle, w\right) = \sum_{a=1}^{\dim\,\mathfrak{g}} :J_a(w)J^a(w): =: \sum_k \mathfrak{S}_k w^{-k-1} \tag{22}$$

where the normal ordering $:J_a[i]J^a[j]:$ is given by $J_a[i]J^a[j]$ if $i<0$ and $J^a[j]J_a[i]$ if $i \geq 0$. Each $\mathfrak{S}_k$ can be viewed as an element of the (completed) enveloping algebra of $\mathfrak{g}^{(1)}$ in the homogeneous realization.

Recall that we denote by ξ in Equation (20) the isomorphism as vector spaces between $\mathrm{V}_{\mathrm{crit}}(\mathfrak{g})$ and a universal enveloping algebra. Now let $\mathfrak{S}_{\mathfrak{g}}^{(1)}, \cdots, \mathfrak{S}_{\mathfrak{g}}^{(r)}$ be a complete set of Segal-Sugawara operators. The element $\mathfrak{S}_{\mathfrak{g}}^{(1)}$ is known to always satisfy

$$\xi\left(\mathfrak{S}_{\mathfrak{g}}^{(1)}\right) = d\cdot\sum_{a=1}^{\dim\,\mathfrak{g}} J_a[-1]J^a[-1] \tag{23}$$

for a non-zero scalar d. It follows that

$$\mathfrak{S}_{\mathfrak{g},k}^{(1)} = d\cdot\mathfrak{S}_k \tag{24}$$

For a clean statement of our results it is useful to fix the scalar in the following manner:

Definition 5. A complete set of $\mathfrak{g}$ Segal-Sugawara operators is called normalized if the constant in Equation (23) satisfies $d = 1/(2(1+h^\vee))$ for the dual Coxeter number $h^\vee$ of $\mathfrak{g}$.

For each $r+1$ tuple $\mathbf{s} = (s_0, \cdots, s_r)$ of non-negative integers, which are not all zero, there is an associated realization $\mathfrak{g}_{\mathbf{s}}^{(1)}$ of type $\mathbf{s}$ of the affine algebra $\mathfrak{g}^{(1)}$. See [10] for details. The two crucial examples for our considerations are

$$\mathbf{s}_{\mathrm{hom}} = (1, 0, \cdots, 0)$$

$$\mathbf{s}_{\mathrm{pri}} = (1, 1, \cdots, 1)$$

These correspond to the standard loop (or homogeneous) realization of $\mathfrak{g}^{(1)}$ as well as the principal realization, respectively. Let

$$h_{\mathbf{s}} := \sum_{i=0}^{r} a_i s_i$$

where the a_i's are the Kac labels, see [10], of $\mathfrak{g}^{(1)}$. Each realization $\mathfrak{g}_{\mathbf{s}}^{(1)}$ has a family of derivations $d_i^{\mathbf{s}}$ (i in $\mathbb{Z}$) satisfying the commutation relations

$$\left[d_i^{\mathbf{s}}, d_j^{\mathbf{s}}\right] = h_{\mathbf{s}}\cdot(j-i)\cdot d_{i+j}^{\mathbf{s}}$$

$$[d_i^{\mathbf{s}}, g[j]] = j\cdot g\,[j+ih_{\mathbf{s}}] \quad \text{for } g\in\mathfrak{g}, j\in\mathbb{Z}$$

The Virasoro algebra commutation relations are obtained for the normalized derivations

$$\mathrm{d}_i^{\mathbf{s}} := -\frac{1}{h_{\mathbf{s}}}\cdot d_i^{\mathbf{s}}$$

For each highest weight representation of $\mathfrak{g}^{(1)}$ of level not equal to $-h^\vee$ (where $h^\vee$ denotes the dual Coxeter number) there is a canonical way to extend the $\mathfrak{g}^{(1)}$ action to an action of the Virasoro algebra. This is described in detail by Wakimoto in [15] and Kac-Peterson [12] and proceeds by defining elements $S_i^{\mathbf{s}}$ in the enveloping algebra corresponding to the $\mathrm{d}_i^{\mathbf{s}}$'s.

Fix $\mathbf{s}$ as before and fix a primitive $h_{\mathbf{s}}$'th root of unity ζ. As described in [10], there is an associated automorphism σ of $\mathfrak{g}$ of order $h_{\mathbf{s}}$. Consider the corresponding $\mathbb{Z}/h_{\mathbf{s}}\mathbb{Z}$-gradation on $\mathfrak{g}$ given by $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/h_{\mathbf{s}}\mathbb{Z}}\mathfrak{g}_i$ where σ acts via multiplication by ζ^i on $\mathfrak{g}_i$. For each i choose a basis $\{J_{i,j}\}_j$ of $\mathfrak{g}_i$ such that $J_{i,j}$ is dual under $(-,-)$ to $J_{-i,j}$. Fix a scalar κ (it will correspond to the level of a representation of $\mathfrak{g}^{(1)}$) and (for $n \neq 0$) define

$$S^{\mathbf{s}}_n = \frac{1}{2h_{\mathbf{s}}(\kappa + h^\vee)} \cdot \sum_{i\in\mathbb{Z}} \sum_{k=1}^{h_{\mathbf{s}}} \sum_{j=1}^{\dim\, \mathfrak{g}_{-(i+k)}} J_{-(i+k),j}[-(i+k)] \cdot J_{i+k,j}[i+k+nh_{\mathbf{s}}] \tag{25}$$

These are the Segal-Sugawara operators described by Kac and Peterson (Proposition 2.27) in [12] and by Wakimoto in [15], for the type $\mathbf{s}$ realization of the affine algebra $\mathfrak{g}^{(1)}$. The relation of the operators $S^{\mathbf{s}}_n$ to the Fourier coefficients of the vertex algebra $\mathrm{V}_{\mathrm{crit}}(\mathfrak{g})$ given in Equation (22) is most easily seen in the case of the homogeneous gradation $\mathbf{s} = \mathbf{s}_{\mathrm{hom}}$. In this case one has $h_{\mathbf{s}} = 1$ and the Segal-Sugawara operators can be written as

$$S^{\mathbf{s}_{\mathrm{hom}}}_n = \frac{1}{2(\kappa + h^\vee)} \cdot \sum_{i\in\mathbb{Z}} \sum_{j=1}^{\dim\, \mathfrak{g}} J_j[-i] \cdot J_j[i+n] = \frac{1}{2(\kappa + h^\vee)} \cdot \mathfrak{S}_{1+n}$$

where the J_a's are as in the beginning of the section.

Therefore, up to the change of indexing by 1 and up to a multiplication by a fixed non-zero scalar, the Fourier coefficients of the first Segal-Sugawara operator of any complete set of Segal-Sugawara operators are described for any choice of $\mathbf{s}$ (not just the homogeneous one) by Equation (25). This holds in particular for the principal realization $\mathbf{s} = \mathbf{s}_{\mathrm{pri}} = (1, \cdots, 1)$. The action of these principal realizations of the Segal-Sugawara operators in the so-called basic representation of $\mathfrak{g}^{(1)}$ is known: Since $\mathfrak{g}$ is simply laced, the basic representation of $\mathfrak{g}^{(1)}$ (in the principal realization) has a particularly simple description due to Kac-Kazhdan-Lepowsky-Wilson [11] with underlying vector space $\mathbb{C}[\![\mathbf{t}]\!]$, where the set $\mathbf{t}$ is as in Equation (1). In particular, see [15] (Theorem 5.1), the action of $S^{\mathbf{s}_{\mathrm{pri}}}_{-1}$ on $\mathbb{C}[\![\mathbf{t}]\!]$ is given by the operator

$$\sum_{i\in \mathrm{E}(\mathfrak{g})^{>0}} \frac{i+h}{h} \cdot t_{i+h}\partial_{t_i} + \frac{1}{2h} \cdot \sum_{\substack{i,j\in \mathrm{E}(\mathfrak{g})^{>0} \\ i+j=h}} ij \cdot t_i t_j$$

Comparing this with the definition in Equation (2) of the Witten-Kontsevich point one sees that the Witten-Kontsevich point can be described in terms of Segal-Sugawara operators by the equation

$$S^{\mathbf{s}_{\mathrm{pri}}}_{-1}\, \tau(\mathbf{t}) = \partial_{t_1}\tau(\mathbf{t}) \tag{26}$$

Note that the basic representation is of level $\kappa = 1$ and therefore

$$S^{\mathbf{s}_{\mathrm{hom}}}_{-1} = \frac{\mathfrak{S}_0}{2(\kappa + h^\vee)} = \frac{\mathfrak{S}_0}{2(1 + h^\vee)}$$

where $\mathfrak{S}_0$ is as in Equation (22). Note that ${}^{(2)}\mathfrak{S}^{(1)}_{\mathfrak{g},0}$ as defined in Equation (6) corresponds to the principal realization in the basic representation of the operator $d\cdot\mathfrak{S}_0$ where d is as in Equation 24. Hence, if the complete set of Segal-Sugawara operators is chosen to be normalised in the sense of Definition 5, then in fact

$${}^{(2)}\mathfrak{S}^{(1)}_{\mathfrak{g},0} = S^{\mathbf{s}_{\mathrm{pri}}}_{-1}$$

Now, if a tau function τ in $\mathrm{Tau}_{\mathfrak{g}}$ satisfies

$$\left(-\partial_{t_1} + {}^{(2)}\mathfrak{S}^{(1)}_{\mathfrak{g},0}\right)(\tau) = 0$$

then by Equation (26) it is the tau function of the Witten-Kontsevich point. It follows from Theorem 1 that

$${}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k}(\tau) = \begin{cases} 1 & \text{if } (i,k) = (r,-2) \\ \\ 0 & \text{otherwise} \end{cases}$$

Recall that it follows from the definition of the Drinfeld-Sokolov time flows that $\partial_{t_1} = \partial_x$. One deduces from the discussion by Frenkel in [9] (Section 4.3.1) that for $m \geq 0$

$$(-\partial_{t_1})^m \left({}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},-1} \right) = m! \cdot {}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},-1-m}$$

One deduces that

$$(-\partial_{t_1}) \left({}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k} \right) = (-k) \cdot {}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k-1}$$

It follows that

$$\left(-\partial_{t_1} {}^{(1)}\mathfrak{S}^{(i)}_{\mathfrak{g},k} \right)(\tau) = \begin{cases} 1 \ \text{ if } i = r \text{ and } k = -1 \\ \\ 0 \text{ otherwise} \end{cases}$$

and this completes the proof of Theorem 2.

Remark 5. We conclude by remarking that the Segal-Sugawara Fourier coefficients $\mathfrak{S}^{(r)}_{\mathfrak{g},k}$ that come up in the above result can frequently be described rather explicitly. See for example the work of Chervov and Molev [3] in the case $\mathfrak{g} = \mathfrak{sl}_n$.

Acknowledgments: It is a pleasure to thank Edward Frenkel and the referees for helpful comments and exchanges.

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