Normal forms of Heisenberg connections

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Abstract

We study irregular connections on a punctured disc associated to elements of principal Heisenberg algebras of affine Lie algebras. In many cases we determine the Levelt-Turrittin normal form of these connections. The proof uses some ideas from quantum field theory.

1 Introduction

Let $\mathfrak{g}$ denote a complex finite-dimensional simple Lie algebra. An interesting class of elements of $\mathfrak{g}$ can be obtained as perturbations of nilpotent elements in the following manner. Given a non-zero nilpotent element e, include it in an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ and consider the $\mathbb{Z}$-grading on $\mathfrak{g}$ coming from the adjoint action of h. Consider perturbations of e of the form $\Lambda = e + F$ where F is non-zero and of minimal degree with respect to the $\mathbb{Z}$-grading. A small variant gives the corresponding notion for the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$ associated to $\mathfrak{g}$. Namely, fix a loop parameter z and realize $\widehat{\mathfrak{g}}$ concretely as an extension of the loop algebra $\mathfrak{g}[z, z^{-1}]$. Define the affine analogue $\Lambda_{\widehat{\mathfrak{g}}}$ of Λ by

$$\Lambda_{\widehat{\mathfrak{g}}} = e + z \cdot F$$

where e and F are as before. Such elements $\Lambda_{\widehat{\mathfrak{g}}}$ are the (generalized) cyclic elements of the affine Lie algebra $\widehat{\mathfrak{g}}$. The most studied case is when e is a principal nilpotent element and we restrict for the remainder of this work to this situation (see the work of Elashvili, Jibladze, Kac [5] and of Elashvili, Kac, Vinberg [6] for the classification of generalized cyclic elements).

Cyclic elements are interesting from a purely algebraic point of view but they also have important connections to integrable systems as described in the work of Drinfeld and Sokolov [4]. The flows of the Drinfeld-Sokolov integrable hierarchy associated to $\widehat{\mathfrak{g}}$ do not simply correspond to the cyclic element $\Lambda_{\widehat{\mathfrak{g}}}$ but more generally to elements in the centralizer $\mathrm{Heis}_{\widehat{\mathfrak{g}}}$ of $\Lambda_{\widehat{\mathfrak{g}}}$. This centralizer is called a principal Heisenberg algebra and in the present work we study connections (in general irregular ones) on a formal punctured disc attached to its elements. After choosing a complex representation ξ of $\mathfrak{g}$ (with underlying vector space denoted by V_ξ) and after choosing a realization (homogeneous, principal, $\cdots$) of $\widehat{\mathfrak{g}}$ one can ask what the eigenvalues of an element H of the Heisenberg algebra are. We investigate a non-linear variant of this question. View $V_\xi(\!(1/z)\!)$ as a bundle on the formal disc around $z = \infty$. Instead of calculating the (Jordan) normal form of H viewed as an endomorphism of $V_\xi(\!(1/z)\!)$ we want to calculate the (Levelt-Turrittin) normal form of H viewed as a connection on the same bundle. Concretely, we are considering the meromorphic differential operator $\partial_z + H$ on a formal punctured disc around $z = \infty$. We call such an object a Heisenberg connection. In Theorem 1 we calculate in rather general circumstances the Levelt-Turrittin normal forms of these connections. The result is similar to the corresponding eigenvalue calculation but with a subtle shift in the regular singular term.

This shift is related to the hidden role quantum field theory plays in our considerations. Through their relation with Drinfeld-Sokolov hierarchies the cyclic elements play a role in quantum field theory since partition functions of some quantum field theories can be expressed as tau functions of special points in the Drinfeld-Sokolov phase space. For example the Witten-Kontsevich partition function of 2d quantum gravity corresponds to a point in phase space singled out uniquely by a certain differential equation of the tau function. Alternatively, the dressing operator of this point relates the Heisenberg connection of a cyclic element to a connection with a shifted regular singular term, as shown by Cafasso and Wu in [3]. This latter connection is essentially the Kac-Schwarz operator whose action on a suitable vector Sato-Grassmannian gives yet another characterization of the Witten-Kontsevich point of $\widehat{\mathfrak{g}}$. A generalization of the result of Cafasso and Wu is our key tool to reduce normal form calculations to those where $\mathfrak{g}$ is of type A and to actually calculate the normal form in type A.

2 First examples

We introduce in this section the main ideas of how to calculate the Levelt-Turrittin normal form of Heisenberg connections. The strategy is to first deal with Lie algebras of type A and then reduce the general case to this. Before doing so we describe the corresponding much simpler calculations for the Jordan normal form of elements in the Heisenberg algebra. Note that for both types of normal form calculations there is an inherent ambiguity coming from the non-uniqueness of cyclic elements. For the linear-algebraic problem it follows from a result of Kostant [11] that the ambiguity is very mild since all cyclic elements are conjugate up to a non-zero scalar. In section 3 we describe how to adapt Kostant's arguments to show that for Heisenberg connections the ambiguity is again mild. In the present section we do not touch on this issue and simply discuss calculations for specific choices of cyclic elements.

For the standard h-dimensional representation of $\mathfrak{sl}_h$ and the homogeneous realization of the affine algebra one can take $\Lambda_{\widehat{\mathfrak{sl}}_h} = \sum_{i=1}^{h-1} e_{i+1,i} + z \cdot e_{1,h}$ where z denotes the choice of loop variable and $e_{i,j}$ denotes the matrix with zero entries everywhere except a 1 at the (i,j) entry. The eigenvalues are then given by $\zeta^i \cdot z^{1/h}$ where $1 \leq i \leq h$ and ζ is a primitive h'th root of unity. To calculate the spectrum of $\Lambda_{\widehat{\mathfrak{g}}}$ for more general Lie algebras $\widehat{\mathfrak{g}}$ one can compare the spectrum of the cyclic element $\Lambda_{\widehat{\mathfrak{g}}}$ with the spectrum of $\Lambda_{\widehat{\mathfrak{sl}}_h}$ where h is the Coxeter number of $\mathfrak{g}$. Consider as an example $\mathfrak{g}$ to be of type B. For $\mathfrak{so}_{2n+1}$ in its standard $2n+1$-dimensional representation one can take as cyclic element $\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} = \sum_{i=1}^{2n} e_{i+1,i} + (z/2) \cdot (e_{1,2n} + e_{2,2n+1})$. Let us describe a simple coordinate change that relates it to $\Lambda_{\widehat{\mathfrak{sl}}_{2n}}$. Namely, let $c_1, \cdots, c_{2n+1}$ denote the standard basis of $\mathbb{C}^{2n+1}$ and view it as a basis of $\mathbb{C}((1/z))^{2n+1}$ as a $\mathbb{C}((1/z))$-vector space. Let γ be the change of coordinates to the basis

$$d_1 = \frac{c_1}{2} + \frac{c_{2n+1}}{z} \quad , \quad d_i = c_i \quad \text{for } 2 \leq i \leq 2n \quad , \quad d_{2n+1} = \frac{c_1}{2} - \frac{c_{2n+1}}{z} \tag{1}$$

Under this change of coordinates one has

$$\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} \mapsto \Lambda_{\widehat{\mathfrak{sl}}_{2n}} \oplus 0 \tag{2}$$

and hence the spectrum of $\Lambda_{\widehat{\mathfrak{so}}_{2n+1}}$ can be obtained from the previous type A calculation.

We now return to the non-linear variant, namely the study of normal forms of Heisenberg connections. By the classical work of Levelt [12] and Turrittin [14] there exists an analogue of the Jordan normal form for meromorphic differential operators. We would like to apply this theory to differential operators of the form $\partial_z + \Lambda_{\widehat{\mathfrak{g}}}$. The first thing to say is that one should not consider this operator on a disc around $z = 0$. The connection is holomorphic at 0 and after a coordinate change is simply given by ∂_z (in contrast to the corresponding linear-algebraic problem). However, on a disc around $z = \infty$ the theory is interesting. We now give the precise definitions to deal with such connections. Let $t = 1/z$ be a coordinate on the formal punctured disc $D^\times$ around $z = \infty$. Let

$$\mathrm{Conn}(D^\times) = \left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \text{ a finite-dimensional } \mathbb{C}((t))\text{-vector space, } \nabla : V \to V \\ \\ \nabla(f \cdot v) = f'(t) \cdot v + f \cdot \nabla(v) \text{ for all } f \text{ in } \mathbb{C}((t)) \text{ and } v \text{ in } V \end{array} \right\}$$

denote the collection of connections on $D^\times = \mathrm{Spec}\ \mathbb{C}((t))$. Morphisms between (V_1, ∇_1) and (V_2, ∇_2) are those $\mathbb{C}$-linear homomorphisms λ from V_1 to V_2 such that $\lambda \circ \nabla_1$ equals $\nabla_2 \circ \lambda$. By choosing a basis for the underlying vector space V of a connection ∇ can be written as $\partial_t + A(t)$ for a matrix $A(t)$. It will be convenient to write connections on $D^\times$ in terms of $z = 1/t$ and then $\partial_t + A(t)$ equals $\partial_z - A(1/z)/z^2$.

Let ξ as before denote a complex representation of $\mathfrak{g}$ with underlying vector space V_ξ. We consider in the present section connections on $D^\times$ associated to cyclic elements in the following manner. The underlying vector space is $V_\xi((1/z))$ and ∇ is given by

$$\nabla_{\widehat{\mathfrak{g}}} = \partial_z + \Lambda_{\widehat{\mathfrak{g}}}$$

We often denote this connection simply by $\nabla_{\widehat{\mathfrak{g}}}$ and our aim is to calculate its normal form. The natural strategy is to proceed as in the spectrum calculation of cyclic elements. However, as can be seen from Equation (1), the coordinate change that relates cyclic elements of type B and type A depends (slightly) on z. Hence, in contrast to the linear-algebraic situation, one picks up a gauge term which makes it non-obvious how the normal forms of $\nabla_{\widehat{\mathfrak{g}}}$ and $\nabla_{\widehat{\mathfrak{sl}}_h}$ are related. Our approach to overcome this issue turns out to be related to work of Kac and Schwarz [10] on mathematical formulations of two-dimensional quantum gravity. Namely, the arguments of Cafasso and Wu in the proof of [3] (Lemma 3.9) show that there is a gauge transformation that maps

$$\nabla_{\widehat{\mathfrak{g}}} = \partial_z + \Lambda_{\widehat{\mathfrak{g}}} \;\;\mapsto\;\; \partial_z + \frac{\rho^\vee_{\mathfrak{g}}}{hz} + \Lambda_{\widehat{\mathfrak{g}}} \tag{3}$$

where h is the Coxeter number and $\rho^\vee_{\mathfrak{g}}$ is half the sum of suitable positive co-roots in $\mathfrak{g}$. Put differently, for an integer k let $\mathrm{d}_k^{\mathrm{hom}} = z^{k+1}\partial_z$ and let $\mathrm{d}^{\mathrm{pri}}_{k,\widehat{\mathfrak{g}}}$ be the corresponding derivation in the principal realization of the Lie algebra, then as connections on the punctured disc near $z=\infty$ Equation (3) yields an isomorphism

$$\mathrm{d}^{\mathrm{hom}}_{-1} + \Lambda_{\widehat{\mathfrak{g}}} \;\;\cong\;\; \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} + \Lambda_{\widehat{\mathfrak{g}}} \tag{4}$$

We use this freedom in shifting the regular singular term for two purposes: To reduce calculations to type A and to actually calculate the normal form of Heisenberg connections in type A. For the remainder of this section we sketch these two applications, full details are given in a more general set-up in Section 3.

2.1 Type A normal form

Consider the standard representation of $\mathfrak{sl}_h$. One can choose Chevalley generators of this algebra such that the result of Cafasso and Wu described in Equation (3) yields an isomorphism of connections on $D^\times$

$$\left(\mathbb{C}(\!(1/z)\!)^h, \partial_z + \Lambda_{\widehat{\mathfrak{sl}}_h}\right) \cong \left(\mathbb{C}(\!(1/z)\!)^h, \partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \Lambda_{\widehat{\mathfrak{sl}}_h}\right) \tag{5}$$

where

$$\rho^\vee_{\mathfrak{sl}_h} = \frac{1}{2}\sum_{i=1}^{h}(-1-h+2i)\cdot e_{i,i}$$

The right-hand side of Equation (5) might appear to be a more complicated description of the isomorphism class. However, the shifting of the regular singular term leads to a great simplification in the calculation of the Levelt-Turrittin normal form. Loosely speaking, the normal form of a connection on a $\mathbb{C}(\!(1/z)\!)$-vector space is obtained by describing its isomorphism class in terms of connections of the form (V,∇) where V is a $\mathbb{C}(\!(1/z)\!)$-vector space of the form $\mathbb{C}(\!(1/\zeta)\!)$ where $\zeta^h = z$ for some positive integer h and where ∇ is given as an operator of the form $\partial_z + f(\zeta)$ for a suitable Laurent series f. It turns out that the presence of $\rho^\vee_{\mathfrak{sl}_h}$ in Equation (5) allows to do just that. Consider as in [10] the isomorphism of $\mathbb{C}(\!(1/z)\!)$-vector spaces between $\mathbb{C}(\!(1/z)\!)^h$ and $\mathbb{C}(\!(1/\zeta)\!)$ given by $g(z)e_i \mapsto g(\zeta^h)\zeta^{h-i}$ where e_i is the standard i'th basis element in $\mathbb{C}(\!(1/z)\!)^h$ with 0's everywhere except a 1 in the i'th entry and where g is a Laurent series. Under this so-called blending map one has

$$\partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \Lambda_{\widehat{\mathfrak{sl}}_h} \;\;\mapsto\;\; \partial_z + \frac{1-h}{2h}\cdot z^{-1} + \zeta \tag{6}$$

where $\partial_z = (\zeta^{1-h}/h)\cdot\partial_\zeta$. This yields the Levelt-Turrittin normal form as desired. We describe this in more detail in Section 3.

As indicated before, the difference between the Jordan and Levelt-Turrittin normal form calculations corresponds to the occurrence of the regular singular term and hence this difference is given by the coefficient $(1-h)/2h$ in the above example. This coefficient has quantum field theoretic meaning as can be seen from a calculation in fermionic algebra given by Kac and Schwarz in [10]. We recall the statement. Fix fermionic operators ψ_i and $\psi_i^\dagger$ (with i in $\mathbb{Z}$)

satisfying the canonical anti-commutation relations

$$\psi_i \psi_j + \psi_j \psi_i = 0 \quad , \quad \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0 \quad , \quad \psi_i^\dagger \psi_j + \psi_j \psi_i^\dagger = \delta_{i,j}$$

Define the corresponding fermionic fields as

$$\psi(\zeta) = \sum_{i \in \mathbb{Z}} \psi_i \, \zeta^{-i-1/2} \quad , \quad \psi^\dagger(\zeta) = \sum_{i \in \mathbb{Z}} \psi_{-i}^\dagger \, \zeta^{-i-1/2}$$

Define the normal ordering $:\psi_i^\dagger \psi_j:$ as $\psi_i^\dagger \psi_j$ if $j > 0$ and $-\psi_j \psi_i^\dagger$ otherwise. Define Ψ via

$$:\partial_\zeta \psi^\dagger(\zeta)\, \psi(\zeta): - :\psi^\dagger(\zeta)\, \partial_\zeta \psi(\zeta): \quad = \quad \cdots + 2h \cdot \Psi \cdot \zeta^{h-2} + \cdots$$

where h is as before a positive integer such that $\zeta^h = z$. The regular singular term of Equation (6) now appears in this fermionic calculation: As discussed for example in [7], [10] one has

$$[\Psi \, , \, \psi(\zeta)] = \left(\partial_z + \frac{1-h}{2h} \cdot z^{-1} \right) \cdot \psi(\zeta)$$

One sees that the shift from considering cyclic elements as endomorphisms of a bundle to viewing them as connections on the same bundle brings genuinely new mathematics and physics into the picture.

2.2 Reduction to type A

Let as before h denote the Coxeter number of $\mathfrak{g}$. To compare the normal forms of the connections $\nabla_{\widehat{\mathfrak{g}}}$ and $\nabla_{\widehat{\mathfrak{sl}}_h}$ it suffices by the result of Cafasso and Wu to compare the corresponding right-hand sides of Equation (4). This turns out to be simpler than the original problem due to the following observation that holds at least for the cases considered in the present work, namely for Lie algebras of type B, C, D, G.

Observation. *There is a coordinate change that relates the cyclic elements* $\Lambda_{\widehat{\mathfrak{g}}}$ *and* $\Lambda_{\widehat{\mathfrak{sl}}_h}$ *and that at the same time maps*

$$\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} + \text{gauge term} \;\; \mapsto \;\; \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{sl}}_h} + \text{simple expression}$$

We illustrate this for Lie algebras of type B in their standard representation. For $\mathfrak{g} = \mathfrak{so}_{2n+1}$ one has $h = 2n$ and one can choose Chevalley generators such that

$$\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} = \sum_{i=1}^{2n} e_{i+1,i} + \frac{z}{2} \cdot (e_{1,2n} + e_{2,2n+1}) \quad , \quad \rho^\vee_{\mathfrak{so}_{2n+1}} = \sum_{i=1}^{2n+1} (-n-1+i) \cdot e_{i,i} \tag{7}$$

Under the same coordinate change γ as before, see Equation (1), one has

$$\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} \mapsto \sum_{i=1}^{2n-1} e_{i+1,i} + z \cdot e_{1,2n} = \Lambda_{\widehat{\mathfrak{sl}}_{2n}}$$

as well as

$$\begin{aligned}
\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{so}}_{2n+1}} + \text{gauge term} &= \partial_z + \frac{\rho^\vee_{\mathfrak{so}_{2n+1}}}{hz} + \gamma\partial_z(\gamma^{-1}) \\
&\mapsto \partial_z + \frac{1}{2nz}\cdot\left(-n\cdot(e_{1,2n+1}+e_{2n+1,1}) + \sum_{i=2}^{2n}(-n-1+i)\cdot e_{i,i}\right) \\
&\quad + \frac{1}{2z}\cdot\left(-e_{1,1}+e_{2n+1,1}+e_{1,2n+1}-e_{2n+1,2n+1}\right) \\
&= \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{sl}}_h} - \frac{1}{2hz}\cdot\sum_{i=1}^{h} e_{i,i} - \frac{1}{2z}\cdot e_{2n+1,2n+1}
\end{aligned}$$

Note the key cancellations between the gauge term and the term involving the half-sum of positive co-roots. As we show later on this phenomenon holds for all Lie algebras of type B, C, D, G. We can now compare the connections $\nabla_{\widehat{\mathfrak{so}}_{2n+1}}$ and $\nabla_{\widehat{\mathfrak{sl}}_{2n}}$ on the disc around $z=\infty$.

For a connection (V,∇) of dimension m we define for any scalar λ the shifted connection

$$(V,\nabla)[\lambda] := (V, \nabla + \frac{\lambda}{mz}\cdot\mathrm{Id})$$

Our previous calculations then imply

$$\begin{aligned}
\partial_z + \Lambda_{\widehat{\mathfrak{so}}_{2n+1}} &\cong \partial_z + \frac{\rho^\vee_{\mathfrak{so}_{2n+1}}}{2nz} + \Lambda_{\widehat{\mathfrak{so}}_{2n+1}} && (8)\\
&\cong \left(\partial_z + \frac{\rho^\vee_{\mathfrak{sl}_{2n}}}{2nz} + \Lambda_{\widehat{\mathfrak{sl}}_{2n}}\right)[-1/2] \oplus (\partial_z + 0)[-1/2] && (9)\\
&\cong (\partial_z + \Lambda_{\widehat{\mathfrak{sl}}_{2n}})[-1/2] \oplus (\partial_z+0)[-1/2] && (10)
\end{aligned}$$

As endomorphisms of a suitable bundle the cyclic elements in type A and B are related as seen in Equation (2) by

$$\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} \cong \Lambda_{\widehat{\mathfrak{sl}}_{2n}} \oplus 0 \tag{11}$$

By Equation (10) the connections associated to $\Lambda_{\widehat{\mathfrak{so}}_{2n+1}}$ and $\Lambda_{\widehat{\mathfrak{sl}}_{2n}}$ are related in almost the same way, the one distinction being the shift by $-1/2$ (and this shift cannot be gauged away by a further suitable coordinate change, in general). We show in Theorem 1 that for all Lie algebras of type A, B, C, D, G the shift can expressed in terms of κ where κ is the following standard Lie theoretic invariant: Let $(-,-)_0$ denote the standard invariant bilinear form on $\mathfrak{g}$, see [9], normalized so that the long root has squared length 2. Then κ can be described as the constant such that

$$(A,B)_0 = \kappa\cdot\mathrm{Tr}\left(\xi(A)\,\xi(B)\right) \tag{12}$$

for all A, B in $\mathfrak{g}$, where ξ denotes the first fundamental representation, hence the standard defining representation for the classical algebras. It is known that κ is $1/2$ for $\mathfrak{g}$ of type B, D, and G and κ is 1 for $\mathfrak{g}$ of type A and C.

In Section 3 we generalize the calculations of the present section to all affine Lie algebras of type A, B, C, D, G and we consider connections associated to more general elements of the principal Heisenberg algebra, not just those associated to the cyclic element $\Lambda_{\widehat{\mathfrak{g}}}$.

3 Heisenberg connections

In Section 3.1 we define the notion of Heisenberg connection. In section 3.2 we discuss a coarser variant of the Levelt-Turrittin classification for connections on a punctured disc. In Section 3.3 we calculate the coarse Levelt-Turittin normal form of Heisenberg connections.

3.1 Basic definitions

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra. Let r denote its rank, let h denote its Coxeter number, and let $\widehat{\mathfrak{g}}$ denote the corresponding untwisted affine Lie algebra. Fix a set $\{E_i, F_i \mid 1 \leq i \leq r\}$ of Chevalley generators of $\mathfrak{g}$. Hence

$$H_i := [E_i, F_i] \quad , \quad [H_i, E_j] = A_{ij} E_j \quad , \quad [H_i, F_j] = -A_{ij} F_j$$

where $A = (A_{ij})$ is the Cartan matrix of $\mathfrak{g}$. Let E_0 be a generator of the lowest root space of $\mathfrak{g}$. Fix an indeterminate z, the loop variable of $\widehat{\mathfrak{g}}$, and let $e_0 = z \cdot E_0$ and $e_i = E_i$ for $1 \leq i \leq r$. We view the e_i's as elements of $\widehat{\mathfrak{g}}$ when realized in terms of its standard loop realization.

Definition 1. Let $\Lambda_{\widehat{\mathfrak{g}}} = e_0 + \cdots + e_r$ be the cyclic element associated to our choice of Chevalley generators. Let

$$\mathrm{Heis}_{\widehat{\mathfrak{g}}} := \mathrm{Cent}(\widehat{\mathfrak{g}}, \Lambda_{\widehat{\mathfrak{g}}})$$

denote its centralizer within the loop part $\mathfrak{g}[z, z^{-1}]$ of $\widehat{\mathfrak{g}}$. This is (up to multiples of the central term) the principal Heisenberg subalgebra of $\widehat{\mathfrak{g}}$ associated to our choice of $\Lambda_{\widehat{\mathfrak{g}}}$.

Let ξ be a finite-dimensional complex representation of $\mathfrak{g}$ with underlying vector space V_ξ. For every H in the Heisenberg algebra $\mathrm{Heis}_{\widehat{\mathfrak{g}}}$ one can define a connection on $D^\times$ given by

$$(V_\xi((1/z)), \partial_z + \xi(H)) \tag{13}$$

As indicated before, a subtlety about this definition is to classify the dependence of this connection on the choice of cyclic element. We now specialize to the situation of Theorem 1, meaning $\mathfrak{g}$ is of type A, B, C, D, or G. Furthermore ξ denotes the first fundamental representation of $\mathfrak{g}$ and hence in particular is simply the standard representation if $\mathfrak{g}$ is one of the classical algebras. It will be useful to have a more concrete description of the image under ξ of the principal Heisenberg algebra. For the cyclic element $\Lambda_{\widehat{\mathfrak{g}}}$ itself it is known, see for example [1], that the image under ξ can be described by the following $d \times d$ matrices where $e_{i,j}$ denotes a $d \times d$ matrix with 0's everywhere except a 1 at the (i, j) entry.

	cyclic element	d
$\mathfrak{sl}_n$	$z \cdot e_{1,n} + \sum_{i=1}^{n-1} e_{i+1,i}$	n
$\mathfrak{so}_{2n+1}$	$\sum_{i=1}^{2n} e_{i+1,i} + \frac{z}{2} \cdot (e_{1,2n} + e_{2,2n+1})$	$2n+1$
$\mathfrak{sp}_{2n}$	$z \cdot e_{1,n} + \sum_{i=1}^{2n-1} e_{i+1,i}$	$2n$
$\mathfrak{so}_{2n}$	$\frac{1}{2} \cdot (e_{n+1,n-1} + e_{n+2,n}) + \sum_{i=1}^{n-1} (e_{i+1,i} + e_{2n+1-i,2n-i}) + \frac{z}{2}(e_{1,2n-1} + e_{2,2n})$	$2n$
$\mathfrak{g}_2$	$\sum_{i=1}^{6} e_{i+1,i} + \frac{z}{2} \cdot (e_{1,6} + e_{2,7})$	7

Let $\mathrm{E}(\widehat{\mathfrak{g}})$ denote the multi-set of exponents of $\widehat{\mathfrak{g}}$ (it is obtained from the finite multi-set of exponents of $\mathfrak{g}$ by adding an arbitrary integer multiple of the Coxeter number h). If $\mathfrak{g}$ is not isomorphic to $\mathfrak{so}_{2n}$ then it is known, see [1], that for the above choice of cyclic element $\Lambda_{\widehat{\mathfrak{g}}}$ the corresponding principal Heisenberg algebra (in the standard loop realization of $\widehat{\mathfrak{g}}$) has a $\mathbb{C}$-basis $\{\Lambda_i\}_{i\in \mathrm{E}(\widehat{\mathfrak{g}})}$ such that Λ_i is of degree i in the principal gradation and $\Lambda_1 = \Lambda_{\widehat{\mathfrak{g}}}$ and

$$\xi(\Lambda_i) = \begin{cases} \xi(\Lambda_1)^i & \text{if } i > 0 \\ \\ \left(z^{-1}\xi(\Lambda_1)^{h-1}\right)^{-i} & \text{if } i < 0 \end{cases} \tag{14}$$

In contrast, if $\mathfrak{g} = \mathfrak{so}_{2n}$ and $i \equiv n-1 \mod h$ then the subspace of $\mathrm{Heis}_{\widehat{\mathfrak{g}}}$ of principal degree i is two-dimensional rather than one-dimensional, with an additional basis element Ψ_i. We do not consider this part of the type D Heisenberg algebra in the present work.

Kostant has shown in [11] that any two cyclic elements are conjugate up to a non-zero scalar. In particular it follows that (again except for Lie algebras of type D) for any choice of cyclic element $\Lambda_{\widehat{\mathfrak{g}}}$ the image under ξ of the positively graded elements in the Heisenberg algebra (with respect to the principal gradation) has a power basis $\{\xi(\Lambda_{\widehat{\mathfrak{g}}})^i\}_{i\in \mathrm{E}(\widehat{\mathfrak{g}})}$.

We now adapt Kostant's arguments to the setting of the connections associated to the Heisenberg algebra as in Equation (13).

Proposition 1. *Let a_i for $i \in \mathbb{Z}^{>0}$ be a collection of complex scalars. For any two choices of cyclic elements $\Lambda_{\widehat{\mathfrak{g}}}$ and $\Lambda'_{\widehat{\mathfrak{g}}}$ there is a non-zero constant λ such that*

$$\left(V_\xi(\!(1/z)\!), \partial_z + \sum_{i\in\mathbb{Z}^{>0}} a_i \cdot \xi(\Lambda_{\widehat{\mathfrak{g}}})^i\right) \cong \left(V_\xi(\!(1/z)\!), \partial_z + \sum_{i\in\mathbb{Z}^{>0}} a_i \cdot \lambda^i \cdot \xi(\Lambda'_{\widehat{\mathfrak{g}}})^i\right)$$

Proof. The main observation is due to Kostant and concerns the eigenvalues of the cyclic elements: In [11] (Theorem 6.2) Kostant shows that the eigenvalues, up to overall multiplication by a non-zero scalar, are independent of the choice of the positive Chevalley generators e_i, for any choice of representation and in particular for ξ. We now show that with some care Kostant's arguments can be generalized to the connections that we are considering.

In [11] (Lemma 6.2) it is shown that for any choice of non-zero scalars a_i and b_i (with $0 \le i \le r$) there is an element a in $\mathfrak{g}$ such that

$$\exp(\mathrm{ad}\ a)\left(\sum_{i=0}^{r} a_i e_i\right) = \lambda \cdot \sum_{i=0}^{r} b_i e_i \tag{15}$$

for some non-zero scalar λ. One can see from the proof in [11] that the dependency of a on a_0 and b_0 is only through the quotient a_0/b_0. Therefore, switching to the affine situation, for two choices of affine cyclic elements

$$\Lambda_{\widehat{\mathfrak{g}}} = a_0 z e_0 + \sum_{i=1}^{r} a_i e_i \quad , \quad \Lambda'_{\widehat{\mathfrak{g}}} = b_0 z e_0 + \sum_{i=1}^{r} b_i e_i$$

one sees that a is constant, meaning independent of z. Hence the corresponding gauge transformation $\exp(\mathrm{ad}\ a)$ is simply conjugation and by using Equation (15) one obtains

$$\begin{aligned} \exp(\mathrm{ad}\ \xi(a))\left(\partial_z + \sum_{i\in\mathbb{Z}^{>0}} a_i \cdot \xi(\Lambda_{\widehat{\mathfrak{g}}})^i\right) &= \partial_z + \exp(\mathrm{ad}\ \xi(a))\left(\sum_{i\in\mathbb{Z}^{>0}} a_i \cdot \xi(\Lambda_{\widehat{\mathfrak{g}}})^i\right) \\ &= \partial_z + \sum_{i\in\mathbb{Z}^{>0}} a_i \cdot \lambda^i \cdot \xi(\Lambda'_{\widehat{\mathfrak{g}}})^i \end{aligned}$$

As in the proof of [11] (Theorem 6.2) the same result holds for any two sets of choices of Chevalley generators. ☐

Proposition 1 allows to describe the dependency on the choice of cyclic element of many Heisenberg connections.

Definition 2. We say $\mathbf{a} = (a_i)_{i\in\mathbb{Z}^{>0}}$ is of $\widehat{\mathfrak{g}}$-type if $a_i = 0$ for $i \gg 0$ and $a_i = 0$ if i is not an exponent of $\widehat{\mathfrak{g}}$.

Definition 3. Let ξ be a finite-dimensional complex representation of $\mathfrak{g}$ with underlying vector space V_ξ. Let $\Lambda_{\widehat{\mathfrak{g}}}$ be a cyclic element. The Heisenberg connection associated to $\mathbf{a}$ of $\widehat{\mathfrak{g}}$-type is the object of $\mathrm{Conn}(D^\times)$ given by

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(\xi, D^\times)_{\mathbf{a}} = \left(V_\xi((1/z)), \partial_z + \sum_{i \in \mathbb{Z}^{>0}} a_i \cdot \xi(\Lambda_{\widehat{\mathfrak{g}}})^i \right) \tag{16}$$

The main focus of the present work concerns Heisenberg connections obtained via the first fundamental representation of the simple Lie algebra $\mathfrak{g}$ (hence via the standard representation in case of the classical algebras). Nonetheless, we also consider a variant related to the folding of Lie algebras. If $\mathfrak{g}$ is of type B, C, F, G it is known how to realize $\mathfrak{g}$ as the fixed point set of an automorphism of a Lie algebra $\widetilde{\mathfrak{g}}$ of type A, D, E. The cases of the folding process $\widetilde{\mathfrak{g}} \rightsquigarrow \mathfrak{g}$ that we treat in this work are

$$\mathfrak{so}_{2n+2} \rightsquigarrow \mathfrak{so}_{2n+1} \quad , \quad \mathfrak{sl}_{2n} \rightsquigarrow \mathfrak{sp}_{2n} \quad , \quad \mathfrak{so}_8 \rightsquigarrow \mathfrak{g}_2$$

We are interested, when applicable, in comparing the Heisenberg connections associated to $\mathfrak{g}$ via the first fundamental representation of $\mathfrak{g}$ and via the corresponding construction for $\widetilde{\mathfrak{g}}$.

Definition 4. For the first fundamental representation ξ of $\mathfrak{g}$ and for $\mathbf{a}$ of $\widehat{\mathfrak{g}}$-type define

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},0} := \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(\xi, D^\times)_{\mathbf{a}}$$

and if $\mathfrak{g}$ is of type B, C, F, G also define

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},1} := \mathrm{Conn}^{\mathrm{Heis}}_{\widetilde{\mathfrak{g}}}(D^\times)_{\mathbf{a},0}$$

Note that the definition makes sense: The set of exponents $\mathrm{E}(\widehat{\mathfrak{g}})$ is a subset of $\mathrm{E}(\widehat{\widetilde{\mathfrak{g}}})$ as can be seen from the following table that lists the congruence conditions on integers i that define the set of exponents, see [9] for details.

	$\mathfrak{sl}_n$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{g}_2$
$\mathrm{E}(\widehat{\mathfrak{g}})$	$i \not\equiv 0 \mod n$	$i \equiv 1 \mod 2$	$i \equiv 1 \mod 2$	$i \equiv 1 \mod 2$	$i \equiv 1,5 \mod 6$
$\mathrm{E}(\widehat{\widetilde{\mathfrak{g}}})$	–	$i \equiv 1 \mod 2$	$i \not\equiv 0 \mod 2n$	–	$i \equiv 1,3,5 \mod 6$
$\mathrm{E}(\widehat{\mathfrak{sl}}_h)$	$i \not\equiv 0 \mod n$	$i \not\equiv 0 \mod 2n$	$i \not\equiv 0 \mod 2n$	$i \not\equiv 0 \mod 2n-2$	$i \equiv 1,3,5 \mod 6$

It follows that if $\mathbf{a}$ is of $\widehat{\mathfrak{g}}$-type then it is in particular also of $\widehat{\widetilde{\mathfrak{g}}}$-type and Definition 4 is well defined. From the table of exponents one can also observe that $\mathrm{E}(\widehat{\mathfrak{g}})$ is a subset of $\mathrm{E}(\widehat{\mathfrak{sl}}_h)$ for the Coxeter number h of $\mathfrak{g}$ (for $\mathfrak{g} = \mathfrak{so}_{2n}$ there are some exponents of multiplicity 2 in contrast to $\mathfrak{sl}_h$ however one can see from the definition of the Heisenberg connection that this does not impact the following arguments). Hence, if $\mathbf{a}$ is of $\widehat{\mathfrak{g}}$-type then it is also of $\widehat{\mathfrak{sl}}_h$-type. Therefore it makes sense to compare the connections $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},0}$ and $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}$ and we do so in Theorem 1.

3.2 Coarse Levelt-Turrittin normal form

Consider connections on a formal punctured disc $D^\times$, as described in Section 2. Levelt [12] and Turrittin [14] showed that isomorphism classes of such connections have certain normal forms, somewhat reminiscent of the Jordan normal form. The aim of the present work is to explicitly calculate the normal forms of the connections $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},i}$ where $i = 0, 1$. As already indicated, the choice of Chevalley generators can change the isomorphism class. Nonetheless, one can obtain a normal form calculation that is independent of the Lie theoretic choices, if one works with a notion of equivalence of connections slightly coarser than isomorphisms. We make this precise in the present section.

We start by recalling the Levelt-Turrittin normal form for connections on $D^\times = \operatorname{Spec} \mathbb{C}((t))$. Let n be a positive integer and let $f = \sum_i a_i t^{i/n}$ be an element of $\mathbb{C}((t^{1/n}))$ that is not an element of $\mathbb{C}((t^{1/m}))$ for $1 \leq m < n$. Write $f^{\text{hol}} = \sum_{i>0} a_i t^{i/n}$, $f^{\text{reg}} = a_0$, and $f^{\text{irreg}} = \sum_{i<0} a_i t^{i/n}$. The basic building blocks of the Levelt-Turrittin normal form are connections of the form

$$\mathcal{E}(f,n) := (\mathbb{C}((t^{1/n})), \partial_t + f(t)/t)$$

with $f^{\text{hol}} = 0$. This connection is irreducible and f^{irreg} is uniquely determined by the isomorphism class of the connection, up to a coordinate change $t^{1/n} \mapsto \zeta_n t^{1/n}$ for an n'th root of unity ζ_n and f^{reg} is determined up to adding an element in $\mathbb{Z} \cdot (1/n)$. The second type of connection needed to describe the normal form is given by

$$\text{Nil}_N := (\mathbb{C}((t))^N, \partial_t + (\sum_{i=1}^{N-1} e_{i,i+1})/t)$$

where $e_{i,i+1}$ is the matrix with zeros everywhere except a 1 at the $(i, i+1)$ entry. The following is the Levelt-Turrittin classification, see for example [8] for further references:

Theorem (Levelt-Turrittin)**.** *For every connection ∇ on $D^\times$ there exists $r \geq 1$ and f_i's, n_i's, and N_i's with*

$$\nabla \cong \bigoplus_{i=1}^{r} (\mathcal{E}(f_i, n_i) \otimes \text{Nil}_{N_i})$$

where the n_i's, and N_i's are unique up to permutation and the f_i's are unique up to adding elements in $\mathbb{Z} \cdot (1/n_i)$ and up to a substitution $t^{1/n_i} \mapsto \zeta_{n_i} \cdot t^{1/n_i}$ for a n_i'th root of unity ζ_{n_i}.

As indicated above, the isomorphism class of a connection of the form $\mathcal{E}(f,n)$ does not change under a coordinate change

$$t^{1/n} \mapsto \zeta_n \cdot t^{1/n}$$

One can work with a coarser notion than isomorphism, corresponding to allowing substitutions

$$t^{1/n} \mapsto c \cdot t^{1/n}$$

for an arbitrary non-zero constant c. We show in Theorem 1 that this is precisely the right notion of equivalence of connections that is invariant under changing the choice of Chevalley generators for the Heisenberg connections introduced in Definition 16.

Definition 5. For two connections ∇_1 and ∇_2 on $D^\times$ with Levelt-Turrittin normal forms

$$\nabla_1 \cong \bigoplus_{i=1}^{r} (\mathcal{E}(f_i, n_i) \otimes \text{Nil}_{N_i}) \quad , \quad \nabla_2 \cong \bigoplus_{j=1}^{s} \left(\mathcal{E}(g_j, m_j) \otimes \text{Nil}_{M_j}\right)$$

We write $\nabla_1 \sim \nabla_2$ if $r = s$ and there is a permutation σ and non-zero scalars $c_1, \cdots, c_r$ such that

$$n_i = m_{\sigma(i)} \ \ , \ \ N_i = M_{\sigma(i)} \ \ , \ \ f_i(z^{1/n_i}) = g_{\sigma(i)}(c_i \cdot z^{1/n_i}) + \mathbb{Z} \cdot \frac{1}{n_i}$$

for all i.

One sees that $\sim$ is an equivalence relation that only depends on the isomorphism classes of ∇_1 and ∇_2.

3.3 Main result

In the present section we carry out the determination of the coarse Levelt-Turrittin normal form for Heisenberg connections associated to an affine Lie algebra $\widehat{\mathfrak{g}}$. As indicated in the introduction, these Heisenberg connections are related to the $\widehat{\mathfrak{sl}}_h$ Heisenberg connections, where h is the Coxeter number of $\mathfrak{g}$, in a manner similar to the relation of eigenvalues of the Lie algebra elements defining the connection. The subtle difference between the linear algebraic and the non-linear algebraic problem manifests itself in a shift for the regular singular term. This shift is given by κ, see Equation (12) for the Lie theoretic meaning of this constant.

In Definition 4 we introduced the connections $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},j}$ where $j=0$ for $\mathfrak{g}$ of type A and D and $j \in \{0,1\}$ for $\mathfrak{g}$ of type B, C, G. We now determine the coarse Levelt-Turrittin normal forms of these connections. In the following let $\mathbf{0} = (\mathbb{C}(\!(t)\!), \partial_t)$ denote the trivial 1-dimensional connection on $D^\times$.

Theorem 1. *Let $\mathfrak{g}$ be a simple complex Lie algebra of type* A, B, C, D, *or* G. *Let h be the Coxeter number, let n be the dimension of the first fundamental representation, and let $\mathbf{a} = (a_i)_i$ be of $\widehat{\mathfrak{g}}$-type such that the largest i with $a_i \neq 0$ is co-prime to h. Then*

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},j} \quad \sim \quad \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}\,[\kappa] \oplus S_j \tag{17}$$

$$\sim \quad \mathcal{E}\left((2\kappa - h - 1)/(2h) + \sum a_i z^i, h\right) \oplus S_j \tag{18}$$

where the correction term S_j and the value of κ is given for the various types of Lie algebras $\mathfrak{g}$ by

	A	B	C	D	G
κ	1	1/2	1	1/2	1/2
S_0	–	$\mathbf{0}[\kappa]$	–	$\mathbf{0}[\kappa] \oplus \mathbf{0}$	$\mathbf{0}[\kappa]$
S_1	–	$\mathbf{0}[\kappa] \oplus \mathbf{0}$	–	–	$\mathbf{0}[\kappa] \oplus \mathbf{0}$

Remark 1. The regular singular term of the first direct summand in Equation (18) corresponds to $(2\kappa - h - 1)/(2h)$. As discussed in Section 3.2, this term has an ambiguity by adding an arbitrary integer multiple of $1/h$. Hence if $\kappa = 1$ then

$$(2\kappa - h - 1)/(2h) \sim (-h-1)/2h$$

and one can hence ignore the κ contribution to the regular singular term. However, if $\kappa = 1/2$ then

$$(2\kappa - h - 1)/(2h) \not\sim (-h-1)/2h$$

and the κ contribution to the regular singular term is genuine and cannot be gauged away.

Proof. (of the theorem) As remarked before, if $\mathbf{a}$ is of $\widehat{\mathfrak{g}}$-type then it is also of $\widehat{\mathfrak{sl}}_h$-type and therefore it makes sense to compare $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},j}$ and $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}$. For the first fundamental representation ξ of $\mathfrak{g}$ we write from now on for simplicity g instead of $\xi(g)$ for any element g of $\widehat{\mathfrak{g}}$.

Note that in Equation (17) we did not specify a choice of Chevalley generators for the two Lie algebras involved. In fact, as part of the proof, we show that the $\sim$ equivalence class, see Definition 5, of the Heisenberg connections are independent of the choice of Chevalley generators. Our strategy to establish the theorem is to first show that for some choice of Chevalley generators of $\widehat{\mathfrak{g}}$ and some choice of Chevalley generators of $\widehat{\mathfrak{sl}}_h$ there is an isomorphism

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},j} \;\cong\; \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}\,[\kappa - 1] \oplus S_j \tag{19}$$

We then proceed to show that

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}\,[\kappa - 1] \oplus S_j \cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}\,[\kappa] \oplus S_j \tag{20}$$

and we show that independent of the choice of Chevalley generators the right-hand side of Equation (20) is $\sim$ equivalent to the expression in Equation (18). Proposition 1 then implies that for any choice of Chevalley generators of $\widehat{\mathfrak{g}}$ the $\sim$ equivalence class of $\mathrm{Conn}_{\mathfrak{g}}^{\mathrm{Heis}}(D^\times)_{\mathbf{a},j}$ is given by Equation (17).

We start the proof of Equation (19). We will see that a simple coordinate change relates the cyclic element of $\widehat{\mathfrak{g}}$ to a cyclic element of $\widehat{\mathfrak{sl}}_h$, as discussed for Lie algebras of type B in Section 2. This allows the calculation of the eigenvalues. However, the coordinate change has z dependency and therefore when looking at the Heisenberg connections one picks up an extra gauge term. To deal with this issue, our next step of the proof is to show that the Heisenberg connections are isomorphic to generalized Kac-Schwarz operators. This amounts to the change

$$\partial_z \mapsto \partial_z + \frac{\rho_{\mathfrak{g}}^\vee}{hz}$$

where $\rho_{\mathfrak{g}}^\vee$ is the element of $\mathfrak{g}$ defined via $[\rho_{\mathfrak{g}}^\vee, e_i] = e_i$ for all $1 \leq i \leq r$. Here the e_i's denote the positive Chevalley generators of $\mathfrak{g}$. It will turn out that a simple coordinate change relating the cyclic element of $\widehat{\mathfrak{g}}$ to a cyclic element of $\widehat{\mathfrak{sl}}_h$ happens to have the property that the change of $\rho_{\mathfrak{g}}^\vee$ interacts very well with the gauge term. Denote by $e_{i,j}$ the $n \times n$ matrix whose entries are all 0 apart from the (i,j) entry which is 1. Cafasso and Wu show in [3] that there is an element μ in the Lie group of $\widehat{\mathfrak{g}}$ with

$$\mu \left(\partial_z + \frac{\rho_{\mathfrak{g}}^\vee}{hz} - \Lambda_1 \right) \mu^{-1} = \partial_z - \Lambda_1 \tag{21}$$

In fact, μ is the dressing operator of the Witten-Kontsevich point of the $\mathfrak{g}$ Drinfeld-Sokolov hierarchy. We now prove that an analogue of Equation (21) holds for all Heisenberg connections, see Equation (26) for the precise statement.

Note that, up to the central term, one has for every choice of cyclic element a decomposition

$$\widehat{\mathfrak{g}} = \ker \operatorname{ad} \Lambda_1 \oplus \operatorname{Im} \operatorname{ad} \Lambda_1 \tag{22}$$

Let j be a positive exponent of $\widehat{\mathfrak{g}}$ that is co-prime to the Coxeter number h. We claim that for $\mathfrak{g}$ of type A, B, C, D, G and any choice of cyclic element Λ_1 one has

$$\ker \operatorname{ad} \Lambda_1^j = \ker \operatorname{ad} \Lambda_1 \quad ; \quad \operatorname{Im} \operatorname{ad} \Lambda_1^j = \operatorname{Im} \operatorname{ad} \Lambda_1 \tag{23}$$

Before proving this, recall that there is some choice of Chevalley generators such that for any positive exponent i the element Λ_1^i is in the principal degree i part of the Heisenberg algebra associated to Λ_1. It follows from the proof of Proposition 1 that this statement then in fact holds for any choice of Chevalley generators. We now come back to the proof of Equation (23). Recall Λ_1 can be written as $z \cdot E_0 + \sum_{i=1}^r E_i$. Let $\overline{\Lambda}_1$ be the element of $\mathfrak{g}$ given by evaluation of Λ_1 at $z = 1$. The map ν from the homogeneous to the principal realization of the untwisted affine Lie algebra associated to $\mathfrak{g}$ can be described on the level of the loop algebra $\mathfrak{g}[z, z^{-1}]$ in the following manner. For X in $\mathfrak{g}$ of principal degree d one has

$$z^i \otimes X \mapsto z^{hi+d} \otimes X$$

Since it is known that Λ_1^i is of principal degree i one can deduce that $\nu(\Lambda_1^i) = z^i \otimes \overline{\Lambda}_1^i$. One can verify by direct calculation that for the choices of Λ_1 listed in Section 3.1 one has $\overline{\Lambda}_1^{h+1} = \overline{\Lambda}_1$ and hence more generally $\overline{\Lambda}_1^{ah+b} = \overline{\Lambda}_1^b$ for non-negative integers a and b. Therefore, if j is co-prime to h then some power of $\overline{\Lambda}_1^j$ equals $\overline{\Lambda}_1$. It follows that

$$\operatorname{Im} \operatorname{ad} \overline{\Lambda}_1 \subseteq \operatorname{Im} \operatorname{ad} \overline{\Lambda}_1^j \quad , \quad \ker \operatorname{ad} \overline{\Lambda}_1^j \subseteq \ker \operatorname{ad} \overline{\Lambda}_1 \tag{24}$$

Since also clearly

$$\operatorname{Im} \operatorname{ad} \overline{\Lambda}_1^j \subseteq \operatorname{Im} \operatorname{ad} \overline{\Lambda}_1 \quad , \quad \ker \operatorname{ad} \overline{\Lambda}_1 \subseteq \ker \operatorname{ad} \overline{\Lambda}_1^j \tag{25}$$

one obtains that the image and kernel of the adjoint action of $\overline{\Lambda}_1$ and $\overline{\Lambda}_1^j$ agree. Switching back to the homogeneous realization of $\widehat{\mathfrak{g}}$ one obtains Equation (23).

We now use Equation (22) and Equation (23) to show that there is $Y = \sum_{j \leq 0} Y_j$ with Y_j in $\widehat{\mathfrak{g}}$ of principal degree

j, such that

$$\exp(\mathrm{ad}\ Y)\left(\partial_z + \frac{\rho_{\mathfrak{g}}^\vee}{hz} + \sum_{i\le m} a_i\Lambda_i\right) = \partial_z + \sum_{i\le m} a_i\Lambda_i \tag{26}$$

with $a_m \neq 0$ (our arguments in fact imply the corresponding result with $\rho_{\mathfrak{g}}^\vee$ replaced by any element of Cartan algebra of $\mathfrak{g}$). Let $\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} = \partial_z + \rho_{\mathfrak{g}}^\vee/(hz)$. It is known that ad $\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}}$ maps the Heisenberg algebra to itself and for any element g of $\widehat{\mathfrak{g}}$ (in the homogeneous realization) of principal degree j one has

$$\left[\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}}\ ,\ g\right] = \frac{j}{hz}\cdot g \tag{27}$$

Note that the left-hand side of Equation (26) is given by

$$\exp(\mathrm{ad}\ Y)\left(\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} + \sum_{i\le m} a_i\Lambda_i\right) = \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} + \sum_{i\le m} a_i\Lambda_i + [Y, \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} + \sum_{i\le m} a_i\Lambda_i] + \frac{[Y,[Y,\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}} + \sum_{i\le m} a_i\Lambda_i]]}{2!} + \cdots$$

Set $Y_j = 0$ for $-h-m < j \le 0$. We solve Equation (26) recursively with respect to decreasing principal gradation. The first non-trivial equation is in degree $-h$ and is given by

$$[Y_{-h-m}, a_m\Lambda_m] = -\frac{\rho_{\mathfrak{g}}^\vee}{hz} \tag{28}$$

Between degree $-h-1$ and $-2h-m+1$ the equations are given by

$$[Y_{-h-m-i}, a_m\Lambda_m] + \sum_{j=1}^{i}[Y_{-h-m-i+j}, a_{m-j}\Lambda_{m-j}] = 0 \tag{29}$$

where $1 \le i \le h+m-1$. In degree $-2h-m$ the equation is

$$[Y_{-2h-2m}, a_m\Lambda_m] + [Y_{-h-m}\ ,\ \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}}] + \frac{[Y_{-h-m},[Y_{-h-m}, a_m\Lambda_m]]}{2!} + \sum_{j=1}^{h+m}[Y_{-2h-2m+j}, a_{m-j}\Lambda_{m-j}] = 0 \tag{30}$$

More generally, for degree $-2h-m-i$ with $i \ge 0$ the equation is

$$[Y_{-2h-2m-i}, a_m\Lambda_m] + [Y_{-h-m-i}\ ,\ \mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}}] + \cdots = 0 \tag{31}$$

where the $\cdots$ expression does not contain any triply or higher iterated brackets involving Y_{-h-m-i}. Namely, since we set $Y_j = 0$ for $j > -h-m$ it follows that the highest degree of a triply iterated bracket involving Y_{-h-m-i} at least once is

$$(-h-m-i) + (-h-m) + (-h-m) + m = -3h-2m-i < -2h-m-i$$

We now show that there exists Y solving these equations. Via Equation (22) every element A of $\widehat{\mathfrak{g}}$ can be written uniquely as a sum $B+C$ with B in the Heisenberg algebra and C in the image of the adjoint action of Λ_1. We call B the Heisenberg part of A. Consider now first Equation (28). Since $-h$ is not an exponent, the Heisenberg part of the right-hand side is 0 and it follows from Equation (23) that there is a solution Y_{-h-m} to Equation (28). To show that there is a choice of Y_{-h-m-i} that solves Equation (29) note that since for every $r \ge 1$

$$\mathrm{Im\ ad}\ \Lambda_r = \mathrm{Im\ ad}\ \Lambda_1^r \subseteq \mathrm{Im\ ad}\ \Lambda_1 \tag{32}$$

there is g in $\widehat{\mathfrak{g}}$ with

$$\sum_{j=1}^{i} [Y_{-h-m-i+j}, a_{m-j}\Lambda_{m-j}] = [\Lambda_1, g]$$

Since by Equation (23) $\mathrm{Im}\ \mathrm{ad}\ \Lambda_1 = \mathrm{Im}\ \mathrm{ad}\ \Lambda_m$ it follows that Equation (29) has a solution. Since the Heisenberg algebra is commutative, both for Equation (28) and Equation (29) we can add an arbitrary element in the Heisenberg algebra to our choice of solution while still solving those equations. This freedom will now be used to guarantee a solution to Equation (30) and more generally Equation (31). We start with Equation (30). Let H denote the Heisenberg part of the last three summands of the left-hand side of Equation (30). Then

$$\tilde{H} := H \cdot \frac{h}{-2h-m} \cdot z$$

is again an element of the Heisenberg algebra and of degree $-h-m$. We now modify Y_{-h-m} by subtracting $\tilde{H}$. This does not affect the validity of the equations in degree bigger than $-2h-m$. Furthermore, by Equation (27) the new Heisenberg part of $[\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}}\ ,\ Y_{-h-m}]$ is obtained by subtracting H. Also, the Heisenberg part of

$$\frac{[Y_{-h-m}, [Y_{-h-m}, a_m\Lambda_m]]}{2!} \tag{33}$$

does not change because by the commutativity of the Heisenberg algebra subtracting an element $\tilde{H}$ of the Heisenberg algebra to Y_{-h-m} changes the quantity in Equation (33) by

$$-\frac{[\tilde{H}, [Y_{-h-m}, a_m\Lambda_m]]}{2!} = \frac{[a_m\Lambda_m, [\tilde{H}, Y_{-h-m}]]}{2!}$$

This is in the image of $\mathrm{ad}\ \Lambda_m$ and hence by Equation (23) is in the image of $\mathrm{ad}\ \Lambda_1$ and hence has 0 Heisenberg part by Equation (22). Hence, the Heisenberg part of

$$\sum_{j=1}^{h+m} [Y_{-2h-2m+j}, a_{m-j}\Lambda_{m-j}] + \frac{[Y_{-h-m}, [Y_{-h-m}, a_m\Lambda_m]]}{2!} + [\mathrm{d}^{\mathrm{pri}}_{-1,\widehat{\mathfrak{g}}}\ ,\ Y_{-h-m}]$$

can be made 0 (using again the commutativity of the Heisenberg algebra for the first $h+m$ summands), and hence there is Y_{-2h-2m} such that Equation (30) has a solution.

More generally we now show how to add a suitable element of the Heisenberg algebra to Y_{-h-m-i} to guarantee that the equation in principal degree $-2h-m-i$, namely Equation (31) has a solution. A similar argument as for Equation (30) applies. Using Equation (32), an argument as for Equation (30) shows that any bracket of the form

$$[Y_{-h-m-i}, [Y_j, a_k\Lambda_k]] \quad \text{or} \quad [Y_j, [Y_{-h-m-i}, a_k\Lambda_k]]$$

does not change its Heisenberg part as the Heisenberg part of Y_{-h-m-i} varies. Finally, note that there is no degree 1 bracket involving Y_{-h-m-i} that contributes in degree $-2h-m-i$ since $-h$ is not an exponent and hence $a_{-h} = 0$. The analogous argument as for Equation (30) now shows that there is a solution to Equation (31). It follows that there is Y solving Equation (26).

For each $\mathfrak{g}$ of type A, B, C, D, G we now make concrete choices of Chevalley generators and prove Equation (19) for those choices. Recall that for $\mathfrak{g}$ of type A, B, C, D the first fundamental representation is isomorphic to the defining vector representation. For $\mathfrak{g}$ of type $\mathfrak{g}_2$ the first fundamental representation is the 7-dimensional representation. We refer to [1] and [4] for tables of Chevalley generators for the algebras that we consider.

For $\mathfrak{g} = \mathfrak{sl}_n$ we choose the Chevalley generators such that

$$\Lambda_{\widehat{\mathfrak{sl}}_n} = z \cdot e_{1,n} + \sum_{i=1}^{n-1} e_{i+1,i} \quad , \quad \rho^{\vee}_{\mathfrak{sl}_n} = \frac{1}{2}\sum_{i=1}^{n} (-1-n+2i) \cdot e_{i,i} \tag{34}$$

The algebra $\mathfrak{sp}_{2n}$ can be obtained via folding from $\mathfrak{sl}_{2n}$ and one can choose Chevalley generators such that $\Lambda_{\widehat{\mathfrak{sp}}_{2n}} = \Lambda_{\widehat{\mathfrak{sl}}_{2n}}$ and $\rho^\vee_{\mathfrak{sp}_{2n}} = \rho^\vee_{\mathfrak{sl}_{2n}}$. It follows that

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sp}_{2n}}(D^\times)_{\mathbf{a},0} \cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sp}_{2n}}(D^\times)_{\mathbf{a},1} \cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_{2n}}(D^\times)_{\mathbf{a},0}$$

as desired. The next case we consider is $\mathfrak{g} = \mathfrak{so}_{2n+1}$. In this case the calculations are only a slight generalization of those described in Section 2. One can choose Chevalley generators such that

$$\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} = \sum_{i=1}^{2n} e_{i+1,i} + \frac{z}{2}\cdot(e_{1,2n}+e_{2,2n+1}) \quad , \quad \rho^\vee_{\mathfrak{so}_{2n+1}} = \sum_{i=1}^{2n+1}(-n-1+i)\cdot e_{i,i} \tag{35}$$

Let $c_1, \cdots, c_{2n+1}$ denote the standard basis of $\mathbb{C}^{2n+1}$ and view it as a basis of $\mathbb{C}(\!(1/z)\!)^{2n+1}$ as a $\mathbb{C}(\!(1/z)\!)$-vector space. Consider the change of coordinates to the basis

$$d_1 = \frac{c_1}{2} + \frac{c_{2n+1}}{z} \quad , \quad d_i = c_i \quad \text{for } 2 \le i \le 2n \quad , \quad d_{2n+1} = \frac{c_1}{2} - \frac{c_{2n+1}}{z}$$

In terms of the new basis one obtains $\Lambda_{\widehat{\mathfrak{so}}_{2n+1}} = \sum_{i=1}^{2n-1} e_{i+1,i} + z\cdot e_{1,2n}$ as well as

$$\frac{\rho^\vee_{\mathfrak{so}_{2n+1}}}{hz} = \frac{1}{2nz}\cdot\left(-n\cdot(e_{1,2n+1}+e_{2n+1,1}) + \sum_{i=2}^{2n}(-n-1+i)\cdot e_{i,i}\right)$$

The gauge term of the coordinate change is $\gamma\partial_z(\gamma^{-1})$ where

$$\gamma^{-1} = \sum_{i=2}^{2n} e_{i,i} + \frac{1}{2}(e_{1,1}+e_{1,2n+1}) + \frac{1}{z}(e_{2n+1,1} - e_{2n+1,2n+1})$$

$$\gamma = \sum_{i=2}^{2n} e_{i,i} + e_{1,1} + e_{2n+1,1} + \frac{z}{2}(e_{1,2n+1} - e_{2n+1,2n+1})$$

One obtains

$$\gamma\partial_z(\gamma^{-1}) = \frac{1}{2z}\cdot(-e_{1,1} + e_{2n+1,1} + e_{1,2n+1} - e_{2n+1,2n+1})$$

It follows that the Heisenberg connection $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{so}_{2n+1}}(D^\times)_{\mathbf{a},0}$ with the Lie theoretic choices as in Equation (35) is isomorphic to the connection (W,∇) with underlying vector space $W = \mathbb{C}^{2n+1}(\!(1/z)\!)$ and $\nabla : W \to W$ given by

$$\partial_z + \frac{1}{2nz}\cdot\sum_{i=1}^{2n}(-n-1+i)\cdot e_{i,i} + \sum_{j\in\mathbb{Z}^{>0}} a_j\left(\sum_{i=1}^{2n-1} e_{i+1,i} + z\cdot e_{1,2n}\right)^j - \frac{1}{2z}\cdot e_{2n+1,2n+1}$$

Note that

$$\frac{1}{2nz}\cdot\sum_{i=1}^{2n}(-n-1+i)\cdot e_{i,i} = \frac{1}{2hz}\cdot\sum_{i=1}^{h}(-h-2+2i)\cdot e_{i,i} = \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} - \frac{1}{2h}\cdot\sum_{i=1}^{h} e_{i,i}$$

and

$$\sum_{j\in\mathbb{Z}^{>0}} a_j\left(\sum_{i=1}^{2n-1} e_{i+1,i} + z\cdot e_{1,2n}\right)^j = \sum_{j\in\mathbb{Z}^{>0}} a_j\Lambda^j_{\widehat{\mathfrak{sl}}_h}$$

where $\Lambda_{\widehat{\mathfrak{sl}}_h}$ and $\rho^\vee_{\mathfrak{sl}_h}$ are chosen as in Equation (34). It follows that

$$\begin{aligned} \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},0} &\cong \left(\mathbb{C}(\!(1/z)\!)^h \ , \ \partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \sum_{j\in\mathbb{Z}^{>0}} a_j \Lambda^j_{\widehat{\mathfrak{sl}}_h} - \frac{1}{2h}\cdot\sum_{i=1}^h e_{i,i} \right) \oplus \mathbf{0}[-1/2] \\ &\cong \left(\mathbb{C}(\!(1/z)\!)^h \ , \ \partial_z + \sum_{j\in\mathbb{Z}^{>0}} a_j \Lambda^j_{\widehat{\mathfrak{sl}}_h} - \frac{1}{2h}\cdot\sum_{i=1}^h e_{i,i} \right) \oplus \mathbf{0}[-1/2] \\ &\cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}\,[-1/2] \oplus \mathbf{0}[-1/2] \\ &\cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}\,[-1/2] \oplus \mathbf{0}[1/2] \end{aligned}$$

as desired. Note that the second isomorphism follows from Equation (26) since $1/(2h)\cdot\sum_{i=1}^h e_{i,i}$ is invariant under conjugation and the fourth isomorphism holds since $(-1/2)-1/2$ is an integer and hence, see Section 3.2, one has $\mathbf{0}[-1/2]\cong\mathbf{0}[1/2]$. The theorem for $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}}(D^\times)_{\mathbf{a},1}$ will be obtained later on. Assume now that $\mathfrak{g}=\mathfrak{so}_{2n}$. In this case one can choose Chevalley generators such that

$$\Lambda_{\widehat{\mathfrak{so}}_{2n}} = \frac{1}{2}\cdot(e_{n+1,n-1}+e_{n+2,n}) + \sum_{i=1}^{n-1}(e_{i+1,i}+e_{2n+1-i,2n-i}) + \frac{z}{2}(e_{1,2n-1}+e_{2,2n}) \tag{36}$$

as well as

$$\rho^\vee_{\mathfrak{so}_{2n}} = \sum_{i=1}^{n-1}(-n+i)e_{i,i} + \sum_{i=n+2}^{2n}(-n-1+i)e_{i,i} \tag{37}$$

Let $c_1,\cdots,c_{2n}$ denote the standard basis of $\mathbb{C}^{2n}$. Consider the change of coordinates to the basis

$$d_1 = \frac{c_1}{2}+\frac{c_{2n}}{z} \quad , \quad d_i = c_i \quad \text{for } 2\le i\le n-1 \quad , \quad d_n = c_n + \frac{c_{n+1}}{2}$$

$$d_i = c_{i+1} \quad \text{for } n+1\le i\le 2n-2 \quad , \quad d_{2n-1} = \frac{c_1}{2}-\frac{c_{2n}}{z} \quad , \quad d_{2n} = c_n - \frac{c_{n+1}}{2}$$

In terms of the new basis one obtains $\Lambda_{\widehat{\mathfrak{so}}_{2n}} = \sum_{i=1}^{2n-3} e_{i+1,i} + z\cdot e_{1,2n-2}$ as well as

$$\rho^\vee_{\mathfrak{so}_{2n}} = (-n+1)(e_{2n-1,1}+e_{1,2n-1}) + \sum_{i=2}^{n-1}(-n+i)e_{i,i} + \sum_{i=n+1}^{2n-2}(-n+i)e_{i,i}$$

The gauge term of the coordinate change is $\gamma\partial_z(\gamma^{-1})$ where

$$\gamma^{-1} = \sum_{i=2}^{n-1} e_{i,i} + \sum_{i=n+1}^{2n-2} e_{i+1,i} + \frac{1}{z}(e_{2n,1}-e_{2n,2n-1}) + \frac{1}{2}(e_{1,1}+e_{n+1,n}-e_{n+1,2n}+e_{1,2n-1}) + e_{n,n} + e_{n,2n}$$

$$\gamma = \sum_{i=2}^{n-1} e_{i,i} + \sum_{i=n+1}^{2n-2} e_{i,i+1} + \frac{z}{2}(e_{1,2n}-e_{2n-1,2n}) + \frac{1}{2}(e_{n,n}+e_{2n,n}) + e_{1,1} + e_{2n-1,1} + e_{n,n+1} - e_{2n,n+1}$$

One obtains

$$\gamma\partial_z(\gamma^{-1}) = \frac{1}{2z}\cdot(-e_{1,1}-e_{2n-1,2n-1}+e_{1,2n-1}+e_{2n-1,1})$$

It follows that $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{so}_{2n}}(D^\times)_{\mathbf{a},0}$ with the Lie theoretic choices as in Equation (36) and Equation (37) is isomorphic to

$\partial_z + A + B + C$ where

$$A = \frac{1}{(2n-2)z} \cdot \left((-n+1)(e_{2n-1,1} + e_{1,2n-1}) + \sum_{i=2}^{2n-2} (-n+i) \cdot e_{i,i} \right)$$

$$B = \sum_{j \in \mathbb{Z}^{>0}} a_j \left(\sum_{i=1}^{2n-3} e_{i+1,i} + z \cdot e_{1,2n-2} \right)^j$$

$$C = \frac{1}{2z} \cdot (-e_{1,1} - e_{2n-1,2n-1} + e_{1,2n-1} + e_{2n-1,1})$$

Hence there are again some important cancellations in the sum $A + B + C$ and one obtains

$$\begin{aligned} &\partial_z + A \quad + \quad B + C \\ &\qquad\qquad = \\ \partial_z + \frac{1}{(2n-2)z} \cdot \left(\sum_{i=1}^{2n-2} (-n+i) e_{i,i} \right) \quad &+ \quad \sum_{j \in \mathbb{Z}^{>0}} a_j \left(\sum_{i=1}^{2n-3} e_{i+1,i} + z \cdot e_{1,2n-2} \right)^j - \frac{1}{2z} \cdot e_{2n-1,2n-1} \end{aligned} \tag{38}$$

Furthermore, one has

$$\frac{1}{2n-2} \sum_{i=1}^{2n-2} (-n+i) e_{i,i} = \frac{1}{2h} \cdot \sum_{i=1}^{h} (-h+2i-2) \cdot e_{i,i} \quad = \quad \rho^\vee_{\mathfrak{sl}_h} - \frac{1}{2h} \cdot \sum_{i=1}^{h} e_{i,i}$$

It follows from Equation (26) and Equation (38) that Equation (19) for $j = 0$ holds. One can also deduce the $j = 1$ result for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ since $\mathfrak{so}_{2n+1}$ is obtained via folding from $\mathfrak{so}_{2n+2}$. It then follows from our calculations that

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{so}_{2n+1}}(D^\times)_{\mathbf{a},1} \cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{so}_{2n+1}}(D^\times)_{\mathbf{a},0} \oplus \mathbf{0}$$

as desired.

The last case to consider is $\mathfrak{g} = \mathfrak{g}_2$. The first fundamental representation is 7-dimensional and one can choose Chevalley generators, see for example [1] (note however that we choose the dual convention compared to loc. cit. in order to align with our choices of Chevalley generators of the other Lie algebras that we consider), such that

$$\Lambda_{\widehat{\mathfrak{g}}_2} = \sum_{i=1}^{6} e_{i+1,i} + \frac{z}{2} \cdot (e_{1,6} + e_{2,7})$$

as well as

$$H_1 = -e_{1,1} + e_{2,2} - 2e_{3,3} + 2e_{5,5} - e_{6,6} + e_{7,7} \quad , \quad H_2 = -e_{2,2} + e_{3,3} - e_{5,5} + e_{6,6}$$

Write $\rho^\vee_{\mathfrak{g}_2} = n_1 H_1 + n_2 H_2$ for scalars n_1 and n_2. Solving $\alpha_i(\rho^\vee) = 1$ for $i = 1, 2$ and using $\alpha_i(H_j) = A_{i,j}$, where $A = (A_{i,j})$ is the Cartan matrix, yields $n_1 = 5$ and $n_2 = 3$. It follows that

$$\frac{\rho^\vee_{\mathfrak{g}_2}}{h_{\mathfrak{g}_2}} \cdot \frac{1}{z} = (-3e_{1,1} - 2e_{2,2} - e_{3,3} + e_{5,5} + 2e_{6,6} + 3e_{7,7}) \cdot \frac{1}{6z}$$

One sees that

$$\Lambda_{\widehat{\mathfrak{g}}_2} = \Lambda_{\widehat{\mathfrak{so}}_7} \quad , \quad \frac{\rho^\vee_{\mathfrak{g}_2}}{h_{\mathfrak{g}_2}} \cdot \frac{1}{z} = \frac{\rho^\vee_{\mathfrak{so}_7}}{h_{\mathfrak{so}_7}} \cdot \frac{1}{z}$$

where the Lie theoretic choices for $\widehat{\mathfrak{so}}_7$ are as in Equation (35). It then follows from the earlier type B calculations that

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}_2}(D^\times)_{\mathbf{a},0} \cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_6}(D^\times)_{\mathbf{a},0} \oplus \mathbf{0}$$

The Lie algebra $\mathfrak{g}_2$ is obtained via folding from $\mathfrak{so}_8$. One deduces from the earlier type D calculations that

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}_2}(D^\times)_{\mathbf{a},1} \cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{g}_2}(D^\times)_{\mathbf{a},0} \oplus \mathbf{0}$$

This gives the desired result for $\mathfrak{g} = \mathfrak{g}_2$ by noting that $\kappa = 1/2$, as indicated earlier. For completeness we calculate κ directly. When decomposing the Cartan matrix A as $A = DB$ with D diagonal and B symmetric, there is a non-zero constant c such that the lower right entry of D is $3c$ and the corresponding entry of B is $2/(3c)$. For each choice of c one obtains a standard invariant bilinear form $(-,-)_c$ on $\mathfrak{g}$ as defined by Kac in [9]. It satisfies $(\alpha_2, \alpha_2)_c = 2/(3c)$ and $H_2/(3c)$ is dual to α_2. In the normalization that the long root α_2 has square length 2 it follows that $c = 1/3$, H_2 is dual to α_2. One then has

$$\kappa \cdot \mathrm{Tr}(H_2^2) = 2 \quad \rightsquigarrow \quad \kappa = \frac{1}{2}$$

In order to complete the prove of the theorem. we first show that for our choice of Chevalley generators one has

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}[\kappa - 1] \cong \mathcal{E}\left((2\kappa - h - 1)/(2h) + \sum_{i \in \mathbb{Z}^{>0}} a_i z^i, h\right) \tag{39}$$

$$\cong \mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}[\kappa] \tag{40}$$

As shown before, there is an isomorphism

$$\left(\mathbb{C}(\!(1/z)\!)^h, \partial_z + \sum_{i \in \mathbb{Z}^{>0}} a_i \Lambda_i\right) \cong \left(\mathbb{C}(\!(1/z)\!)^h, \partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \sum_{i \in \mathbb{Z}^{>0}} a_i \Lambda_i\right)$$

Let ζ be such that $\zeta^h = z$ and consider as in [10] an isomorphism of $\mathbb{C}(\!(1/z)\!)$-vector spaces between $\mathbb{C}(\!(1/z)\!)^h$ and $\mathbb{C}(\!(1/\zeta)\!)$ given by $f(z)e_i \mapsto f(\zeta^h)\zeta^{h-i}$ where e_i is the standard i'th basis element in $\mathbb{C}(\!(1/z)\!)^h$ with 0's everywhere except a 1 in the i'th entry. Under this map one has for any scalar s

$$\partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \frac{s}{hz} \cdot \sum_{i=1}^h e_{i,i} + \sum_{i \in \mathbb{Z}^{>0}} a_i \Lambda_i \quad \mapsto \quad \partial_{\zeta^h} + \frac{1 - h + 2s}{2h} \cdot \zeta^{-h} + \sum_{i \in \mathbb{Z}^{>0}} a_i \zeta^i$$

Letting $s = \kappa - 1$ one obtains Equation (39). Letting $s = \kappa$ one obtains

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)_{\mathbf{a},0}[\kappa] \cong \mathcal{E}\left((2\kappa - h + 1)/(2h) + \sum_{i \in \mathbb{Z}^{>0}} a_i z^i, h\right)$$

$$\cong \mathcal{E}\left((2\kappa - h - 1)/(2h) + \sum_{i \in \mathbb{Z}^{>0}} a_i z^i, h\right)$$

where the second isomorphism holds since the difference of the two regular singular terms is in $\mathbb{Z}/h$. Therefore one obtains Equation (40). Suppose now that a different choice of Chevalley generators for $\widehat{\mathfrak{sl}}_h$ is made. Repeating the above calculations with the a_i's replaced by $a_i \lambda^i$ (for some non-zero scalar λ) and using Proposition 1 it follows that the corresponding Heisenberg connection is isomorphic to

$$\partial_{\zeta^h} + \frac{2\kappa - h - 1}{2h} \zeta^{-h} + \sum_{i \in \mathbb{Z}^{>0}} a_i \lambda^i \zeta^i$$

for a suitable non-zero scalar λ. The $\sim$ equivalence class of this connection is independent of λ and therefore the $\sim$ equivalence class of the Heisenberg connection is independent of the choice of Chevalley generators. Therefore Equation (39) implies Equation (18). This completes the proof of the theorem. □

Remark 2. There are several points of interaction between the above calculations of normal forms of Heisenberg connections associated to $\widehat{\mathfrak{g}}$ and the Drinfeld-Sokolov hierarchy associated to $\widehat{\mathfrak{g}}$. A close relation is natural since the Heisenberg algebra is intricately related to the flows of the hierarchy.

First, Equation (26) is a key tool in the proof of Theorem 1 and as indicated before this equation is in fact a generalization of a central result employed by Cafasso and Wu [1](Lemma 3.8 and Theorem 3.10) in their work on Witten-Kontsevich points of Drinfeld-Sokolov hierarchies. The reason that Equation (26) is relevant for Witten-Kontsevich points is that via this type of isomorphism Heisenberg connections describe certain Virasoro constraints on tau functions when the Drinfeld-Sokolov phase space is described in terms of a suitable Sato Grassmannian.

Another close relation between Heisenberg connections and Drinfeld-Sokolov hierarchies concerns the role of folding constructions for Lie algebras. It is shown by Cafasso and Wu in [1] (Section 4.2) that the generalized Witten-Kontsevich points of type A, D, E yield, after a suitable restriction on the flow variables, the corresponding points for algebras of type B, C, F, G that are obtained via folding. See also the work of Liu, Ruan, Zhang [13] on folding constructions for Drinfeld-Sokolov hierarchies. Similarly, we show in Theorem 1, that the difference between the Heisenberg connection of a B, C, G type algebra via its first fundamental representation and via the folding construction is almost negligible.

Lastly, the shift involving κ in Theorem 1 resembles a shift by κ for tau functions shown by Cafasso and Wu in [2]. Given a tau function τ of the Drinfeld-Sokolov hierarchy of $\widehat{\mathfrak{g}}$, there is an associated point of a vector Sato Grassmannian whose isomonodromic tau function τ_{SSW} is related to τ by

$$\log \tau = \kappa \cdot \log \tau_{\mathrm{SSW}}$$

In this Grassmannian approach to Drinfeld-Sokolov tau functions as well as in our set-up for Heisenberg connections one realizes the simple Lie algebra $\mathfrak{g}$ via its first fundamental representation.

Remark 3. The change of basis that was employed in the proof of Theorem 1 to relate the Heisenberg connections for Lie algebras of type D and type A was used before in a somewhat different context by Vakulenko [15]. In loc. cit. the type A results of Kac and Schwarz [10] are generalized to type D by describing the point in the Sato Grassmannian giving the generalized Witten-Kontsevich point.

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