Normal forms of Heisenberg connections

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Abstract

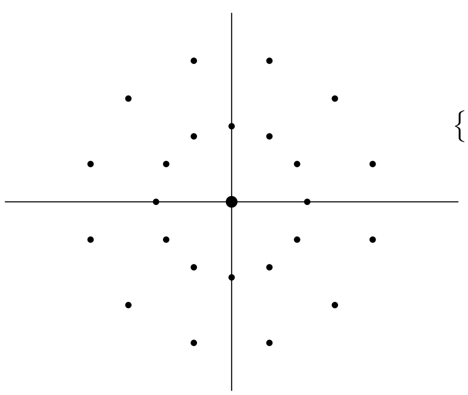
Kostant introduced the notion of a cyclic element in a finite-dimensional complex simple Lie algebra. Its spectrum has beautiful trigonometric expressions, thanks to work of Coxeter and others that relates the linear algebra of Coxeter elements and Cartan matrices. Instead of viewing cyclic elements as linear operators, we consider the associated linear differential operators, viewed as connections on a formal punctured disc. After introducing a variant of the Levelt-Turrittin normal form of such connections, we calculate them for cyclic elements on simple Lie algebras of type ABCDG.

1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$, let $\alpha_1, \cdots, \alpha_r$ be a set of simple roots and e_i generators of the corresponding root spaces with respect to $\mathfrak{h}$. Let e_0 denote the lowest root space. Kostant introduced in [10] the cyclic element

$$\Lambda = e_0 + \cdots + e_r$$

After fixing a representation of $\mathfrak{g}$, the cyclic element can be viewed as a linear operator. What is its spectrum? Note that by [10] (Theorem 6.2), this spectrum is independent of any of the above Lie theoretic choices, up to an overall scaling. Furthermore, by loc. cit. (Lemma 6.3) Λ is diagonalizable. The spectrum turns out to be very interesting. Consider for example $\mathfrak{e}_6$ in its 27-dimensional first fundamental representation. Let $h = 12$ denote the Coxeter number and for each k let $\zeta_k = e^{2\pi i/k}$. Up to an overall scaling, it turns out that the spectrum equals the following multi-set:



$$\left\{\zeta_h^i \;\middle|\; 0 \le i < 12\right\} \sqcup \left\{2\cos\left(\tfrac{\pi}{h}\right)\zeta_{2h}\cdot\zeta_h^i \;\middle|\; 0 \le i < 12\right\} \sqcup \{0,0,0\}$$

Such surprising trigonometric formulas for the spectrum can be understood via the "Cartan-Coxeter" correspondence. This is a beautiful relation between the linear algebra of Coxeter elements and Cartan matrices, initiated in the work of Coxeter [3]. Kostant showed in [10] (Lemma 6.4B) that the centralizer of Λ is Cartan algebra $\mathfrak{h}'$ (different than $\mathfrak{h}$). Let $\alpha_1', \cdots, \alpha_r'$ denote a set of simple roots, viewed as functionals on $\mathfrak{h}'$. Choose a bi-coloration of the corresponding Dynkin diagram, so that adjacent vertices are of opposite color. Let $c_j = \pm 1$ depending on the color of the j'th vertex and let h denote the Coxeter number of $\mathfrak{g}$. The work of Coxeter was refined by Fring-Liao-Olive in [5] to show that (up to scaling Λ)

$$\Lambda \cdot \alpha_j' = c_j \exp\left(-c_j \cdot \frac{i\pi}{2h}\right) \cdot x_j \tag{1}$$

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where the x_j's are the entries of a (right) Perron-Frobenius eigenvector of the Cartan matrix of $\mathfrak{g}$. This allows the spectrum calculation of Λ with respect to any representation, by expressing the weights in terms of the simple roots. For example, the previously mentioned spectrum for $\mathfrak{e}_6$ can be deduced from Equation (1) and the calculation of a Perron-Frobenius eigenvector of the Cartan matrix (with indexing convention as in [1]) as

$$\left(\cos\left(\frac{4\pi}{12}\right), \cos\left(\frac{3\pi}{12}\right), \cos\left(\frac{\pi}{12}\right), 2\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{3\pi}{12}\right), \cos\left(\frac{\pi}{12}\right), \cos\left(\frac{4\pi}{12}\right)\right)^{\mathrm{T}}$$

It should be noted that for most of the simple Lie algebras, the spectrum (say with respect to the first fundamental representation) can be calculated also in a purely elementary manner. Namely, let $e_{i,j}$ denote the matrix with 0's everywhere except a 1 in the (i,j) entry. Using the same choices of cyclic elements as in [2] (Appendix A) and [4] (Appendix 1), one obtains the following:

	cyclic element	spectrum
$\mathfrak{sl}_n$	$e_{1,n} + \sum_{i=1}^{n-1} e_{i+1,i}$	$1, \zeta_n, \cdots, \zeta_n^{n-1}$
$\mathfrak{so}_{2n+1}$	$\frac{1}{2} \cdot (e_{1,2n} + e_{2,2n+1}) + \sum_{i=1}^{2n} e_{i+1,i}$	$1, \zeta_{2n}, \cdots, \zeta_{2n}^{2n-1}, 0$
$\mathfrak{sp}_{2n}$	$e_{1,2n} + \sum_{i=1}^{2n-1} e_{i+1,i}$	$1, \zeta_{2n}, \cdots, \zeta_{2n}^{2n-1}$
$\mathfrak{so}_{2n}$	$\frac{1}{2}(e_{1,2n-1} + e_{2,2n}) + \frac{1}{2}(e_{n+1,n-1} + e_{n+2,n}) + \sum_{i=1}^{n-1}(e_{i+1,i} + e_{2n+1-i,2n-i})$	$1, \zeta_{2n-2}, \cdots, \zeta_{2n-2}^{2n-1}, 0, 0$
$\mathfrak{g}_2$	$\frac{1}{2} \cdot (e_{1,6} + e_{2,7}) + \sum_{i=1}^{6} e_{i+1,i}$	$1, \zeta_6, \cdots, \zeta_6^5, 0$

In the present work we consider a non-linear variant of the above discussion. In fact, we also switch to the affine version of the cyclic element:

$$\Lambda_{\mathfrak{g}} = z \cdot e_0 + e_1 + \cdots + e_r$$

This affine cyclic element is central for example in the definition of Drinfeld-Sokolov integrable hierarchies in [4]. Instead of viewing $\Lambda_{\mathfrak{g}}$ as a linear operator, we consider the differential operator

$$\partial_z + \Lambda_{\mathfrak{g}}$$

The most interesting behavior of this operator occurs near $z = \infty$. When viewed as a connection on the formal punctured disc $D^\times$ around ∞, we call it the Heisenberg connection $\mathrm{Conn}_{\mathfrak{g}}^{\mathrm{Heis}}(D^\times)$, since the centralizer of $\Lambda_{\mathfrak{g}}$ is a Heisenberg algebra in the affine algebra $\widehat{\mathfrak{g}}$. Our notation suppresses the dependency of the connection on the Lie theoretic choices involved in the definition of $\Lambda_{\mathfrak{g}}$, we justify this in Section 2. After fixing a finite-dimensional complex representation of $\mathfrak{g}$, the Heisenberg connection has a Levelt-Turrittin normal form. What is it? It has been observed in different contexts that after passing to a suitable finite cover of the disc, the problem reduces again to the spectrum of $\Lambda_{\mathfrak{g}}|_{z=1}$. See for example [13] (Lemma 2.1) for a result of this flavor. Nonetheless, what happens without the base change is interesting, and this is what we study in the present work.

Remark 1. Connections similar to $\mathrm{Conn}_{\mathfrak{g}}^{\mathrm{Heis}}(D^\times)$ have been studied for example by Frenkel-Gross [6], Kamgarpour-Sage [8], and Masoero-Raimondo-Valeri [13].

2 Coarse Levelt-Turrittin normal form

Fix an indeterminate t. By definition, a connection (V, ∇) on a formal punctured disc $D^\times = \operatorname{Spec} \mathbb{C}(\!(t)\!)$ consists of a finite-dimensional $\mathbb{C}(\!(t)\!)$-vector space V together with ∇ in $\operatorname{End}_{\mathbb{C}}(V)$ such that $\nabla(fv) = f'v + f\nabla(v)$ for all f in $\mathbb{C}(\!(t)\!)$ and all v in V. Levelt [11] and Turrittin [15] showed that isomorphism classes of such connections have certain normal forms, reminiscent of the Jordan normal form. The aim of the present work is to explicitly calculate the normal form of $\partial_z + \Lambda_{\mathfrak{g}}$, viewed as a connection on $\operatorname{Spec} \mathbb{C}(\!(1/z)\!)$. Observe that the choice of Chevalley generators involved in the definition of $\Lambda_{\mathfrak{g}}$ can change the isomorphism class of the connection. Hence, we now introduce a slightly coarser variant of the Levelt-Turrittin normal form which, for Heisenberg connections, is independent of the Lie theoretic choices.

Let n be a positive integer and let $f = \sum_i a_i t^{i/n}$ be an element of $\mathbb{C}(\!(t^{1/n})\!)$ that is not an element of $\mathbb{C}(\!(t^{1/m})\!)$ for $1 \leq m < n$. View $\mathbb{C}(\!(t^{1/n})\!)$ as an n-dimensional $\mathbb{C}(\!(t)\!)$-vector space. The basic building blocks of the Levelt-Turrittin classification are connections of the form

$$\mathcal{E}(f, n) = \left(\mathbb{C}(\!(t^{1/n})\!) \ , \ \partial_t + \frac{f(t^{1/n})}{t} \right)$$

with $a_i = 0$ for all $i > 0$. This connection is irreducible, see for example [14] (Proposition 3.1). (Since $\mathcal{E}(f, n)$ is n-dimensional, it will be convenient later on to allow the notation $\mathcal{E}(f, 0)$ to denote a 0-dimensional connection.)

Given $\mathcal{E}(f, n)$, there are two obvious modifications to f that do not change the isomorphism class of the connection. First, let k in $\mathbb{Z}$ and consider the $\mathbb{C}(\!(t)\!)$-linear automorphism of $\mathbb{C}(\!(t^{1/n})\!)$ defined via $v \mapsto t^{k/n} \cdot v$. This shows that

$$\mathcal{E}(f, n) \cong \mathcal{E}(f + \frac{k}{n}, n)$$

for all integers k. Second, let ζ_n be an n'th root of unity, and consider the $\mathbb{C}(\!(t)\!)$-linear automorphism of $\mathbb{C}(\!(t^{1/n})\!)$ defined via the substitution

$$t^{1/n} \mapsto \zeta_n \cdot t^{1/n} \tag{2}$$

This shows that

$$\mathcal{E}(f(t^{1/n}), n) \cong \mathcal{E}(f(\zeta_n t^{1/n}), n)$$

In the Levelt-Turrittin normal form, these two modifications are essentially the only ambiguity. To make this precise, we restrict now to semi-simple connections (this is all we will need), so connections that are the direct sum of irreducible ones. Levelt [11] and Turrittin [15] then show the following: For every semi-simple connection ∇ on $D^\times$ there exists $r \geq 1$ and f_i's, n_i's such that

$$\nabla \cong \bigoplus_{i=1}^{r} \mathcal{E}(f_i, n_i)$$

This description is essentially unique: Consider

$$\nabla_1 \cong \bigoplus_{i=1}^{r} \mathcal{E}(f_i, n_i) \quad , \quad \nabla_2 \cong \bigoplus_{j=1}^{s} \mathcal{E}(g_j, m_j)$$

Then

$$\nabla_1 \cong \nabla_2$$

if and only if $r = s$ and, possibly after a permutation of the indices, one has $n_i = m_i$ and for each i there is an n_i'th root of unity ζ_{n_i} such that

$$f_i(z^{1/n_i}) - g_i(\zeta_{n_i} \cdot z^{1/n_i}) \in \mathbb{Z} \cdot \frac{1}{n_i}$$

We refer to the work of Sabbah [14] for a description in modern language of the results of Levelt and Turrittin.

We now introduce a coarser notion of equivalence of connections: Instead of the substitutions in Equation (2), we now consider

$$t^{1/n} \mapsto c \cdot t^{1/n}$$

for an arbitrary non-zero constant c, not just n'th roots of unity. More precisely:

Definition 1. Consider two semi-simple connections ∇_1 and ∇_2 on $D^\times$ with Levelt-Turrittin normal forms

$$\nabla_1 \cong \bigoplus_{i=1}^{r} \mathcal{E}(f_i, n_i) \quad , \quad \nabla_2 \cong \bigoplus_{j=1}^{s} \mathcal{E}(g_j, m_j)$$

We write

$$\nabla_1 \sim \nabla_2$$

if $r = s$ and, possibly after a permutation of the indices, one has $n_i = m_i$ and there are non-zero scalars $c_1, \cdots, c_r$ such that

$$f_i(z^{1/n_i}) - g_i(c_i \cdot z^{1/n_i}) \in \mathbb{Z} \cdot \frac{1}{n_i}$$

for all i.

One sees that $\sim$ is an equivalence relation, and isomorphic semi-simple connections ∇_1 and ∇_2 are always equivalent. We show in Theorem 1 that with this coarser variant of the Levelt-Turrittin normal form, the Lie theoretic choices involved in the definition of cyclic elements do not change the equivalence class of Heisenberg connections.

As indicated before, we view the Heisenberg connection $\partial_z + \Lambda_{\mathfrak{g}}$ near $z = \infty$, hence as a connection on the punctured disc Spec $\mathbb{C}((t))$ with $t = 1/z$. For notational reasons we often write various connections on this disc in terms of z and its fractional powers. For example, if $\zeta^h = z$ for some positive integer h, and c is a constant, then

$$\mathcal{E}(c + \zeta^{h+1}, h) = \left(\mathbb{C}((t^{1/h})), \partial_t - \frac{c + t^{-(h+1)/h}}{t}\right) \tag{3}$$

To state our main result, we introduce the notion of a shifted connection: If Conn $= (V, \nabla)$ is a connection on $D^\times = $ Spec $\mathbb{C}((t))$ with $d = \dim_{\mathbb{C}((t))}(V)$, define the shifted connection for a (complex, say) scalar i by

$$\mathrm{Conn}[i] = (V, \nabla + \frac{i}{d} \cdot \frac{1}{t})$$

Here $1/t$ stands for the $\mathbb{C}((t))$-linear map $v \mapsto (1/t) \cdot v$, where v is in V.

Theorem 1. *Let $\mathfrak{g}$ be a simple complex Lie algebra of type* ABCDG *and consider the associated Heisenberg connections* $\mathrm{Conn}_{\mathfrak{g}}^{\mathrm{Heis}}(D^\times)$ *with respect to the first fundamental representation. These connections are semi-simple, and their $\sim$ equivalence class is independent of the choice of cyclic element. Define κ via*

type of $\mathfrak{g}$	A	B	C	D	G
κ	0	1	0	1	1

Let h be the Coxeter number and ζ such that $\zeta^h = z$. Then

$$\mathrm{Conn}_{\mathfrak{g}}^{\mathrm{Heis}}(D^\times) \sim \mathrm{Conn}_{\mathfrak{sl}_h}^{\mathrm{Heis}}(D^\times)\left[\frac{\kappa}{2}\right] \oplus \mathcal{E}(\frac{\kappa}{2}, \kappa) \sim \mathcal{E}\left(\frac{-1 + h - \kappa}{2h} + \zeta^{h+1}, h\right) \oplus \mathcal{E}(\frac{\kappa}{2}, \kappa) \tag{4}$$

Proof. We start by relating the Heisenberg connections for two different choices of cyclic elements on $\mathfrak{g}$. Kostant shows in [10] (Theorem 6.2) that any two cyclic elements are conjugate, up to a non-zero scalar. We adapt these argument to the corresponding Heisenberg connections. Fix a set of simple roots $S = \{\alpha_1, \cdots, \alpha_r\}$ and corresponding lowest root α_0. Fix a Cartan algebra $\mathfrak{h}$ and view all roots as element of $\mathfrak{h}$ via the Killing form. For each i choose a generator

e_i of the α_i root space with respect to $\mathfrak{h}$. Choose two cyclic elements

$$\Lambda = a_0 z e_0 + \sum_{i=1}^{r} a_i e_i$$

$$\Lambda' = b_0 z e_0 + \sum_{i=1}^{r} b_i e_i$$

for non-zero scalars a_i and b_i. Let ϵ_i be the fundamental weights, viewed as elements of $\mathfrak{h}$, and define

$$y = \sum_{i=1}^{r} \log\left(\frac{b_i}{a_i}\right) \epsilon_i$$

Define b via $\exp(\mathrm{ad}\ y)e_0 = be_0$. Let q denote the height of the highest root and define c via

$$e^{-c(q+1)} = \frac{b_0}{ba_0}$$

Let $\rho_{\mathfrak{g}}^{\vee}$ be the element of $\mathfrak{h}$ so that $[\rho_{\mathfrak{g}}^{\vee}, e_i] = e_i$ for all $1 \leq i \leq r$. Kostant shows in the proof of [10] (Theorem 6.2) that

$$\exp(\mathrm{ad}(y + c\rho_{\mathfrak{g}}^{\vee}))\Lambda = \lambda\Lambda'$$

for $\lambda = e^c$. Let

$$a := y + c\rho_{\mathfrak{g}}^{\vee}$$

From the definition of y, b, and c it follows that the coefficients of e_0 in Λ and Λ' only contribute to a through their quotient. Hence, a has no z-dependency. Therefore

$$\begin{aligned}\exp(\mathrm{ad}\ a)\left(\partial_z + \Lambda\right) &= \partial_z + \exp(\mathrm{ad}\ a)(\Lambda) \\ &= \partial_z + \lambda\Lambda'\end{aligned}$$

For any other set $\tilde{S}$ of simple roots, choose a Weyl group element σ such that $\tilde{S} = \{\sigma\alpha_1, \cdots, \sigma\alpha_r\}$. Now choose a finite-order inner automorphism $\exp(\mathrm{ad}\ r)$ of $\mathfrak{g}$ that restricts to σ on $\mathfrak{h}$, see [7]. The new root space generators can be chosen as $\exp(\mathrm{ad}\ r)e_i$. It follows that for every cyclic element $\tilde{\Lambda}$ with respect to $\tilde{S}$, there is a cyclic element Λ with respect to S, such that

$$\exp(\mathrm{ad}\ r)(\partial_z + \Lambda) = \partial_z + \tilde{\Lambda}$$

Finally, since all Cartan algebras are conjugate, we have shown that for any two cyclic elements Λ and $\tilde{\Lambda}$ on $\mathfrak{g}$, there is a non-zero scalar λ such that

$$\exp(\mathrm{ad}\ x)\left(\partial_z + \Lambda\right) = \partial_z + \lambda\tilde{\Lambda} \tag{5}$$

for some x in $\mathfrak{g}$.

Having established this close relation of Heisenberg connections associated to different cyclic elements, we now show that one can perturb these connections in a very specific manner, without leaving the isomorphism class. Let $\Lambda_{\mathfrak{g}}$ be an arbitrary cyclic element on $\mathfrak{g}$. We claim that for every non-zero scalar λ there is Y in $\mathfrak{g}[\![1/z]\!]$ such that

$$\exp(\mathrm{ad}\ Y)\left(\partial_z + \frac{\rho_{\mathfrak{g}}^{\vee}}{hz} + \lambda\Lambda_{\mathfrak{g}}\right) = \partial_z + \lambda\Lambda_{\mathfrak{g}} \tag{6}$$

In particular, with respect to any representation ξ of $\mathfrak{g}$, one obtains isomorphic connections on $\mathbb{C}(\!(1/z)\!)$. A convenient reference for Equation (6) is the work of Cafasso and Wu [2] (Lemma 3.9 and Theorem 3.11). The idea is to decompose $Y = \sum_{j \leq 0} Y_j$ with Y_j in $\widehat{\mathfrak{g}}$ of principal degree j, and then to solve the equation recursively with respect to the principal degree. Note that in loc. cit. Equation (6) is obtained with $-\Lambda_{\mathfrak{g}}$ instead of $\lambda\Lambda_{\mathfrak{g}}$. However, this is simply due to the intended application to the construction of Drinfeld-Sokolov integrable hierarchies. The proof can be seen to be independent of the choice of root space generators $e_0, e_1, \cdots, e_r$ involved in the definition of $\Lambda_{\mathfrak{g}}$. In particular, after working with $-\lambda e_i$ for all i, one obtains Equation (6).

For $\mathfrak{g} = \mathfrak{sl}_h$ we choose the Chevalley generators such that

$$\Lambda_{\mathfrak{sl}_h} = z \cdot e_{1,h} + \sum_{i=1}^{h-1} e_{i+1,i} \quad , \quad \rho^\vee_{\mathfrak{sl}_h} = \frac{1}{2}\sum_{i=1}^{h}(-1-h+2i)\cdot e_{i,i} \tag{7}$$

All matrix realizations of Lie algebras that we use in the proof of Theorem 1 can be found for example in [2] (Appendix A) and [4]. Let ζ be such that $\zeta^h = z$ and consider as in work by Kac and Schwarz [8] an isomorphism ν of $\mathbb{C}((1/z))$-vector spaces between $\mathbb{C}((1/z))^h$ and $\mathbb{C}((1/\zeta))$ given by $f(z)v_i \mapsto f(\zeta^h)\zeta^{i-h}$ where v_i with $1 \le i \le h$ is the standard i'th basis element in $\mathbb{C}((1/z))^h$, with 0's everywhere except a 1 in the i'th entry. Note that

$$(\nu\partial_z\nu^{-1})(f(\zeta^h)\zeta^{i-h}) = \partial_{\zeta^h}(f(\zeta^h)\zeta^{i-h}) - \frac{i-h}{h\zeta^h}\cdot f(\zeta^h)\zeta^{i-h}$$

Exploiting the second summand and its partial cancellation with the $\rho^\vee_{\mathfrak{sl}_h}$ contribution, one sees that via ν, for any scalar s

$$\partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \frac{s}{hz}\cdot\sum_{i=1}^{h} e_{i,i} + \lambda\Lambda_{\mathfrak{sl}_h} \quad\mapsto\quad \partial_{\zeta^h} + \frac{-1+h+2s}{2h}\cdot\zeta^{-h} + \lambda\zeta \tag{8}$$

Crucially, the coarse Levelt-Turrittin normal form is independent of λ: The connection on the right-hand side is nothing but

$$\mathcal{E}(\frac{-1+h+2s}{2h} + \lambda\zeta^{h+1}, h)$$

Changing λ corresponds exactly to a substitution

$$\zeta \mapsto \lambda^{1/(h+1)}\zeta$$

that does not change the coarse normal form.

Let Y be as Equation (6) for $\mathfrak{g} = \mathfrak{sl}_h$. Then, for any complex scalar μ

$$\exp(\mathrm{ad}(-Y))\left(\partial_z + \frac{\mu}{z}\cdot\sum_{i=1}^{r} e_{i,i} + \lambda\Lambda_{\mathfrak{sl}_h}\right) = \partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} + \frac{\mu}{z}\cdot\sum_{i=1}^{r} e_{i,i} + \lambda\Lambda_{\mathfrak{sl}_h} \tag{9}$$

Note that as in Equation (3), there is a sign change in the regular singular term when switching from the description of a connection in terms of z to $1/z$. Hence, Equation (9) with $\mu = -\kappa/2h$ and $\lambda = 1$ implies together with Equation (8) that

$$(\partial_z + \Lambda_{\mathfrak{sl}_h})\left[\frac{\kappa}{2}\right] \cong \mathcal{E}\left(\frac{-1+h-\kappa}{2h} + \zeta^{h+1}, h\right)$$

Choose now an arbitrary cyclic element for $\mathfrak{sl}_h$ and let $\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)$ be the corresponding Heisenberg connection. By Equation (5), Equation (6), and Equation (8) it follows that

$$\mathrm{Conn}^{\mathrm{Heis}}_{\mathfrak{sl}_h}(D^\times)\left[\frac{\kappa}{2}\right] \sim \mathcal{E}\left(\frac{-1+h-\kappa}{2h} + \zeta^{h+1}, h\right) \tag{10}$$

In particular, in type A, the coarse Levelt-Turrittin normal form of is independent of the choice of cyclic element.

Consider now again the case of general $\mathfrak{g}$ in its first fundamental representation, and let h denote the Coxeter number. It follows from Equation (5) and Equation (10), that in order to prove the theorem, it suffices to show that there is some choice of cyclic element $\Lambda_{\mathfrak{g}}$ such that for every non-zero scalar λ

$$\partial_z + \lambda\Lambda_{\mathfrak{g}} \cong \left(\mathbb{C}((1/z))^h, \partial_z + \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz} - \frac{\kappa}{2hz} \cdot \sum_{i=1}^{h} e_{i,i} + \lambda\Lambda_{\mathfrak{sl}_h}\right) \oplus \mathcal{E}(\frac{\kappa}{2}, \kappa) \tag{11}$$

We establish this on a case-by-case basis. Consider first $\mathfrak{g} = \mathfrak{so}_{2n+1}$, with Coxeter number $h = 2n$ and $(2n+1)$-dimensional first fundamental representation. Following [2] (Appendix A), one can choose Chevalley generators such that for $1 \leq i \leq n$

$$e_i = e_{i+1,i} + e_{2n+2-i,2n+1-i}$$

and

$$e_0 = \frac{1}{2}(e_{1,2n} + e_{2,2n+1})$$

Hence

$$\Lambda_{\mathfrak{g}} = \sum_{i=1}^{2n} e_{i+1,i} + \frac{z}{2} \cdot (e_{1,2n} + e_{2,2n+1}) \tag{12}$$

$$\rho^\vee_{\mathfrak{g}} = \sum_{i=1}^{2n+1} (-n-1+i) \cdot e_{i,i} \tag{13}$$

Note that the right-hand side of Equation (13) has trace 0 and commutator 1 with e_i for all $1 \leq i \leq n$, as desired.

Let $c_1, \cdots, c_{2n+1}$ denote the standard basis of $\mathbb{C}^{2n+1}$ and view it as a basis of $\mathbb{C}((1/z))^{2n+1}$ as a $\mathbb{C}((1/z))$-vector space. Consider the change of coordinates to the basis

$$d_1 = \frac{c_1}{2} + \frac{c_{2n+1}}{z}$$
$$d_i = c_i \quad \text{for } 2 \leq i \leq 2n$$
$$d_{2n+1} = \frac{c_1}{2} - \frac{c_{2n+1}}{z}$$

Remark 2. A very similar coordinate change was used by Vakulenko in [16] to describe a type D integrable hierarchy of Drinfeld-Sokolov type.

In terms of the new basis, one obtains

$$\Lambda_{\mathfrak{g}} = \sum_{i=1}^{2n-1} e_{i+1,i} + z \cdot e_{1,2n}$$

as well as

$$\frac{\rho^\vee_{\mathfrak{g}}}{hz} = \frac{1}{2nz} \cdot \left(-n \cdot (e_{1,2n+1} + e_{2n+1,1}) + \sum_{i=2}^{2n}(-n-1+i) \cdot e_{i,i}\right) \tag{14}$$

The gauge term of the coordinate change is $\gamma\partial_z(\gamma^{-1})$ where

$$\gamma^{-1} = \sum_{i=2}^{2n} e_{i,i} + \frac{1}{2}(e_{1,1} + e_{1,2n+1}) + \frac{1}{z}(e_{2n+1,1} - e_{2n+1,2n+1})$$

$$\gamma = \sum_{i=2}^{2n} e_{i,i} + e_{1,1} + e_{2n+1,1} + \frac{z}{2}(e_{1,2n+1} - e_{2n+1,2n+1})$$

One obtains

$$\gamma\partial_z(\gamma^{-1}) = \frac{1}{2z}\cdot(-e_{1,1}+e_{2n+1,1}+e_{1,2n+1}-e_{2n+1,2n+1})$$

Observe the crucial cancellation when adding this term with the term in Equation (14). This is at the heart of our argument!

We obtain that for any non-zero scalar λ, the connection $\partial_z+\lambda\Lambda_{\mathfrak{g}}$ is isomorphic to the connection (W,∇) with underlying vector space $W=\mathbb{C}(\!(1/z)\!)^{2n+1}$ and $\nabla: W\to W$ given by

$$\partial_z+\frac{1}{2nz}\cdot\sum_{i=1}^{2n}(-n-1+i)\cdot e_{i,i}+\lambda\cdot\left(\sum_{i=1}^{2n-1}e_{i+1,i}+z\cdot e_{1,2n}\right)-\frac{1}{2z}\cdot e_{2n+1,2n+1}$$

Note that

$$\begin{aligned}\frac{1}{2nz}\cdot\sum_{i=1}^{2n}(-n-1+i)\cdot e_{i,i} &= \frac{1}{2hz}\cdot\sum_{i=1}^{h}(-h-2+2i)\cdot e_{i,i}\\ &= \frac{\rho^\vee_{\mathfrak{sl}_h}}{hz}-\frac{1}{2hz}\cdot\sum_{i=1}^{h}e_{i,i}\end{aligned}$$

and

$$\sum_{i=1}^{2n-1}e_{i+1,i}+z\cdot e_{1,2n}=\Lambda_{\mathfrak{sl}_h}$$

where $\Lambda_{\mathfrak{sl}_h}$ and $\rho^\vee_{\mathfrak{sl}_h}$ are chosen as in Equation (7). It follows that for $\mathfrak{g}=\mathfrak{so}_{2n+1}$

$$\partial_z+\lambda\Lambda_{\mathfrak{g}}\cong\left(\mathbb{C}(\!(1/z)\!)^h\ ,\ \partial_z+\frac{\rho^\vee_{\mathfrak{sl}_h}}{hz}-\frac{1}{2hz}\cdot\sum_{i=1}^{h}e_{i,i}+\lambda\Lambda_{\mathfrak{sl}_h}\right)\oplus\left(\mathbb{C}(\!(1/z)\!),\partial_z-\frac{1}{2z}\right)\tag{15}$$

As before, there is a sign change in the regular singular term when switching from a description of a connection in terms of z to $1/z$. Hence the second summand in Equation (15) is exactly $\mathcal{E}(\frac{\kappa}{2},\kappa)$, and we have shown Equation (11).

Assume now that $\mathfrak{g}=\mathfrak{so}_{2n}$, with Coxeter number $h=2n-2$ and $2n$-dimensional first fundamental representation. As in [2] (Appendix A) choose Chevalley generators such that for $1\le i\le n-1$

$$e_i=e_{i+1,i}+e_{2n-i+1,2n-i}$$

and

$$\begin{aligned}e_n &= \frac{1}{2}(e_{n+1,n-1}+e_{n+2,n})\\ e_0 &= \frac{1}{2}(e_{1,2n-1}+e_{2,2n})\end{aligned}$$

Hence

$$\Lambda_{\mathfrak{so}_{2n}}=\frac{1}{2}\cdot(e_{n+1,n-1}+e_{n+2,n})+\sum_{i=1}^{n-1}(e_{i+1,i}+e_{2n+1-i,2n-i})+\frac{z}{2}(e_{1,2n-1}+e_{2,2n})\tag{16}$$

as well as

$$\rho^\vee_{\mathfrak{so}_{2n}}=\sum_{i=1}^{n-1}(-n+i)e_{i,i}+\sum_{i=n+2}^{2n}(-n-1+i)e_{i,i}\tag{17}$$

Note that the right-hand side of Equation (17) has trace 0 and commutator 1 with e_i for all $1\le i\le n$, as desired.

Let $c_1,\cdots,c_{2n}$ denote the standard basis of $\mathbb{C}^{2n}$ and view it as a basis of $\mathbb{C}(\!(1/z)\!)^{2n}$ as a $\mathbb{C}(\!(1/z)\!)$-vector space.

Consider the change of coordinates to the basis

$$d_1 = \frac{c_1}{2} + \frac{c_{2n}}{z}$$

$$d_i = c_i \quad \text{for } 2 \le i \le n-1$$

$$d_n = c_n + \frac{c_{n+1}}{2}$$

$$d_i = c_{i+1} \quad \text{for } n+1 \le i \le 2n-2$$

$$d_{2n-1} = \frac{c_1}{2} - \frac{c_{2n}}{z}$$

$$d_{2n} = c_n - \frac{c_{n+1}}{2}$$

In terms of the new basis one obtains

$$\Lambda_{\mathfrak{g}} = \sum_{i=1}^{2n-3} e_{i+1,i} + z \cdot e_{1,2n-2}$$

as well as

$$\rho_{\mathfrak{g}}^{\vee} = (-n+1)(e_{2n-1,1} + e_{1,2n-1}) + \sum_{i=2}^{n-1}(-n+i)e_{i,i} + \sum_{i=n+1}^{2n-2}(-n+i)e_{i,i}$$

The gauge term of the coordinate change is $\gamma\partial_z(\gamma^{-1})$ where

$$\gamma^{-1} = \sum_{i=2}^{n-1} e_{i,i} + \sum_{i=n+1}^{2n-2} e_{i+1,i} + \frac{1}{z}(e_{2n,1} - e_{2n,2n-1}) + \frac{1}{2}(e_{1,1} + e_{n+1,n} - e_{n+1,2n} + e_{1,2n-1}) + e_{n,n} + e_{n,2n}$$

$$\gamma = \sum_{i=2}^{n-1} e_{i,i} + \sum_{i=n+1}^{2n-2} e_{i,i+1} + \frac{z}{2}(e_{1,2n} - e_{2n-1,2n}) + \frac{1}{2}(e_{n,n} + e_{2n,n}) + e_{1,1} + e_{2n-1,1} + e_{n,n+1} - e_{2n,n+1}$$

One obtains

$$\gamma\partial_z(\gamma^{-1}) = \frac{1}{2z} \cdot (-e_{1,1} - e_{2n-1,2n-1} + e_{1,2n-1} + e_{2n-1,1})$$

It follows that for every non-zero scalar λ

$$\partial_z + \lambda\Lambda_{\mathfrak{g}} \cong \partial_z + A + \lambda B + C$$

where

$$A = \frac{1}{(2n-2)z} \cdot \left((-n+1)(e_{2n-1,1} + e_{1,2n-1}) + \sum_{i=2}^{2n-2} (-n+i) \cdot e_{i,i} \right)$$

$$B = \sum_{i=1}^{2n-3} e_{i+1,i} + z \cdot e_{1,2n-2}$$

$$C = \frac{1}{2z} \cdot (-e_{1,1} - e_{2n-1,2n-1} + e_{1,2n-1} + e_{2n-1,1})$$

Hence, there are again some important cancellations in the sum $A + \lambda B + C$, and $\partial_z + A + \lambda B + C$ equals

$$\partial_z + \frac{1}{(2n-2)z} \cdot \left(\sum_{i=1}^{2n-2} (-n+i) e_{i,i} \right) + \lambda \left(\sum_{i=1}^{2n-3} e_{i+1,i} + z \cdot e_{1,2n-2} \right) - \frac{1}{2z} \cdot e_{2n-1,2n-1} \tag{18}$$

Furthermore

$$\frac{1}{2n-2}\sum_{i=1}^{2n-2}(-n+i)e_{i,i} = \frac{1}{2h}\cdot\sum_{i=1}^{h}(-h+2i-2)\cdot e_{i,i}$$

$$= \frac{\rho^\vee_{\mathfrak{sl}_h}}{h} - \frac{1}{2h}\cdot\sum_{i=1}^{h} e_{i,i}$$

Hence, in the same way as for type B, one obtains Equation (11).

Suppose now $\mathfrak{g} = \mathfrak{sp}_{2n}$, with Coxeter number $h = 2n$ and $2n$-dimensional first fundamental representation. As in [2] (Appendix A), one can choose the cyclic element so that

$$\Lambda_{\mathfrak{sp}_{2n}} = \Lambda_{\mathfrak{sl}_{2n}}$$

Hence the theorem follows from the already treated type A case. Note that this relation between type A and C cyclic elements holds more generally in the context of folding, see for example [2] and [12].

The last case to consider is $\mathfrak{g} = \mathfrak{g}_2$, with Coxeter number $h = 6$ and 7-dimensional first fundamental representation. One can choose Chevalley generators such that

$$\Lambda_{\mathfrak{g}_2} = \sum_{i=1}^{6} e_{i+1,i} + \frac{z}{2}\cdot(e_{1,6} + e_{2,7}) = \Lambda_{\mathfrak{so}_7}$$

Hence, the result follows from the $\mathfrak{so}_7$-case treated earlier. □

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