KP Flows and Quantization

Martin T. Luu

Abstract

The quantization of a pair of commuting differential operators is a pair of non-commuting differential operators. Both at the classical and quantum level the flows of the KP hierarchy are defined and further one can consider switching, up to a sign, the ordering of the operators. We discuss the interaction of these operations with the quantization.

1 Introduction

Fix an indeterminate x. For a parameter $\hbar$, consider pairs of differential operators (P, Q) with P and Q in $\mathbb{C}[\![x]\!][\partial_x]$ solutions to the generalized string equation $[P, Q] = \hbar$. For $\hbar = 0$ such operators are related by the Krichever construction to classical curves and therefore if $\hbar \neq 0$ such operators are also called a quantum curve. The quantization of commuting differential operators developed by Schwarz [5], [10], [11] concerns the process of varying the parameter $\hbar$ from 0 to non-zero values. It turns out that the ordering of the operators matters in the quantization process. Furthermore, both, the pairs of commuting operators and their quantization, each sit in a moduli space on which the flows of the KP integrable hierarchy are defined. In the present work we discuss the interaction of the KP flows and the choice of ordering of the operators with the quantization.

The quantization is in particular well defined on pairs (P, Q) of operators with P and Q of positive co-prime degree and such that P is monic, meaning of the form

$$\partial^i + a_{i-1}\partial^{i-1} + \cdots + a_0,$$

where we have dropped the subscript x from ∂_x. We will continue to do this throughout the paper. Note that starting with a monic differential operator the condition that a_{i-1} vanishes can be achieved by conjugating the operator by a function. Such a monic operator with vanishing subleading terms is called normalized. We assume from now on that P and Q are indeed of co-prime degrees and that P is monic. It is useful to look at these pairs (P, Q) from a slightly different perspective: They give rise to an element of the space $\mathrm{Conn}_\hbar(D^\times)$ of $\hbar$-connections on the formal punctured disc $D^\times = \mathrm{Spec}\ \mathbb{C}(\!(t)\!)$. We denote the subset of $\mathrm{Conn}_\hbar(D^\times)$ obtained in this manner by $\mathrm{String}_\hbar$. The quantization procedure developed by Schwarz yields a map

$$\mathrm{Quant} : \mathrm{String}_0 \longrightarrow \mathrm{String}_1.$$

An important feature is the presence of flows of the KP integrable hierarchy, both at the classical and quantum level, meaning for String_0 and String_1. The equations for the Lax operator

$$L = \partial + c_1\partial^{-1} + c_2\partial^{-2} + \cdots$$

of the hierarchy are given by

$$\partial_{t_k} L = [(L^k)_+, L]$$

where $t_1, t_2, \cdots$ denote the KP times and the subscript $+$ indicates taking the differential part of a pseudodifferential operator. An important point is the isospectrality of the KP flows, we refer to [8] for a more detailed description and relation to the relevant spectral curves. It is important to note that there are two ways to define the flows on a pair

(P, Q), depending on whether the Lax operator L is chosen to be a p'th root of P or a q'th root of Q, where p and q are the degrees of P and Q. Comparing these two ways to flow turns out to be an important problem related to dualities of 2D quantum field theories in the case $\hbar = 1$, see for example [3], [6], [7]. From now on we make the convention that given an ordered pair (P, Q) with P of degree p, the associated Lax operator is chosen to be a p'th root of P. In order to not break the symmetry between P and Q, we then also consider the map

$$\iota : \mathrm{String}'_\hbar \longrightarrow \mathrm{String}_\hbar$$

corresponding to

$$(P, Q) \mapsto (Q, -P)$$

where $\mathrm{String}'_\hbar$ denotes the subset of $\mathrm{String}_\hbar$ corresponding to (P, Q) with both P and Q monic. In order to understand the interaction of the KP flows as well as the map ι with the quantization one should describe properties of the diagram given in Figure 1. Here $t_1, t_2, \cdots$ and $\hat{t}_1, \hat{t}_2, \cdots$ denote two sets of KP times and the maps from String_0 to String_1 are given by Quant.

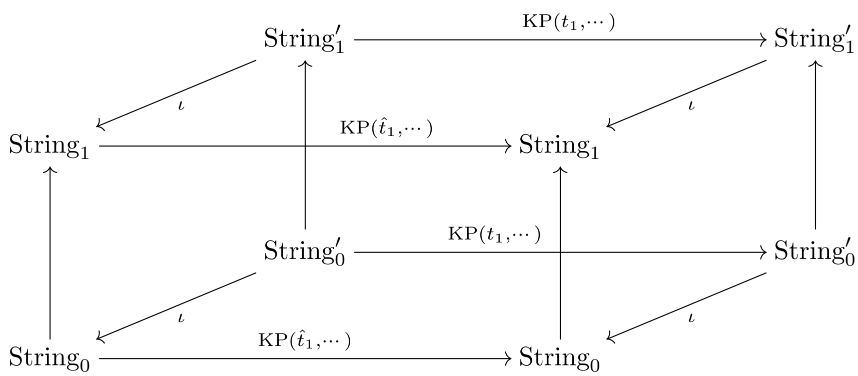

Figure 1: Quantization and KP flows

It turns out that the diagram is not commutative, and the following are natural questions to answer:

(i) Describe how ι changes the isomorphism class of an element of String'_1.

(ii) Describe how ι changes the isomorphism class of an element of String'_0.

(iii) Compare $\mathrm{Quant} \circ \iota$ and $\iota \circ \mathrm{Quant}$.

(iv) Decide if quantization commutes with KP flows.

Note that question (i) has a reformulation in terms of how ι interacts with the flows of the KP hierarchy, see [7] where this question is answered in joint work with Albert Schwarz and applications to duality of 2D quantum gravity are given. In the present work we address the remaining three questions.

2 Aspects of the quantization of commuting differential operators

The aim is to construct interesting families, indexed by a parameter $\hbar$, of pairs of operators

$$(P_\hbar, Q_\hbar) \in \mathbb{C}[\![x]\!][\partial_x] \times \mathbb{C}[\![x]\!][\partial_x]$$

satisfying the generalized string equation

$$[P_\hbar, Q_\hbar] = \hbar.$$

The scheme developed by Schwarz [10] constructs such families by fixing the degrees p and q of $P_\hbar$ and $Q_\hbar$ and fixing furthermore the so called companion matrix in each family: This is an element of the Lie algebra $\mathfrak{gl}_p\mathbb{C}[u]$, for an indeterminate u. To define it, we recall the definition of the Sato Grassmannian:

For an indeterminate z, the index zero big cell Gr of the Sato Grassmannian consists of complex subspaces of $\mathbb{C}(\!(1/z)\!)$ whose projection to $\mathcal{H}^+ := \mathbb{C}[z]$ is an isomorphism. One defines an action of elements in $\mathbb{C}[\![x]\!][\partial_x]$ on $\mathbb{C}(\!(1/z)\!)$ via

$$(x^i\partial_x^j)f := \left(-\frac{\mathrm{d}}{\mathrm{d}z}\right)^i z^j f$$

for all $f \in \mathbb{C}(\!(1/z)\!)$. Suppose now that A and B are elements of $\mathbb{C}[\![x]\!][\partial_x]$ with A monic of degree p. For an indeterminate u one can view $\mathcal{H}^+$ as a free $\mathbb{C}[u]$-module of rank p by letting u act via A. The companion matrix $M_{A,B}$ is the matrix describing the action of B with respect to the basis $1, z, \cdots, z^{p-1}$.

In the situation where the degree p and q of $P_\hbar$ and $Q_\hbar$ are co-prime, Schwarz [10] has shown that fixing the companion matrix $M_{P_\hbar,-Q_\hbar}$ yields a well defined way to let $\hbar$ vary in the string equation. Consider now the restriction to a formal punctured disc around ∞ of the rank p bundle $\mathcal{H}^+$ on $\mathbb{A}^1 = \text{Spec }\mathbb{C}[u]$ obtained via the $P_\hbar$-action on $\mathcal{H}^+$. One obtains a p-dimensional $\mathbb{C}(\!(t)\!)$ vector space M, where

$$t = \frac{1}{u}$$

and the operator $-Q_\hbar$ introduces a further structure on this vector space: Since $[-Q_\hbar, P_\hbar] = 1$ the $-Q_\hbar$ action on M yields the structure of an $\hbar$-connection on the formal punctured disc

$$D^\times = \text{Spec }\mathbb{C}(\!(t)\!).$$

This means M has a $\mathbb{C}$-linear endomorphism $\nabla_\hbar$, given by the the $-Q_\hbar$ action, such that

$$\nabla_\hbar(fm) = \hbar f'(t)m + f\nabla_\hbar(m)$$

for all $f \in \mathbb{C}(\!(t)\!)$ and all $m \in M$. In the case of $\hbar = 1$ one obtains an object of the category $\text{Conn}(D^\times)$ of connections on $D^\times$ and for $\hbar = 0$ one obtains an object of the category $\text{Higgs}(D^\times)$ of Higgs bundles on $D^\times$. In this sense one can think of the quantization scheme as a way to quantize certain Higgs bundles:

$$\begin{array}{ccc}
\text{Higgs}(D^\times) & & \text{Conn}(D^\times) \\
\uparrow & & \uparrow \\
\text{String}_0 & \xrightarrow{\text{Quant}} & \text{String}_1
\end{array}$$

where for a companion matrix

$$M = M_{P_\hbar,-Q_\hbar} = -M_{P_\hbar,Q_\hbar}$$

one has

$$\text{Quant}: \quad M\mathrm{d}u \longmapsto \partial_u + M.$$

The quantization scheme turns out to be quite natural from the point of view of quantum field theory: As shown by Liu-Schwarz in [5], for co-prime p and q the quantization of the pair of commuting operators (∂^p, ∂^q) yields the τ-function of (p, q) minimal conformal matter coupled to gravity.

Note that since the classical data of the quantization scheme corresponds to commuting differential operators, it is known via the Krichever correspondence, see for example [8] for a detailed exposition, that this data can be described in terms of algebraic curves with additional structure. In the following we describe the previously mentioned Higgs bundle in terms of this algebro-geometric data.

The input of the Krichever correspondence consists of objects of the form

$$\mathcal{X} = (X, s, \mathcal{F}, \text{trvialization data})$$

where X is a curve over $\mathbb{C}$, s is a point of X, $\mathcal{F}$ is a vector bundle and one chooses trivialization of $\mathcal{F}$ in a formal neighborhood of s, see [8] for details. We will from now on focus on the data that will yield points of the index zero part of the big-cell of the Sato Grassmannian, and hence $\mathcal{F}$ is a line bundle. An index 0 Schur pair (A, W) consists of a point W of Gr and a $\mathbb{C}$-subalgebra A of $\mathbb{C}((1/z))$ such that $AW \subseteq W$ and such that A properly contains $\mathbb{C}$. One then attaches such an object to $\mathcal{X}$ in the following manner: The point $W \subseteq \mathbb{C}((1/z))$ is simply the image under the choice of trivializations of the space $\mathrm{H}^0(X\backslash s, \mathcal{F})$ and A corresponds to $\mathrm{H}^0(X\backslash s, \mathcal{O}_X)$. Note that in our conventions, $1/z$ rather than z is a local coordinate at s. Then for an element $\tilde{P}$ of A of the form $z^p +$ lower order terms one obtains a free $\mathbb{C}[u]$-module of rank p by letting u act on W via $\tilde{P}$. By restricting the corresponding bundle on Spec $\mathbb{C}[u]$ to a formal disc around ∞ one obtains a rank p vector bundle $\mathcal{V}$ on the punctured disc $D^{\times}$. Given a second element $\tilde{Q} \in A$ one obtains, since $\tilde{P}$ and $\tilde{Q}$ commute, an endomorphism of $\mathcal{V}$ and hence an endomorphism valued 1-form ϕ. This gives the algebro-geometric formulation of the previously defined Higgs bundles.

2.1 Quantization and KP flows

We now answer question (iv) from the introduction:

Theorem 1. *In general, for (P_0, Q_0) corresponding to* String_0 *one has*

$$(\mathrm{Quant} \circ \mathrm{KP}(t_1, t_2, \cdots))\,(P_0, Q_0) \ncong (\mathrm{KP}(t_1, t_2, \cdots) \circ \mathrm{Quant})\,(P_0, Q_0).$$

One can prove this statement by considering the case $(P_0, Q_0) = (\partial^p, \partial^q)$. In the following we give a numerically more interesting example. Consider the following pair of differential operators related to the 1-soliton solution of the KdV hierarchy:

$$\begin{aligned} P_0(t_1,t_2,t_3) &= \partial^2 + \frac{8e^{2(x+t_3)}}{(1+e^{2(x+t_3)})^2} \\ Q_0(t_1,t_2,t_3) &= \partial^3 + \frac{-1+e^{2(x+t_3)} - 2e^{x+t_3}\cdot \sinh(x+t_3)}{1+e^{2(x+t_3)}} \cdot \partial^2 \\ &\quad - \frac{2e^{x+t_3}(-4e^{x+t_3} + (e^{2(x+t_3)}-1)\cdot \sinh(x+t_3))}{(1+e^{2(x+t_3)})^2} \cdot \partial^1 \\ &\quad - \frac{8e^{2(x+t_3)}(-1+e^{2(x+t_3)} + e^{x+t_3}\cdot \sinh(x+t_3))}{(1+e^{2(x+t_3)})^3}. \end{aligned}$$

One can check that indeed

$$[P_0(t_1,t_2,t_3), Q_0(t_1,t_2,t_3)] = 0.$$

Note that the KdV hierarchy is the 2-reduced KP hierarchy and hence the above operators do not depend on t_2. Note also that by the first Lax equation of the KP hierarchy one knows that up to a constant one has $x = t_1$. Furthermore, we interpret the above operators as elements of $\mathbb{C}[\![x]\!][\partial_x]$ by taking the Taylor series with respect to x of the coefficients.

We now show that

$$(\mathrm{Quant} \circ \mathrm{KP}(0,0,1))\,(P_0(0,0,0), Q_0(0,0,0)) \ncong (\mathrm{KP}(0,0,1) \circ \mathrm{Quant})\,(P_0(0,0,0), Q_0(0,0,0))$$

and hence, in general, the quantization does not commute with KP flows.

We will need to calculate several companion matrices. A simple algorithm is the following: Let M denote the companion matrix of P and Q where P is monic of degree p and Q is monic of degree q: Consider $v_i = z^i$ for $0 \leq i \leq p-1$. The entries of M are determined via

$$Q \cdot v_i = \sum_j M_{ij}(P) \cdot v_j$$

where M_{ij} is a polynomial. One has $Q \cdot v_i = z^{i+q} +$ lower order terms. Writing $i + q = s \cdot p + r$ with $0 \leq r < p$ one sees that

$$Qv_i - P^s v_r = \text{ sum of terms of order at most } i + q - 1 = a_{i+q-1} z^{i+q-1} + \cdots .$$

Writing $i + q - 1 = a \cdot p + b \quad$ where $0 \leq b < p$, one obtains

$$Qv_i - P^s v_r - a_{i+q-1} P^a \cdot v_b = \text{ sum of terms of order at most } i + q - 2.$$

Continuing this process one obtains the entries of the companion matrix.

The companion matrix at $t_1 = t_2 = 0$ and $t_3 = 1$ can be calculated via the previously described algorithm and one obtains that

$$(\text{Quant} \circ \text{KP}(0,0,1)) \left(P_0(0,0,0), Q_0(0,0,0)\right) \cong \partial_u + M_{P_0(0,0,1), -Q_0(0,0,1)}$$

with $M_{P_0(0,0,1), -Q_0(0,0,1)} = (m_{ij})$ given by

$$m_{11} = \frac{4e^2(-1+e^2)}{(1+e^2)^3}$$

$$m_{12} = -\frac{-1 + 2e^2 - e^4 + (1 + 2e^2 + e^4)u}{(1+e^2)^2}$$

$$m_{21} = -\frac{16e^4 + (-1 - 8e^2 - 14e^4 - 8e^6 - e^8)u + (1 + 4e^2 + 6e^4 + 4e^6 + e^8)u^2}{(1+e^2)^4}$$

$$m_{22} = -\frac{4e^2(-1+e^2)}{(1+e^2)^3}.$$

It follows from the Levelt-Turrittin classification that every irreducible connection (V, ∇) on the formal punctured disc is obtained as a pushforward of a one-dimensional connection from a suitable covering $t^{1/i} \mapsto t$. More specifically, it is isomorphic to a connection with $V = \mathbb{C}(\!(t^{1/i})\!)$ and $\nabla = \partial_t + \frac{f}{t}$ for some $f \in \mathbb{C}(\!(t^{1/i})\!)$. We denote this connection as $[i]_*[f/t]$. Two such connections corresponding to f_1 and f_2 in $\mathbb{C}(\!(t^{1/i})\!)$ are isomorphic if and only if f_2 is obtained from f_1 by adding an element of $\frac{\mathbb{Z}}{i} + t^{1/i}\mathbb{C}[\![t^{1/i}]\!]$ and substituting $\zeta_i t^{1/i}$ for $t^{1/i}$ where ζ_i is an i'th root of unity, see for example [4] for more details. For $F \in \mathbb{C}(\!(t^{1/i})\!)$ we also write $[i]_*[F]$ in terms of $u := 1/t$ and z such that $z^i = u$. Namely we denote this by $[i]_*[-F(1/z)/u^2]$.

By using the Levelt-Turrittin algorithm one calculates that the connection $\partial_u + M_{P_0(0,0,1), -Q_0(0,0,1)}$ is isomorphic to

$$[2]_* \left[-\frac{1}{4z^2} - z + z^3 \right].$$

Now note that

$$M_{P_0(0,0,0), -Q_0(0,0,0)} = \begin{bmatrix} 0 & -u \\ -(u-1)^2 & 0 \end{bmatrix}$$

and one calculates that

$$\text{Quant}(P_0(0,0,0), Q_0(0,0,0)) \cong [2]_* \left[-\frac{1}{4t} - \frac{1}{t^{5/2}} + \frac{1}{t^{7/2}} \right] \cong [2]_* \left[-\frac{1}{4z^2} - z + z^3 \right].$$

Note that

$$[p]_* \left[\sum c_i t^{i/p} \right] \cong \partial_u - \frac{1}{u^2} \sum c_i u^{-i/p}$$

and the KP flows with KP times $t_1, t_2, \cdots$ perturb this connection via

$$\partial_u - \frac{1}{u^2} \sum_i c_i u^{-i/p} \mapsto \partial_u - \frac{1}{u^2} \sum_i c_i u^{-i/p} - \frac{1}{p} \sum_i i t_i u^{(i-p)/p}.$$

One deduces that

$$(\mathrm{KP}(0,0,1)\circ \mathrm{Quant})\left(P_0(0,0,0),Q_0(0,0,0)\right)\cong [2]_*\left[-\frac{1}{4z^2}-\frac{1}{2z}-z+z^3\right].$$

This shows that KP flows do not in general commute with quantization.

2.2 Quantization and ι

In this section we answer questions (ii) and (iii) from the introduction. To do so, we adapt the arguments of [3] to the current setting. Suppose (P,Q) correspond to String_0' and P and Q are of co-prime degree p and q. As mentioned before, after possibly conjugating P by a function we can and will assume the subleading coefficient to be zero. Let Γ denote the group of monic degree-zero pseudodifferential operators. Since P is normalized it follows that there is W in Γ such that

$$\tilde{P}:=WPW^{-1}=\partial^p$$

and

$$\tilde{Q}:=WQW^{-1}=\sum_{i\geq 0}c_i\partial^{q-i}.$$

The parameters c_i then determine the isomorphism class of the element of String_0' corresponding to (P,Q) and we will denote it by $(p,q)_0(c_i)_i$. If (P,Q) correspond to String_1' then if P is normalized there is W in Γ such that

$$\tilde{P}:=WPW^{-1}=\partial^p$$

and

$$\tilde{Q}:=WQW^{-1}=\frac{1}{p}x\partial^{1-p}+\frac{1-p}{2p}\partial^{-p}+\sum_{i\geq 0}c_i\partial^{q-i}.$$

The parameters c_i then determine the isomorphism class of the element of String_1' and we will denote it by $(p,q)_1(c_i)_i$.

Theorem 2. *For positive co-prime integers p and q there are isomorphisms*

$$\begin{aligned}\iota\left((p,q)_1(c_i)_i\right) &\cong (q,p)_1(\hat{c}_i)_i\\ \iota\left((p,q)_0(c_i)_i\right) &\cong (q,p)_0(\hat{c}_i)_i\end{aligned}$$

where

$$c_n=-\frac{q}{p}\cdot\sum_{k\geq 1}\frac{1}{k}\binom{(n-p-q)/p}{k-1}\sum_{\substack{m_1,\cdots,m_k\geq 1\\ \sum m_i=n}}(-1)^k\hat{c}_{m_1}\cdots\hat{c}_{m_k}$$

and

$$\hat{c}_n=\frac{p}{q}\cdot\sum_{k\geq 1}\frac{1}{k}\binom{(n-p-q)/q}{k-1}\sum_{\substack{m_1,\cdots,m_k\geq 1\\ \sum m_i=n}}c_{m_1}\cdots c_{m_k}.$$

Furthermore, for (P_0,Q_0) corresponding to String_0' one has

$$(\mathrm{Quant}\circ\iota)(P_0,Q_0)\cong(\iota\circ\mathrm{Quant})(P_0,Q_0).$$

Proof. The statement concerning $(p,q)_1(c_i)_i$ is essentially a reformulation of the results of Fukuma-Kawai-Nakayama [3], see also [7]: Suppose W in Γ is such that

$$\tilde{P}:=WPW^{-1}=\partial^p$$

and

$$\tilde{Q} := WQW^{-1} = \frac{1}{p}x\partial^{1-p} + \frac{1-p}{2p}\partial^{-p} + \sum_{i\geq 0} c_i \partial^{q-i}.$$

Note that $\mathrm{Quant}(P,Q) \cong \partial_u + M_{P,-Q}$. On the other hand, from the form of $\tilde{Q}$, one obtains for $z^p = u$ that

$$\partial_u + M_{P,-Q} \cong [p]_* \left[-\frac{1-p}{2p}\frac{1}{z^p} - \sum_{i\geq 0} c_i z^{q-i} \right].$$

Let

$$\xi := \exp\left(\frac{c_1}{qc_0}x\right).$$

There is V in Γ such that

$$V(\frac{1}{c_0}\xi Q \xi^{-1})V^{-1} = \partial^q$$

and

$$V\left(-c_0 \xi P \xi^{-1}\right)V^{-1} = \frac{1}{q}x\partial^{1-q} + \frac{1-q}{2q}\partial^{-q} + \sum_{i\geq 0} \hat{c}_i \partial^{p-i}.$$

By definition $\iota\left((p,q)_1(c_i)_i\right)$ corresponds to the string equation $[Q,-P]=1$ and therefore

$$\iota\left((p,q)_1(c_i)_i\right) \cong \partial_u + M_{Q,P}.$$

In terms of the $\hat{c}_i$'s this is isomorphic to

$$[q]_* \left[-\frac{1-q}{2q}\frac{1}{z^q} - \sum_{i\geq 0} \hat{c}_i z^{p-i} \right].$$

Under our assumption that P and Q are monic it follows that $c_0 = 1$ and $\hat{c}_0 = -1$. It then follows from [3] that

$$c_n = -\frac{q}{p}\cdot \sum_{k\geq 1} \frac{1}{k} \binom{(n-p-q)/p}{k-1} \sum_{\substack{m_1,\cdots,m_k \geq 1 \\ \sum m_i = n}} (-1)^k \hat{c}_{m_1}\cdots \hat{c}_{m_k}$$

and

$$\hat{c}_n = \frac{p}{q}\cdot \sum_{k\geq 1} \frac{1}{k} \binom{(n-p-q)/q}{k-1} \sum_{\substack{m_1,\cdots,m_k \geq 1 \\ \sum m_i = n}} c_{m_1}\cdots c_{m_k}.$$

We now prove the statement concerning $(p,q)_0(c_i)_i$. Let W be in Γ such that

$$\tilde{P} := WPW^{-1} = \partial^p$$

and

$$\tilde{Q} := WQW^{-1} = \sum_{i\geq 0} c_i \partial^{q-i}.$$

Let $\xi := \exp(\frac{c_1}{qc_0}x)$ and let V in Γ be such that

$$V(\frac{1}{c_0}\xi Q \xi^{-1})V^{-1} = \partial^q$$

and

$$V(-c_0\xi P\xi^{-1})V^{-1} = \sum_{i\geq 0} \hat{c}_i \partial^{p-i}.$$

For $\gamma := W\xi^{-1}V^{-1}$ let $\overline{\partial} := \gamma\partial\gamma^{-1}$. One obtains the system of equations

$$\begin{aligned} \overline{\partial}^q &= \partial^q \left(1 + \sum_{i\geq 1} c_i \partial^{-i}\right) \\ \partial^p &= \overline{\partial}^p \left(1 + \sum_{i\geq 1} (-\hat{c}_i) \overline{\partial}^{-i}\right). \end{aligned}$$

Now, take q'th and p'th roots of the two equations and let α and β be two indeterminates. Note that $\mathbb{C}((\partial^{-1})) \cong \mathbb{C}((1/\alpha))$ and $\mathbb{C}((\overline{\partial}^{-1})) \cong \mathbb{C}((1/\beta))$. One then deduces the following system of equations:

$$\begin{aligned} \beta &= \alpha \left(1 + \sum_{i\geq 1} c_i \alpha^{-i}\right)^{1/q} \\ \alpha &= \beta \left(1 + \sum_{i\geq 1} (-\hat{c}_i) \beta^{-i}\right)^{1/p}. \end{aligned}$$

The explicit description between the c_i's and $\hat{c}_i$'s follows then by the same argument as given in [3], Section 4.

The last part of the theorem follows from the previous calculations together with the observation that for (P_0, Q_0) corresponding to String_0 the eigenvalues of the companion matrix agree with the exponential factors of the corresponding connection up to regular terms. □

We now illustrate Theorem 2 with some numerical examples.

Example 1. We start by giving an example of the way ι changes the isomorphism class of a Higgs bundle. In the notation of Section 2.1 the characteristic polynomial of $M_{P_0(0,0,0),-Q_0(0,0,0)}$ is given by

$$\text{char}(M_{P_0(0,0,0),-Q_0(0,0,0)}, x) = x^2 - u(u-1)^2$$

and in terms of z with $z^2 = u$ one obtains the eigenvalues as $z^3 - z$ and $-z^3 + z$. Considering the second of these two, one therefore knows that the Higgs bundle associated with $(P_0(0,0,0), Q_0(0,0,0))$ is given by $(2,3)_0(c_i)_i$ with $c_0 = 1, c_1 = 0, c_2 = -1$ and $c_i = 0$ for all $i \geq 3$. By Theorem 2

$$\sum_{i\geq 0} \hat{c}_i z^{2-i} = -z^2 - \frac{2}{3} - \frac{1}{9z^2} + \frac{2}{81z^4} + \text{ lower order terms.}$$

Then for example if $f(z) := z^2 + \frac{2}{3} + \frac{1}{9z^2} - \frac{2}{81z^4}$ and if ζ_3 denotes a primitive third root of unity, one obtains

$$(x - f(z))(x - f(\zeta_3 z))(x - f(\zeta_3^2 z)) = x^3 - 2x^2 + x - z^6 + \frac{2x}{243z^6} - \frac{19}{2187z^6} + \frac{8}{531441z^{12}}.$$

This should give an approximation of the characteristic polynomial of $M_{Q_0(0,0,0),P_0(0,0,0)}$. The latter can be calculated independently in an exact manner since the companion matrix can be calculated and indeed one obtains with $u = z^3$ that

$$\text{char}(\begin{bmatrix} 2 & 0 & 1 \\ u & 0 & 0 \\ -1 & u & 0 \end{bmatrix}, x) = x^3 - 2x^2 + x - z^6.$$

Example 2. We now give an example, via the KdV 1-soliton as discussed in Section 2.1, that ι commutes with quantization. In the previously used notation first note that

$$\mathrm{Quant}(P_0(0,0,0), Q_0(0,0,0)) \cong \partial_u - \sum c_i u^{(3-i)/2}$$

with $c_0 = 1, c_1 = 0, c_2 = -1, a_3 = 0, a_4 = 0, c_5 = -1/4$. Via Theorem 2 one obtains with $z = u^3$ that

$$\begin{aligned}(\iota \circ \mathrm{Quant})(P_0(0,0,0), Q_0(0,0,0)) &\cong \partial_u - \sum_i \hat{c}_i u^{(2-i)/3} \\ &\cong \partial_u + \frac{1}{3z^3} + \frac{1}{9z^2} + \frac{2}{3} + z^2.\end{aligned}$$

On the other hand, one has

$$(\mathrm{Quant} \circ \iota)(P_0(0,0,0), Q_0(0,0,0)) \cong \partial_u + M_{Q_0(0,0,0),P_0(0,0,0)}$$

with

$$M_{Q_0(0,0,0),P_0(0,0,0)} = \begin{bmatrix} 2 & 0 & 1 \\ u & 0 & 0 \\ -1 & u & 0 \end{bmatrix}.$$

And the Levelt-Turrittin algorithm yields that

$$(\mathrm{Quant} \circ \iota)(P_0(0,0,0), Q_0(0,0,0)) \cong [3]_* \left[\frac{1}{3z^3} + \frac{1}{9z^2} + \frac{2}{3} + z^2\right].$$

Hence, in this example the quantization does commute with ι.

Example 3. We give another example that ι commutes with quantization. Let

$$P_0 = \partial^3 - 7\partial + 2$$

$$Q_0 = \partial^5 + \partial^3 - 2\partial^2 + \partial + 5.$$

Then clearly the two operators commute. Furthermore

$$M_{P_0,-Q_0} = \begin{bmatrix} 11 - 8u & -57 & 4 - u \\ -8 + 6u - u^2 & 39 - 15u & -57 \\ 57(2-u) & -407 + 6u - u^2 & 39 - 15u \end{bmatrix}.$$

One then calculates

$$\mathrm{Quant}(P_0, Q_0) \cong [3]_* \left[\frac{1}{3z^3} + \frac{1418}{27z^2} - \frac{10483}{81z} + \frac{89}{3} - \frac{562}{9}z + \frac{16}{3}z^2 - \frac{38}{3}z^3 - z^5\right].$$

Hence, via Theorem 2, the composition $\iota \circ \mathrm{Quant}$ applied to (P_0, Q_0) is isomorphic to

$$[5]_* \left[-\frac{2}{5z^5} + \frac{1008}{125z^4} + \frac{223}{125z^3} - \frac{163}{25z^2} + \frac{26}{25z} + \frac{16}{5} - \frac{38}{5}z + z^3\right].$$

We now verify directly that this is isomorphic to applying $\mathrm{Quant} \circ \iota$ to (P_0, Q_0). Note that

$$M_{Q_0,P_0} = \begin{bmatrix} 2 & -7 & 0 & 1 & 0 \\ 0 & 2 & -7 & 0 & 1 \\ -5+u & -1 & 4 & -8 & 0 \\ 0 & -5+u & -1 & 4 & -8 \\ 40 - 8u & 8 & -21 + u & 7 & 4 \end{bmatrix}.$$

Then

$$(\mathrm{Quant} \circ \iota)(P_0, Q_0) \cong \partial_u + M_{Q_0,P_0}$$

and in terms of z with $z^5 = u$ the Levelt-Turrittin algorithm shows that indeed this is isomorphic to

$$[5]_* \left[-\frac{2}{5z^5} + \frac{1008}{125z^4} + \frac{223}{125z^3} - \frac{163}{25z^2} + \frac{26}{25z} + \frac{16}{5} - \frac{38}{5}z + z^3 \right].$$

To give more context to the classical/quantum correspondence of the ι dynamics, we now show that in the theory of the local Fourier transform the same phenomenon occurs:

After choosing a basis, write an n-dimensional connection ∇ on the punctured disc as

$$\nabla = \partial_t + A \quad \text{where} \quad A \in \mathfrak{gl}_n\mathbb{C}(\!(t)\!).$$

The connection can naturally be deformed into an $\hbar$-connection by setting

$$\nabla_\hbar = \hbar\partial_t + A.$$

As discussed for example by Graham-Squire [4], the formulas for the local Fourier transforms, up to the regular term, can be easily understood if the differential part of the connection is dropped. This corresponds to letting $\hbar = 0$ in the above situation. We now give some more details. Consider for example the local Fourier transform $\mathcal{F}^{(\infty,\infty)}$. It can be defined, see [1], in the following manner: Suppose given a connection (V, ∇_t) over $\mathbb{C}(\!(t)\!)$. Let $\hat{t}$ denote the variable of the Fourier dual connection. Then $\mathcal{F}^{(\infty,\infty)}(V, \nabla_t)$ has the same underlying $\mathbb{C}$-vector space as V and the action of $\hat{t}$ and $\nabla_{\hat{t}}$ is given via

$$\hat{t} = (t^2\nabla_t)^{-1} \quad \text{and} \quad \hat{t}^2\nabla_{\hat{t}} = -1/t.$$

Note that indeed $\nabla_{\hat{t}}$ is a connection with respect to the $\mathbb{C}(\!(\hat{t})\!)$ vector space structure: One has

$$[t^{-1}, \hat{t}^{-1}] = 1$$

and hence, as an example, note that for all $v \in V$ one has

$$\nabla_{\hat{t}}(\hat{t}^{-1}v) = -\hat{t}^{-2}t^{-1}\hat{t}^{-1}v = -\hat{t}^{-2}(v + \hat{t}^{-1}t^{-1}v) = -\hat{t}^{-2}v + \hat{t}^{-1}\nabla_{\hat{t}}(v)$$

as desired. Explicit formulas for the transform $\mathcal{F}^{(\infty,\infty)}$ have been obtained independently by Fang [2], Graham-Squire [4], Sabbah [9]:

Let $f \in \mathbb{C}(\!(t^{1/r})\!)$ for some r and denote $[i]_*[f/t]$ simply by E_f. Suppose $\mathrm{ord}(f) = -s/r$, meaning

$$f = \frac{c}{t^{s/r}} + \text{ higher order terms}$$

for some $c \in \mathbb{C}^\times$. Then for irreducible E_f one has

$$\mathcal{F}^{(\infty,\infty)}E_f \cong E_g$$

where g is determined by the following system of equations:

$$f = \frac{1}{t\hat{t}}$$

$$g = -f + \frac{s}{2(s-r)}.$$

The key point to emphasize is that most of the difficulty of these results concerns the regular term. As a heuristic one has: Without the differential parts one has

$$\nabla_t = \frac{f}{t}$$

$$\nabla_{\hat{t}} = \frac{g}{\hat{t}}$$

and one obtains from the previously described defining equations of $\mathcal{F}^{(\infty,\infty)}$ that

$$f = t\nabla_t = \frac{1}{t\hat{t}}$$

$$g = \hat{t}\nabla_{\hat{t}} = -\frac{1}{\hat{t}t} = 1 - \frac{1}{t\hat{t}} = 1 - f.$$

Hence, this correctly describes the local Fourier transform up to changing the regular term. This gives another viewpoint on the classical/quantum correspondence.

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M. Luu, Department of Mathematics, University of Illinois at Urbana-Champaign, IL 61801, USA
E-mail address: mluu@illinois.edu