Local Langlands Duality and a Duality of Conformal Field Theories

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Abstract

We show that the numerical local Langlands duality for GL_n and the T-duality of two-dimensional quantum gravity arise from one and the same symmetry principle. The unifying theme is that the local Fourier transform in both its ℓ-adic and complex incarnation gives rise to symmetries of arithmetic and geometric local Langlands parameters.

1 Introduction

In the 1990's physicists gave rigorous mathematical proofs of the T-duality, also known as p-q duality, of 2D quantum gravity, see [5] and [12]. Recently, we revisited this topic in [18] and in joint work with Albert Schwarz [19], giving a proof of this duality based on the complex local Fourier transform of Bloch-Esnault [2] and Lopez [16]. This transform has an ℓ-adic analogue and it is then natural to ask if there is arithmetic meaning to this new proof of the T-duality. In the present work we explain that this is indeed the case: From a certain perspective the arithmetic analogue is the numerical local Langlands duality for GL_n over local fields.

To describe this passage between physics and arithmetic one should move along the four pillars of Weil's augmented Rosetta stone, see [8]: These pillars are number fields, curves over finite fields, Riemann surfaces, and quantum physics. The analogies are particularly interesting when comparing various dualities that exist within the four different frameworks. Before explaining our results concerning this relation of dualities we recall a global analogue of the passage.

Consider first the arithmetic duality given by the relation between Galois representations and automorphic representations. One can say that the study of this duality started, in its true non-abelian generality, in the work of Eichler [4]. By now, it is a well studied theme and often called a Langlands duality. For example in the context of $\mathrm{GL}_{n,\mathbb{Q}}$ it is a popular conjecture, see for example [24], that every irreducible representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

that is continuous for the ℓ-adic topology on the target and the profinite topology on the source and that is unramified at all but finitely many places and de Rham at ℓ is attached to a cuspidal automorphic representation of $\mathrm{GL}_{n,\mathbb{Q}}$. This has a global geometric analogue: Given a Riemann surface X, the unramified geometric Langlands duality for GL_n associates to every flat GL_n-bundle over X a certain D-module on the moduli stack of GL_n-bundles over X. The fourth aspect of the Rosetta stone, quantum physics, enters through the work of Kapustin and Witten [14] that relates the geometric Langlands duality to the S-duality of 4D gauge theories.

In the present work we show that in analogy with the above global passage between arithmetic and physics, the T-duality of 2D quantum gravity can be related to dualities in the local incarnations of the first three pillars of the Rosetta stone. In the first two pillars, the numerical local Langlands correspondence for GL_n over local fields has been obtained in work of Henniart [10]. To move to the third pillar we introduce a local geometric numerical Langlands correspondence for GL_n. Due to classical results of Levelt and Turrittin one can describe the relevant local geometric Langlands parameters in a very explicit manner. What is interesting is that in this manner the variety $\mathbb{A}^j \times \mathbb{G}_m$ plays an important role and in the arithmetic numerical local Langlands correspondence one knows that the relevant number of local Langlands parameters is given by the number of $\mathbb{F}_q$-points of precisely this variety, where $\mathbb{F}_q$ is the finite residue field of the local field in question. However, in contrast to the geometric case, one arrives at this count via the properties of Laumon's local ℓ-adic Fourier functors and a direct parametrization of the Langlands parameters seems unknown.

We then move to the fourth pillar of the Rosetta stone by explaining how quantum physics enters the picture: The starting point is the association of Virasoro constrained τ-functions of the KP hierarchy to suitable local geometric Langlands parameters. These τ-functions are relevant in quantum field theory, in particular in 2D quantum gravity where they allow to describe the partition functions of the various models of the theory. The question then occurs whether there is an analogue of the local numerical Langlands correspondence in these physical theories. It turns out that, rightly interpreted, there indeed is and it has the same underlying mechanism as the arithmetic duality, namely symmetries of local Langlands parameters coming from the local Fourier transform. We explain how this viewpoint yields the relation between the Langlands duality and the T-duality.

2 Numerical local Langlands correspondence

Up to this point, the numerical local Langlands correspondence (numerical LLC) for GL_n has been known for the local incarnations of the first two pillars of Weil's Rosetta stone, namely fields of the form $\mathbb{F}_q(\!(t)\!)$ and finite extension of $\mathbb{Q}_p$. After reviewing the precise statement in the arithmetic case, we develop in this section a version of the correspondence in the context of the local geometric Langlands correspondence.

2.1 The arithmetic theory

Fix a non-archimedean local field K with finite residue field of size q and residue field characteristic p. For $n \geq 1$ let $\mathcal{A}^0(n)$ denote the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}_n(K)$ and let $\mathcal{G}^0(n)$ denote the set of isomorphism classes of n-dimensional Weil-Deligne representations (r, N) of the Weil group W_K with r irreducible. The local Langlands correspondence predicts a bijection between $\mathcal{A}^0(n)$ and $\mathcal{G}^0(n)$ that preserves interesting arithmetic data such as the conductor. A natural approach to test this bijection is to put more and more constraints on both sets, until one obtains finite sets. These should then have the same size and this is the idea of the numerical local Langlands correspondence. This correspondence plays a crucial role in the original proof of the local Langlands correspondence for GL_n over a p-adic field K by Harris-Taylor [11] since it reduces the task of constructing a suitable bijection between the relevant Weil-Deligne representations and smooth representations to the task of constructing an injection. This is a significant simplification:

Suppose $\tilde{\pi}$ is a smooth admissible representation of $\mathrm{GL}_n(K)$ which is supercuspidal. Due to work of Clozel it can be realized as local component of a suitable global object, an automorphic representation π over a global field F, which factors as a restricted tensor product

$$\pi \cong \otimes'_v \pi_v$$

such that there is a finite place v_0 of F such that $F_{v_0} \cong K$ and $\pi_{v_0} \cong \tilde{\pi}$. As a consequence of the local-global compatibility philosophy for Langlands correspondences, a candidate for the local Langlands correspondence is given in the following manner: For the choice of auxiliary prime ℓ one associates to $\tilde{\pi}$ the Weil-Deligne representation of $W_{F_{v_0}}$ associated to $\rho_{\pi,\ell}|_{\mathrm{Gal}(\overline{F}_{v_0}/F_{v_0})}$ where $\rho_{\pi,\ell}$ is a suitable ℓ-adic representation attached to π. However, it is not clear from the construction what the image of this correspondence is. The great use of the numerical local Langlands correspondence is that it makes it unnecessary to understand the image: Henniart [10] used the numerical correspondence to show that any injection

$$\mathcal{G}^0(n) \hookrightarrow \mathcal{A}^0(n)$$

that is compatible with twists by unramified characters and preserves conductors has to be a bijection. We now recall the statement of the numerical local Langlands correspondence:

For $d \geq 1$ and $k \geq 0$ let $\mathcal{G}_{d,k}$ denote the set of isomorphism classes of Weil-Deligne represenations $(r, 0) \in \mathcal{G}^0(d)$ such that $r|_{I_K}$ is irreducible and $\mathrm{sw}(\sigma) = k$ where I_K denotes the inertia subgroup of W_K and $\mathrm{sw}(-)$ denotes the Swan conductor. Define the set

$$C(n, j)_{\mathrm{arith}} = \bigcup_{d|n} \bigcup_{k \cdot \frac{n}{d} \leq j} (\mathcal{G}_{d,k})_{/\sim}$$

where the equivalence relation $\sim$ identifies representations that differ by a twist by an unramified character. This set of equivalence classes is known to be a finite set by work of Koch and hence it makes sense to define the number

$$c(n,j)_{\mathrm{arith}} := |C(n,j)_{\mathrm{arith}}|.$$

The following was conjectured by Koch, in a slightly different but equivalent form:

Arithmetic Numerical LLC. *Let $n \geq 1$ and $j \geq 0$ be two integers. Then*

$$c(n,j)_{\mathrm{arith}} = (q-1)q^j$$

where q denotes the size of residue field of K.

This is one of several ways to formulate the numerical local Langlands correspondence, see [17] for relevant background and history. After long and laborious calculations, Koch [13] was able to prove the conjecture if the residue characteristic p of $\mathbb{F}_q$ divides n at most linearly. Henniart used completely different methods in [10] to prove the general case. This shift in the approach to the conjecture involves the local Fourier functors and is crucial for us to make the connection with quantum duality.

2.2 The geometric theory

Our aim is now to obtain a numerical local geometric Langlands correspondence. We start by describing arithmetic and geometric local Langlands parameters in a unified manner.

- Let K be a non-archimedean local field and let W_K be the corresponding Weil group. An arithmetic local Langlands parameter $\mathcal{L}$ for GL_n with trivial monodromy operator is a pair (V,ρ) where V is an n-dimensional $\mathbb{C}$-vector space and ρ is a homomorphism
$$\rho : W_K \longrightarrow \mathrm{GL}(V)$$
with open kernel.
- A geometric local Langlands parameter $\mathcal{L}$ for GL_n is simply a connection (V,∇) on the formal punctured disc, see [7]. After choosing a local coordinate, this consists of the data of an n-dimensional $\mathbb{C}((t))$-vector space M together with a $\mathbb{C}$-linear map
$$\nabla : M \longrightarrow M$$
such that
$$\nabla(f\cdot m) = f\cdot\nabla(m) + \frac{\mathrm{d}f}{\mathrm{d}t}\cdot m$$
for all $f \in \mathbb{C}((t))$ and all $m \in M$.

There will be two important integer parameters n and j associated to such arithmetic and geometric Langlands parameters: For an arithmetic Langlands parameter $\mathcal{L} = (V,\rho)$ let

$$\begin{aligned} n(\mathcal{L}) &= \dim_{\mathbb{C}} V \\ j(\mathcal{L}) &= \mathrm{sw}(\rho) \end{aligned}$$

where $\mathrm{sw}(-)$ is the Swan conductor. For a geometric Langlands parameter $\mathcal{L} = (\nabla, V)$ define

$$\begin{aligned} n(\mathcal{L}) &= \dim_{\mathbb{C}((t))} V \\ j(\mathcal{L}) &= \sum \text{slopes of } \nabla. \end{aligned}$$

The integer $j(\mathcal{L})$ is called the irregularity of the connection.

For $n, j \in \mathbb{Z}^{>0}$ let $\mathrm{LL}_{\mathrm{arith}}(n,j)$ denote the set of isomorphism classes of arithmetic local Langlands parameters of dimension n and Swan conductor j, and let $\mathrm{LL}^{\circ}_{\mathrm{arith}}(n,j)$ denote the subset of irreducible parameters. The geometric analogue is the set $\mathrm{LL}_{\mathrm{geom}}(n,j)$ of isomorphism classes of irreducible geometric local Langlands dimension n and irregularity j and the subset $\mathrm{LL}^{\circ}_{\mathrm{geom}}(n,j)$ consisting of irreducible connections.

Our aim is now to find a set of isomorphism classes of local geometric Langlands parameters for GL_n that is defined in a similar manner to $C(n,j)_{\mathrm{arith}}$ and to show that its suitably interpreted size is closely related to $c(n,j)_{\mathrm{arith}}$. Recall that on the arithmetic side one considers a union

$$C(n,j)_{\mathrm{arith}} = \bigcup_{d|n} \bigcup_{k \cdot \frac{n}{d} \leq j} \cdots$$

of equivalence classes of arithmetic local Langlands parameters. The reason one lets d and k vary in the above manner is that this corresponds to counting suitable discrete series representations rather than supercuspidal representations. It turns out that this simplifies matters when combined with the local Jacquet-Langlands correspondence. In the arithmetic setting it is thus a fact that the "discrete series count" gives a simpler formula than the "supercuspidal count" if we make the following notational conventions:

- discrete series count: all pairs (d,k) such that $d \big| n$ and $k \leq j \cdot \frac{d}{n}$
- supercuspidal count: all pairs (d,k) such that $d = n$ and $k \leq j$

Under the analogies of Weil's Rosetta stone, the geometric residue field

$$k_g := \mathbb{C}$$

of $\mathbb{C}(\!(t)\!)$ corresponds to the arithmetic residue field

$$k_a := \mathbb{F}_q$$

of K. Hence one might expect that a geometric analogue of $C(n,j)_{\mathrm{arith}}$ should be described via suitable parameters in the residue field $\mathbb{C}$. Furthermore, given the formula

$$c(n,j)_{\mathrm{arith}} = (q-1)q^j$$

it is natural to try to relate this to a point count of a variety over a finite field. It seems unknown if there is indeed a variety whose points count the set $C(n,j)_{\mathrm{arith}}$ in a natural manner. However, after the fact, meaning with the above formula in hand, the arithmetic numerical local Langlands correspondence can be phrased as follows:

Arithmetic Numerical LLC. *Let K be a non-archimedean local field with finite residue field k_a. Let $n \geq 1$ and $j \geq 0$ be two integers. Then there is a bijection*

$$C(n,j)_{\mathrm{arith}} \xleftrightarrow{1:1} \left(\mathbb{A}^j \times \mathbb{G}_m\right)(k_a)\,.$$

The occurrence of the variety $\mathbb{A}^j \times \mathbb{G}_m$ might seem artificial but something interesting occurs when moving to the geometric pillar of the Rosetta stone: We now show that the same variety occurs in a natural analogue of the numerical local Langlands duality.

Geometric Numerical LLC. *Let $K = \mathbb{C}(\!(t)\!)$ and let $k_g = \mathbb{C}$ denote its residue field. Let $n \geq 1$ and $j \geq 0$ be two integers.*

(i) Define the subspace U of $\mathbb{A}^j = \mathrm{Spec}\ \mathbb{C}[x_1, \cdots, x_j]$ given by complement of the subspaces defined via

$$x_{i_1} = \cdots = x_{i_r} = 0$$

where n and the elements of $\{1, 2, \cdots, j\}\backslash\{i_1, \cdots, i_r\}$ have non-trivial gcd. There is a set $C(n, j)^{\text{cusp}}_{\text{geom}}$ of local geometric Langlands parameters for GL_n whose definition is modeled on the supercuspidal count such that there is a bijection

$$C(n,j)^{\text{cusp}}_{\text{geom}} \stackrel{1:1}{\longleftrightarrow} (U \times \mathbb{G}_m)(k_g)_{/\sim}$$

where $\sim$ indicates that

$$(a_1, \cdots, a_j, x) \sim (a_1\zeta_n, \cdots, a_j\zeta_n^j, x)$$

whenever ζ_n is an n'th root of unity.

(ii) There is a set $C(n, j)^{\text{disc}}_{\text{geom}}$ of local geometric Langlands parameters for GL_n whose definition is modeled on the discrete series count such that there is a bijection

$$C(n,j)^{\text{disc}}_{\text{geom}} \stackrel{1:1}{\longleftrightarrow} (\mathbb{A}^j \times \mathbb{G}_m)(k_g)_{/\sim}$$

where $\sim$ is as before.

Proof. For the proof of both parts of the theorem it will be useful to recall how the classical results of Levelt and Turrittin allow to describe irreducible n-dimensional connections on the formal punctured disc: Given a map

$$\rho : \mathbb{C}[\![t]\!] \longrightarrow \mathbb{C}[\![u]\!]$$

which takes t to some element in $u\mathbb{C}[\![u]\!]$ one can define associated push-forward and pull-back operations on the categories of connections on the formal punctured disc with local coordinate t and u respectively. See for example [21] for more details. For $i \in \mathbb{Z}^{\geq 1}$, denote by $[i]$ the map ρ that takes t to u^i. For a Laurent series $f \in \mathbb{C}(\!(t^{1/n})\!)$ define the $\mathbb{C}(\!(t)\!)$-connection

$$E(f, t^{1/n}, n) := [n]_* \left(\mathbb{C}(\!(t^{1/n})\!), \frac{\mathrm{d}}{\mathrm{d}t^{1/n}} + n \cdot \frac{f}{t^{1/n}} \right).$$

Then

$$\dim_{\mathbb{C}(\!(t)\!)} E(f, t^{1/n}, n) = n.$$

The Levelt-Turrittin classification implies that the gauge equivalence class of the connection $E(f, t^{1/n}, n)$ depends on f precisely up to adding an arbitrary element of

$$\frac{1}{n}\mathbb{Z} + t^{1/n}\mathbb{C}[\![t^{1/n}]\!]$$

and up to substituting $\zeta_n z$ for z where ζ_n is an n'th root of unity, see for example [9].

Furthermore, this connection is irreducible if and only if f is not in $\mathbb{C}(\!(t^{1/m})\!)$ for some $0 < m < n$. Every irreducible n-dimensional connection over $\mathbb{C}(\!(t)\!)$ is isomorphic to some $E(f, t^{1/n}, n)$. Note that if

$$f = \frac{*}{t^{j/n}} + \text{higher order terms}$$

with $* \in \mathbb{C}^\times$ and $j \leq 0$, then the irregularity is given by

$$j(E(f, t^{1/n}, n)) = j.$$

We now apply these considerations to prove both parts of the geometric numerical local Langlands correspondence.

We start with part (i) of the theorem, the supercuspidal count. Hence we are considering n-dimensional connections of irregularity at most j. We now discuss geometric analogues of the other constraints involved in the definition of $C(n, j)_{\text{arith}}$. First consider the process of identifying representations that differ by a twist by an unramified character. A natural geometric analogue is the process of tensoring a connection with a holomorphic one-dimensional connection. However, this does not change the isomorphism class of the connection. Hence, given an irreducible connection

$E(f, t^{1/n}, n)$ we can reduce to the case where f is of the form

$$f = \sum_{i \leq 0} a_i t^{i/n}.$$

The condition that the irregularity is at most j corresponds to

$$a_i = 0 \quad \text{for all} \quad i < -j.$$

Given a Weil-Deligne representation (r, N), the condition that r restricted to inertia is irreducible we simply translate to the condition that $E(f, t^{1/n}, n)$ is irreducible. As indicated before, this is equivalent to the fact that $f \in \mathbb{C}(\!(t^{1/n})\!)$ is not in fact an element of $\mathbb{C}(\!(t^{1/m})\!)$ for $0 < m < n$. In other words one needs that the j-tuple $(a_{-1}, \cdots, a_{-j})$ is not an element of the set

$$\{(b_{-1}, \cdots, b_{-j}) \in \mathbb{C}^j \mid b_i = 0 \text{ except possibly for } i \text{ in a set of indicies whose gcd with } n \text{ is non-trivial}\}.$$

With all these conventions we obtain the geometric analogue of $C(n, j)_{\text{arith}}$ as

$$C(n, j)^{\text{cusp}}_{\text{geom}} := \bigcup_{k \leq j} \text{LL}^{\circ}_{\text{geom}}(n, k).$$

Due to the Levelt-Turrittin classification and the isomorphism $\exp(2\pi\sqrt{-1}n-) : \mathbb{C}/\frac{1}{n}\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^{\times}$ one obtains the desired bijection

$$C(n, j)^{\text{cusp}}_{\text{geom}} \longleftrightarrow (\{(a_{-1}, \cdots, a_{-j}) \in U\} \times \mathbb{G}_m(k_g))_{/\sim}$$

where U is as in the statement of the theorem and the a_i's are the coefficients of f as before.

Now we consider part (ii) of the theorem, the discrete series count. In analogy with the structure of arithmetic Weil-Deligne represenations associated to discrete series representations we define

$$C(n, j)^{\text{disc}}_{\text{geom}} := \{ \text{ indecomposable } \mathcal{L} \in \text{LL}_{\text{geom}}(n, k) \text{ for } k \leq j \ \}$$

We start to describe such parameters in more detail. For $i \geq 1$ let N_i denote the i-dimensional connection over $\mathbb{C}(\!(t)\!)$ with connection map given by

$$\partial_t + \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \frac{1}{t}.$$

With this notation, every indecomposable n-dimensional connection is of the form

$$V \otimes N_d$$

where $d|n$ and V is an irreducible n/d-dimensional connection, see for example [1]. Therefore, in order to count $C(n, j)^{\text{disc}}_{\text{geom}}$ one can simply count the representations V. Note that in analogy to the arithmetic conductor constraints we impose

$$j(V) \cdot d \leq j.$$

We now count such representations. Every irreducible n/d-dimensional connection V can be written as

$$[\frac{n}{d}]_* \, E(f, t^{d/n}, 1)$$

for some $f \in \mathbb{C}(\!(t^{d/n})\!)$ and the isomorphism class of this push-forward depends on f precisely up to adding an element in $\mathbb{Z} + t^{d/n}\mathbb{C}[\![t^{d/n}]\!]$ and substituting $\zeta t^{d/n}$ for $t^{d/n}$ where ζ is an n/d'th root of unity. To such an isomorphism class of connections we associate f. Due to our irregularity constraints one sees that as d varies f is precisely an element of $\mathbb{C}(\!(t^{1/n})\!)$ of the form

$$f = \sum_{r \le i \le 0} a_i t^{i/n}$$

with $|r| \le j$. To incorporate the gauge equivalence of connections in this description one identifies f's that differ from one another by an integer on the constant term and by substituting $\zeta_n t^{1/n}$ for $t^{1/n}$ where ζ_n is an n'th root of unity. Via the coefficients of f one obtains a bijection to $(\mathbb{A}^j \times \mathbb{G}_m)(k_g)_{/\sim}$ as desired. □

Note that the similarity between the arithmetic and geometric numerical LLC in particular gives a complex shadow of the structure of the abelian part of the inertia group of a local field.

One can also observe from the above proof a crucial difference between the arithmetic and geometric side: In the latter, all irreducible parameters are obtained from one-dimensional ones via push-forward. In the former, the question of which parameters of W_K are obtained by induction from a parameter of $W_{K'}$ for some finite extension K'/K is subtle. Indeed, for example in the local Langlands correspondence for GL_2 over p-adic fields, a crucial case is $\mathrm{GL}_2(\mathbb{Q}_2)$, precisely since in this case there exist non-induced parameters.

Consider for example for an odd $c \ge 3$ the set of isomorphism classes of Weil-Deligne representations of W_K that are induced from a proper subgroup of W_K and that have Artin conductor c and trivial determinant on a choice of uniformizer. Here K is a p-adic field and we denote by q the size of the residue field of K. Work of Tunnell [25] shows that this set has size

$$2(q-1)^2 q^{c-3}(1 - X(c)) q^{-[(c+1)/6]}$$

where the function $X(c)$ depends on how c compares to $6 \cdot \mathrm{val}_K(2) + 1$ and also on the congruence of c modulo 3. As an application of the local Jacquet-Langlands correspondence one sees that there is no such subtlety for the smooth representations of GL_2 over p-adic fields. It follows that on the Galois side the number of local Langlands parameters that are not induced also has a subtle behavior that offsets the subtle count of induced parameters.

3 Quantum Physics

The work of Henniart and Laumon [10], [15] in conjunction with the discussion of the previous section implies the existence of a numerical local Langlands duality for the first three pillars of Weil's Rosetta stone. The question then arises whether there exists a version of this duality in the fourth pillar, quantum physics. Furthermore, if it does, what is its relation to known dualities in quantum field theory? As a first step towards our answer, we now explain how quantum physics enters the picture at all, forgetting for now about the desired duality. As a particular case we discuss how one can describe 2D quantum gravity via local geometric Langlands parameters. This is not an obvious consequence of the original formulations of this quantum field theory but its importance has recently been stressed in [18], [19], [22]. The relation between the local parameters and quantum field theory is given via τ-functions of the KP hierarchy of partial differential equations and hence we briefly recall some important aspects of this theory.

Consider a Lax operator

$$L(t_1, t_2, \cdots) = \partial_{t_1} + u_{-1}(t_1, t_2, \cdots)\partial_{t_1}^{-1} + \cdots$$

depending on the infinite set $t_1, \cdots$ of time variables. This is a pseudo-differential operator and the Lax equations are given by

$$\frac{\partial}{\partial t_i} L = [L_+^i, L]$$

where L_+^i denotes the part of L^i that does not involve negative powers of ∂_{t_1}. This yields constraints for the coefficients $u_{-1}, u_{-2}, \cdots$. For example, u_{-1} has to satisfy the KP equation

$$\partial_{t_1}\left(4\partial_{t_3} u_{-1} - 12 u_{-1}\partial_{t_1} u_{-1} - \partial_{t_1}^3 u_{-1}\right) = 3\partial_{t_2}^2 u_{-1}.$$

This type of equation was introduced in the theory of water waves and is a generalization in one higher dimension of the KdV equation. A τ-function of the hierarchy allows to describe the coefficients of the Lax operator in a concise manner, one has for example

$$u_{-1} = \partial_{t_1}^2 \ln \tau$$
$$u_{-2} = \frac{1}{2}(\partial_{t_1}^3 + \partial_{t_1}\partial_{t_3}) \ln \tau$$
$$\vdots$$

The relation with quantum physics comes from the fact that partition functions of certain quantum field theories are expected to be expressible in terms of special KP τ-functions. For example, the work of Kontsevich shows that the partition function of topological gravity is the square of a 2-reduced KP τ-function that satisfies certain Virasoro constraints:

The Virasoro algebra is given by Vir $:= \langle L_n \rangle_{n\in\mathbb{Z}} \oplus \langle c \rangle$ with c a central element and

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{m^3-m}{12}\delta_{m+n,0} \cdot c.$$

Consider the differential operators

$$\mathcal{L}_n = \frac{1}{2}\sum_{i+j=-n} ijt_it_j + \sum_{i-j=-n} it_i\partial_j + \frac{1}{2}\sum_{i+j=n}\partial_i\partial_j.$$

These yield a representation of the Virasoro algebra with central charge 1 on the space $\mathbb{C}[\![t_1, t_2, \cdots]\!]$. In particular, this yields an action on the τ-functions of the KP hierarchy. By a Virasoro constrained τ-function $\tau(t_1, t_2, \cdots)$ of the KP hierarchy we mean a τ-function annihilated by suitable operators $\mathcal{L}_n$ as above.

Let p and q be two positive co-prime integers. For every ordered pair (p,q) there is a corresponding model of 2D quantum gravity, see for example [5], with associated partition function denoted by $\mathrm{Z}_{p,q}(t_1, \cdots, t_{p+q})$. On the level of rigor of physics, it is defined to be

$$\mathrm{Z}_{p,q}(t_1, \cdots, t_{p+q}) = \sum_h \int_g \exp(\dots)$$

where the exponential term is derived from the equations of general relativity and one sums over all possible values of the genus h of the surface and integrates over all metrics g. This is crucial since in quantum gravity the metric is supposed to become dynamical. It is a crucial insight that the partition function satisfies

$$Z_{p,q}(t_1, \cdots) = \tau_{p,q}^2(t_1, \cdots)$$

where $\tau_{p,q}$ is a τ-function of the KP hierarchy that satisfies certain Virasoro constraints. Hence to describe the partition function of the (p,q) models of 2D gravity it suffices to describe the point $W_{p,q}$ of the Sato Grassmannian whose associated τ-function equals $\tau_{p,q}$.

There are two ways that local geometric Langlands parameters enter this picture. The first way is as follows: It is known that there is such a parameter such that out of its flat sections one can construct the desired point $W_{p,q}$. This is a subtle procedure that depends on the parameter itself and not just the gauge equivalence class and the construction is related to the quantization of the pair $(\partial_x^p, \partial_x^q)$ of commuting differential operators as described by Schwarz in [23].

Before explaining the second way that local Langlands parameters are important in this context we give some details for the above arguments:

Let Gr be the big-cell of the Sato Grassmannian. It is the set of complex subspaces of $\mathbb{C}(\!(1/z)\!)$ whose projection onto $\mathbb{C}[z]$ is an isomorphism. By results of Sato, to every such point one has an associated τ-function of the KP hierarchy. In terms of the Sato Grassmannian the function $\tau_{p,q}$ can be described by a point $W_{p,q}(t_1, \cdots, t_{p+q})$ of Gr such that

$$z^p W_{p,q}(t_1, \cdots, t_{p+q}) \subseteq W_{p,q}(t_1, \cdots, t_{p+q})$$
$$\left(\frac{1}{pz^{p-1}}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1-p}{2p}\frac{1}{z^p} - \frac{1}{p}\sum_{i=1}^{p+q} it_iz^{i-p}\right) W_{p,q}(t_1, \cdots, t_{p+q}) \subseteq W_{p,q}(t_1, \cdots, t_{p+q}).$$

We now explain how one can construct the point $W_{p,q}$ via a suitable local geometric Langlands parameter.

Let t be given by

$$z^p = t$$

Suppose given an irreducible connection

$$\mathcal{L} = (\mathbb{C}((t))^p, \frac{\mathrm{d}}{\mathrm{d}t} + A) \in \mathrm{LL}^{\circ}_{\mathrm{geom}}(p, j)$$

for $A \in \mathfrak{gl}_p(\mathbb{C}((t)))$. By the Levelt-Turrittin classification it follows that after extending scalars from $\mathbb{C}((t))$ to $\mathbb{C}((t^{1/p}))$ one can gauge transform A via some $g \in \mathfrak{gl}_p(\mathbb{C}((t^{1/p})))$ into a diagonal connection matrix D. The beautiful observation described in much more detail by Schwarz in [23] is that for suitable A's each column $\mathbf{u}$ of g gives rise to a Virasoro constrained KP τ-function in the following manner:

The equation

$$g^{-1}Ag + g^{-1}\frac{\mathrm{d}}{\mathrm{d}t}g = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{bmatrix}$$

implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} + \lambda(z)\mathbf{u} = A\mathbf{u}$$

for some Laurent series $\lambda(z)$. Let u_i denote the i'th component of $\mathbf{u}$ and for $0 \leq i \leq p-1$ let $v_i = z^i u_i$. If $u_i = \text{constant} + \text{lower order terms}$, then the $\mathbb{C}$-subspace of $\mathbb{C}((1/z))$ given by

$$V := \mathrm{span}_{\mathbb{C}} \{z^{pj} v_i \mid 0 \leq i \leq p-1 \text{ and } j \geq 0\}$$

is a point of the big cell Gr. The point V satisfies

$$z^p V \subseteq V$$

$$\left(\frac{1}{pz^{p-1}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} - \sum c_i z^i\right) V \subseteq V$$

where the c_i's can be explicitly described in terms of the d_i's. In this manner it is known that one can construct the point $W_{p,q}$ of the Grassmannian that describe the partition function of the (p, q) model of 2D quantum gravity. Note that whether or not the assumption on the shape of the u_i's holds depends not just on the gauge equivalence class of the connection. There is however a gauge invariant way in which local Langlands parameters are related to the (p, q) models. We now describe this.

By applying the Boson-Fermion correspondence, the above described stabilization property of V translates to Virasoro constraints for the associated KP τ-function, see for example [6]. Note that the two previously described differential operators stabilizing V have commutator equal to 1. In other terminology, they are solutions to the string equation. As a consequence one can attach a D-module on the formal punctured disc to these Virasoro constraints. See [19] for details and in particular for the relation to the previously described way Langlands parameters play a role in the construction of $W_{p,q}$.

The parameters associated to the Virasoro constraints give rise to a well defined gauge equivalence class of connections and this is the second way that local geometric Langlands parameters enter the picture. In fact, one can write down the parameter that is associated to the (p, q) model of 2D quantum gravity. It is given by

$$\mathcal{L}_{p,q} := [p]_* \left((\mathbb{C}((z)), \frac{\mathrm{d}}{\mathrm{d}z} + z^{p-1} \left(\frac{1-p}{2}\frac{1}{z^p} - \sum_{i=1}^{p+q} i t_i z^{i-p} \right)) \right).$$

The gauge equivalence class of the connection $\mathcal{L}_{p,q}$ carries enough information about the (p, q) model of 2D quantum gravity to allow a description of the T-duality in terms of such local geometric Langlands parameters.

4 Duality: Symmetries of Langlands parameters

In the previous section it was explained how the fourth pillar of Weil's Rosetta stone, quantum physics, enters the picture through Virasoro constrained τ-functions of the KP hierarchy. We now explain that from this point of view, the numerical local Langlands correspondence does indeed correspond to a quantum duality, namely to the well studied T-duality of 2D quantum gravity. We refer to [5] for a detailed description of this quantum field theory and we focus on a description of the T-duality.

The starting point is the matter content of the theory before gravity is introduced into the picture. This matter content is chosen to be a minimal model conformal field theory. It has central charge

$$c(p,q) = 1 - 6\frac{(p-q)^2}{pq}$$

for positive co-prime integers p and q. Note that under this constraint on p and q the value of $c(p,q)$ determines p and q up to exchanging p and q. It is known that there is an associated rational conformal field theory. To introduce gravity one couples this (p,q) minimal model to Liouville gravity.

To explain the duality there is now a crucial observation. Let Σ be a surface with a metric g^{ij} of scalar curvature R. The un-renormalized Liouville action functional is given by

$$\phi \mapsto \int_\Sigma \sqrt{g}\left(\frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi + R\phi + \exp(\mu\phi)\right)$$

for a suitable μ. In contrast, the renormalized Liouville action functional is given by

$$\phi \mapsto \frac{1}{4\pi}\int_\Sigma \sqrt{g}\left(g^{ij}\partial_i\phi\,\partial_j\phi + QR\phi + 4\pi\mu\exp(2b\phi)\right)$$

where b is the coupling constant and the background charge Q satisfies $Q = b + b^{-1}$ and is related to the central charge c_L of the Liouville theory by

$$c_L = 1 + 6Q^2.$$

These quantities, and in particular the value of b, can be related to the central charge $c(p,q)$ of the matter content by an anomaly cancellation calculation and one obtains

$$c_L + c(p,q) = 26.$$

Hence

$$Q = (4 + \frac{(p-q)^2}{pq})^{1/2}.$$

This means that the coupling constant b satisfies the quadratic equation

$$b^2 - \frac{p+q}{(pq)^{1/2}}\cdot b + 1 = 0$$

and the two roots are

$$b_1 = \left(\frac{q}{p}\right)^{1/2}$$

$$b_2 = \left(\frac{p}{q}\right)^{1/2}.$$

The duality of the (p,q) minimal model coupled to gravity can be interpreted as an invariance of the theory with respect of choosing either one of the two values of the coupling constant. Note however that the resulting theories are not identical but rather dual, meaning that one can be expressed in terms of the other one. There is hence an expected duality

$$b \longmapsto b^{-1}$$

that corresponds to the switch of p and q and hence is sometimes called the p-q duality. Due to the way the coupling constant enters the Lagrangian this is also called the T-duality of 2D quantum gravity, see [3].

In the discretized framework one can rigorously prove this duality, as was done first by Fukuma-Kawai-Nakayama [5] and Kharchev-Marshakov [12] in the early 90's.

To make the relation with the numerical local Langlands duality, one has to recast the initial formulations, both on the arithmetic and the quantum side: Koch's direct approach to the Langlands duality had to be modified by Laumon by introducing the local ℓ-adic Fourier transforms. On the physics side, the previous approaches had to be modified in order to phrase the duality in terms of D-modules, as was done by Schwarz and the author in [18], [19]. To put it differently, in order to be able to switch from the arithmetic duality to the quantum duality one should look at the underlying reason that the arithmetic numerical local Langlands correspondence holds: It comes from the symmetries of local Langlands parameters coming from the local Fourier transform. This transform has complex and ℓ-adic incarnations and this gives the bridge between the dualities.

For every point x of $\mathbb{P}^1$ there exists an arithmetic local Fourier transform $\mathcal{F}^{(x,\infty)}_{\text{arith}}$ and a geometric version $\mathcal{F}^{(x,\infty)}_{\text{geom}}$. On the geometric side, these transforms relate connections on various formal punctured discs on the Riemann sphere. For example, the geometric $\mathcal{F}^{(\infty,\infty)}$ transform is described in the following picture:

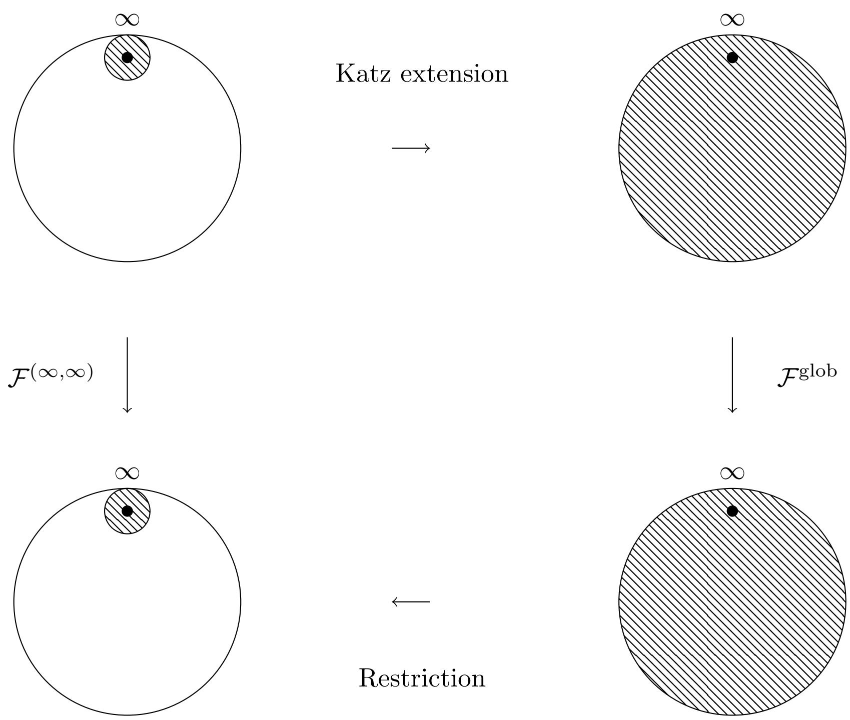


Figure 1: The local Fourier transform $\mathcal{F}^{(\infty,\infty)}$ and its relation to the global Fourier transform $\mathcal{F}^{\text{glob}}$ of D-modules on the plane $\mathbb{P}^1\backslash\{\infty\}$.

In the following statements we suppress the subscripts for local Langlands parameters indicating whether they are arithmetic or geometric. The reason is that one obtains the same statement in both cases: The $\mathcal{F}^{(0,\infty)}$ local Fourier transform gives rise to a map

$$\text{LL}(n, j) \longrightarrow \text{LL}(j + n, j)$$

and for $j > n$ the $\mathcal{F}^{(\infty,\infty)}$ local Fourier transform gives rise to a map

$$\text{LL}(n, j) \longrightarrow \text{LL}(j - n, j).$$

See [15] (Théorème 2.4.3) for the arithmetic case and [2] (Proposition 3.14) for the geometric case. These symmetries are very powerful:

On the arithmetic side, Henniart [10] used them to prove the numerical local Langlands correspondence for $\mathrm{GL}_n(K)$ where K is a non-archimedean local field. We outline his arguments: First he reduces to the case

$$\mathrm{char}(K) = p > 0$$

and

$$n = p^r$$

with r bigger than the power of p dividing j. In this case the properties of the local Fourier transform quickly lead to the desired size count of $c(n,j)_{\mathrm{arith}}$. Namely, one can show that the properties of the transform $\mathcal{F}^{(0,\infty)}$ imply the symmetry

$$c(n,j)_{\mathrm{arith}} - c(n,j-1)_{\mathrm{arith}} = c(n+j,j)_{\mathrm{arith}} - c(n+j,j-1)_{\mathrm{arith}}.$$

A double induction on j and the power of p dividing n then shows that

$$c(n,j)_{\mathrm{arith}} = (q-1)(q^j - q^{j-1} + q^{j-1}) = (q-1)q^j,$$

as desired. This amazingly concise argument should be contrasted with Koch's [13] laborious explicit computations that could only treat the case where the residue characteristic p divides n at most linearly. Henniart's approach via the Fourier transform is completely different: The transform gives rise to a hidden symmetry of local Langlands parameters that allows to count them without having an explicit handle on them. One can sum up Henniart's result as follows: For p a positive integer and $q \geq -p$, the map

$$\mathcal{F}^{(0,\infty)}_{\mathrm{arith}} : \mathrm{LL}_{\mathrm{arith}}(p, p+q) \longrightarrow \mathrm{LL}_{\mathrm{arith}}(2p+q, p+q)$$

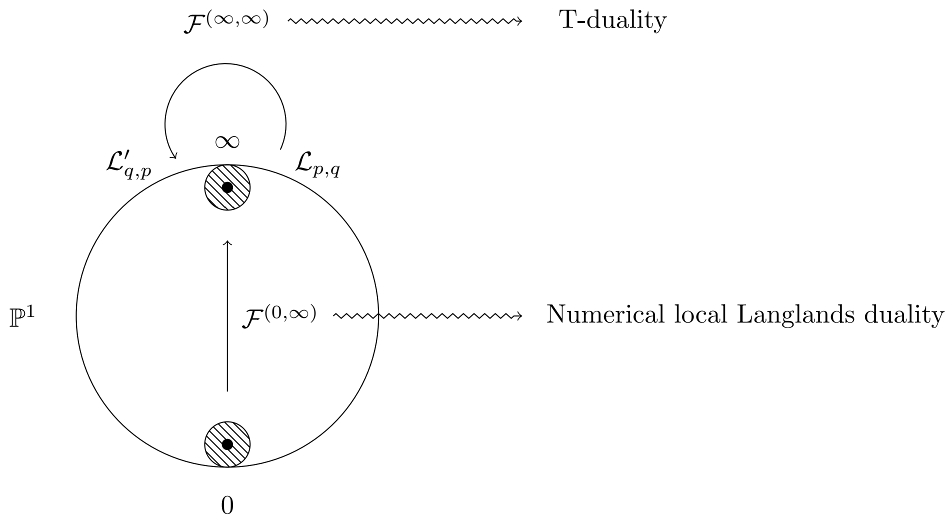
of arithmetic Langlands parameters gives rise to the arithmetic numerical local Langlands duality.

Concerning the T-duality, it follows from our work [18], [19] that when translated to the geometric setting one obtains the duality of 2D quantum gravity. Namely, for p and q positive co-prime integers, the (p,q) model of 2D quantum gravity can be described via the local geometric Langlands parameter $\mathcal{L}_{p,q}$ described earlier in such a way that the map

$$\mathcal{F}^{(\infty,\infty)}_{\mathrm{geom}} : \mathrm{LL}_{\mathrm{geom}}(p, p+q) \longrightarrow \mathrm{LL}_{\mathrm{geom}}(q, p+q)$$

realizes the T-duality of 2D quantum gravity. We refer to [19] for a detailed statement involving control of the dynamics of the KP flows. In particular, $\mathcal{L}_{p,q}$ does not get mapped to $\mathcal{L}_{q,p}$ but to a variant $\mathcal{L}'_{q,p}$ whose definition involves some non-trivial KP time dynamics.

A unified treatment of the above arithmetic and quantum dualities emerges. The local Fourier transforms, in their ℓ-adic and complex version, are a source of symmetries of local Langlands parameters. In the arithmetic setting these symmetries imply the numerical local Langlands duality and in the geometric setting they allow to describe the T-duality:



In this sense, the two dualities are simply specific incarnations of the same symmetry principle.

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