The Toda-Weyl mass spectrum

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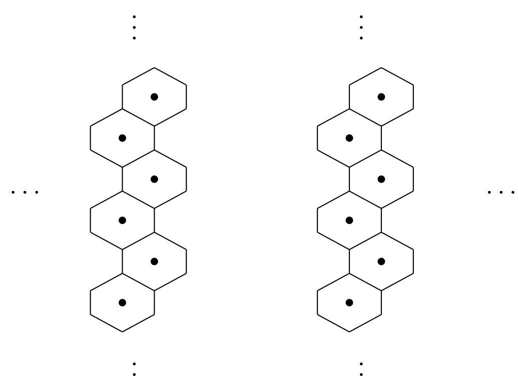
Abstract

The masses of affine Toda theories are known to correspond to the entries of a Perron-Frobenius eigenvector of the relevant Cartan matrix. The Lagrangian of the theory can be expressed in terms of a suitable eigenvector of a Coxeter element in the Weyl group. We generalize this set-up by formulating Lagrangians based on eigenvectors of arbitrary elements in the Weyl group. Under some technical conditions (that hold for many Weyl group elements), we calculate the classical mass spectrum. In particular, we indicate the relation to the relative geometry of special roots, generalizing the affine Toda mass spectrum description in terms of the Cartan matrix. Related questions of three point coupling and integrability are left to be addressed on a future occasion.

1 Introduction

In low temperatures, below around 2.95 K, Cobalt-Niobate $CoNb_2O_6$ is in a magnetically ordered phase, see [20]. The magnetic structure is dominated by the Cobalt ions Co^{2+} since these possess the only unpaired electrons, having 7 electrons in the $3d$ orbital. The solid exhibits quasi-1D behavior due to the chain geometry in which the Cobalt ions assemble. Describing the CoO_6 environment, the oxygen atoms assemble in a slightly deformed hexagonal octahedron around zig-zag chains of Cobalt ions. Schematically:



There is a magnetically preferred easy axis at an angle of around $\pm 30°$ with the Cobalt chain direction, see [15]. As a consequence, Cobalt-Niobate is well modeled by the 1D quantum Ising model. The situation becomes very interesting with an externally applied magnetic field orthogonal to all the Cobalt chain easy axes. For suitable field strength of the external field, the 1D transverse field quantum Ising model has a quantum phase transition and is described by a central charge 1/2 minimal model conformal field theory. From a Lie algebraic perspective, this conformal field theory can be described, see [16], as a Toda theory:

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra of rank r, with a real form $\mathfrak{g}^{*=1}$ of "Hermitian operators", we recall the definition in Section (1.1). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and consider a field $\phi : \mathbb{R}^2 \to \mathfrak{h} \cap \mathfrak{g}^{*=1}$. Let R be a subset of the roots. Let $(-,-)$ denote the Killing form and define the Lagrangian

$$\mathrm{L}_R = \frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) - \sum_{\alpha_i \in R} \exp(\alpha_i \cdot \phi)$$

If $R = \{\alpha_1, \cdots, \alpha_r\}$ is chosen to be a set of simple roots, one obtains the usual conformal and massless Toda theory (we have set all coupling constants to 1, for simplicity). Its quantization, as carried out in [16], describes for $\mathfrak{g} = \mathfrak{sl}_2$

and $\mathfrak{g} = \mathfrak{e}_8$ the previously mentioned minimal model.

The actual magnetic structure of $CoNb_2O_6$ incorporates a non-zero, yet small, interchain coupling. One obtains a Hamiltonian with an additional term corresponding to magnetization along the easy axis. This magnetic deformation of the transverse field Ising model has a beautiful Lie algebraic formulation on the level of Toda theory: Replace the set R of simple roots by $R_{\text{new}} = R \cup \{\alpha_0\}$ for the lowest root α_0. The resulting affine Toda theory acquires a fascinating mass spectrum, that is known to correspond to the entries of a Perron-Frobenius eigenvector of the Cartan matrix of $\mathfrak{g}$, see [12]. Since in the case $\mathfrak{g} = \mathfrak{e}_8$ affine Toda theory describes the quantum phase transition of $CoNb_2O_6$ in the presence of a small interchain coupling, this confirms the mass spectrum predicted in groundbreaking work by Zamolodchikov [22]. Coldea et al. in [6] were able to confirm experimentally for $CoNb_2O_6$ some of these predictions using neutron diffraction, in particular that the ratio of second lightest to lightest mass is given by the golden ratio. As we are about to recall, affine Toda theory is intimately linked to the Coxeter element of the Weyl group of $\mathfrak{g}$. The aim of the present work is to generalize these mass spectrum calculations for Lagrangians based on much more general Weyl group elements. We show that the classical masses can again be expressed in terms of the relative geometry of a special collection of roots, generalizing the mass expression in terms of the Cartan matrix.

The starting point is Freeman's slight reformulation in [9] of affine Toda theory. For R_{new} as before, rewrite the field as $\phi_{\min} + \phi$ where $\phi_{\min}$ is a minimum. The resulting Lagrangian for the new field ϕ is

$$\frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) - \sum_{\alpha_i \in R_{\text{new}}} n_i \exp(\alpha_i \cdot \phi) \tag{1}$$

where n_0 is normalized to be 1 and the n_i's are such that

$$\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i \tag{2}$$

See for example [3] (Section 2) for details. Using the Killing form, identify roots as elements of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For each root α fix a generator E_α of the corresponding root space with respect to $\mathfrak{h}$, normalized so that

$$[E_\alpha, E_{-\alpha}] = \alpha \in \mathfrak{h}$$

Let

$$\Lambda_+ = \sum_{i=0}^{r} a_i \mathrm{E}_{\alpha_i}$$

$$\Lambda_- = \sum_{i=0}^{r} b_i \mathrm{E}_{-\alpha_i}$$

where $a_i b_i = n_i$ for each i. Since α_0 is the lowest root, $[E_{\alpha_i}, E_{-\alpha_j}] = \delta_{i,j}\alpha_i$ for all $0 \leq i, j \leq r$. Hence, the Lagrangian in Equation (1) can be re-written as

$$\frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) - (\exp(\text{ad } \phi)(\Lambda_+), \Lambda_-) \tag{3}$$

One can show, see the work of Kostant [18] (Section 6), that the centralizer of Λ_+ is a Cartan algebra, denote it by $\mathfrak{h}^{\text{Weyl}}$. It follows from Equation (2) that Λ_- is in $\mathfrak{h}^{\text{Weyl}}$. The Weyl group W of $\mathfrak{g}$ acts on this Cartan algebra and Kostant has shown in loc. cit. that there is a Coxeter element σ_{Coxeter} in W such that

$$\sigma_{\text{Coxeter}}(\Lambda_\pm) = e^{\pm \frac{2\pi i}{h}} \cdot \Lambda_\pm$$

where h is the Coxeter number. In this manner, Equation (3) allows to formulate affine Toda theory in terms of the linear algebra of Coxeter elements. We extend this formalism to more general Weyl group elements and calculate the classical mass spectrum.

For usual affine Toda theory, the masses were calculated in the early 90's by Dorey [8], Freeman [9], and Fring-Liao-Olive [12]: They can beautifully be expressed in terms of the Perron-Frobenius eigenvector of the Cartan matrix of $\mathfrak{g}$. More recently, Brillon-Schechtman in [4] considered the case of the Coxeter element but with Λ_+ replaced by an arbitrary eigenvector. Many other works concern various generalizations of Toda theory, see for example [7], [10], [11], [13], [14].

1.1 The Toda-Weyl Lagrangians

In this section we generalize the affine Toda Lagrangians described by Equation (3). As before, let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra of rank r, and fix a Cartan subalgebra $\mathfrak{h}^{\mathrm{Weyl}}$. Let $(-,-)$ denote the Killing form and identify elements in root space as elements of $\mathfrak{h}^{\mathrm{Weyl}}$ by associating to x in $\mathfrak{h}^{\mathrm{Weyl}}$ the functional $(x,-)$. For every root α consider the endomorphism of $\mathfrak{h}^{\mathrm{Weyl}}$ given by

$$r_\alpha(\beta) = \beta - 2 \cdot \frac{(\alpha,\beta)}{(\alpha,\alpha)} \cdot \alpha$$

The Weyl group W of $\mathfrak{g}$ is the group generated by the r_α's. For σ in W we aim to construct an analogue of the Lagrangian in Equation (3). This requires two things: Define the target space of the field ϕ, and generalize the special elements $\Lambda_\pm$ in the Lie algebra $\mathfrak{g}$.

For each root α choose generators $e_{\pm\alpha}$ of the root space of $\pm\alpha$ (with respect to $\mathfrak{h}^{\mathrm{Weyl}}$), normalized so that $[e_\alpha, e_{-\alpha}] = \alpha$. Let $\mathfrak{a}$ be the real subalgebra of $\mathfrak{g}$ given by the $\mathbb{R}$-span of the following elements, as α ranges through the set of all roots:

(i) α

(ii) $e_\alpha + e_{-\alpha}$

(iii) $i(e_\alpha - e_{-\alpha})$

Consider the involution $* : \mathfrak{g} \to \mathfrak{g}$ discussed in detail by Kostant [18] (Section 6): It is given by

$$* : x + iy \mapsto x - iy$$

where x and y are in $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{g}^{*=1}$ can be considered the space of Hermitian operators, it is a real form of the complex algebra $\mathfrak{g}$. The field ϕ will take values in a real subspace of $\mathfrak{g}^{*=1}$. To define this subspace, we appeal to results by Kac.

Starting with σ and the Cartan algebra $\mathfrak{h}^{\mathrm{Weyl}}$, we recall how to construct a second Cartan algebra $\mathfrak{h}^{\mathrm{Kac}}$ as well as a gradation $\mathfrak{g} = \oplus_k \mathfrak{g}_k$ (if σ is a Coxeter element, the two Cartan algebras $\mathfrak{h}^{\mathrm{Kac}}$ and $\mathfrak{h}^{\mathrm{Weyl}}$ are in apposition, in the terminology of Kostant [18]). Kac showed, see [17] (Theorem 8.6), that there is a Cartan algebra $\mathfrak{h}^{\mathrm{Kac}}$ and a collection of non-negative integers $\mathbf{s} = (s_0, \cdots, s_r)$ with the following properties:

(i) There is a finite-order inner automorphism $\tilde{\sigma}$ of $\mathfrak{g}$ (of order $\tilde{n}$, say) such that

$$\tilde{\sigma}|_{\mathfrak{h}^{\mathrm{Weyl}}} = \sigma$$

$$\tilde{\sigma}|_{\mathfrak{h}^{\mathrm{Kac}}} = \mathrm{id}$$

(ii) Let $\zeta = e^{\frac{2\pi i}{\tilde{n}}}$, let $\alpha_1, \cdots, \alpha_r$ be a set of simple roots, and let α_0 denote the lowest root. For all $j = 0, \cdots, r$

$$\tilde{\sigma}(\mathrm{E}_{\alpha_j}) = \zeta^{s_j} \cdot \mathrm{E}_{\alpha_j}$$

where E_{α_j} generates the α_j root space with respect to $\mathfrak{h}^{\mathrm{Kac}}$.

(iii) The s_j's that are non-zero are co-prime.

The collection of integers $\mathbf{s} = (s_0, \cdots, s_r)$ are called Kac coordinates of σ. In general, there are multiple possible coordinates $\mathbf{s}$ associated to a given Weyl group element. In some situations however, for example if σ lies in a regular conjugacy class, they are uniquely defined. See for example [19] for more details. Associate to $\tilde{\sigma}$ a $\mathbb{Z}/\tilde{n}\mathbb{Z}$-gradation on

$\mathfrak{g}$, by letting $\mathfrak{g}_k$ denote the $\tilde{\sigma}$-eigenspace with eigenvalue $e^{2\pi ik/\tilde{n}}$. The space $\mathfrak{g}_0$ then contains $\mathfrak{h}^{\mathrm{Kac}}$ but is in general larger. We can now define the target space of the field ϕ: We assume

$$\phi : \mathbb{R}^2 \longrightarrow \mathfrak{g}_0 \cap \mathfrak{g}^{*=1}$$

Furthermore, let from now on Λ_+ in $\mathfrak{h}^{\mathrm{Weyl}}$ denote an eigenvector of σ with eigenvalue μ, say, and let $\Lambda_- = *(\Lambda_+)$. It follows from the definition of $*$ that it maps $\mathfrak{h}^{\mathrm{Weyl}}$ to itself. Hence Λ_- is again in $\mathfrak{h}^{\mathrm{Weyl}}$ and is in fact an eigenvector of σ with complex conjugate eigenvalue compared to Λ_+. To summarize the situation:

$$\begin{array}{ccccccc}
\mathbb{R}^2 & \xrightarrow{\phi} & \mathfrak{g}_0 \cap \mathfrak{g}^{*=1} & \hookrightarrow & \mathfrak{g}_0 & & \sigma(\Lambda_\pm) = \mu^{\pm 1}\Lambda_\pm \\
 & & & & \uparrow & & \\
 & & & & \mathfrak{h}^{\mathrm{Kac}} & \underset{\text{generalized apposition}}{\longleftrightarrow} & \mathfrak{h}^{\mathrm{Weyl}} \supset \{\Lambda_+, \Lambda_-\}
\end{array}$$

Definition 1. For ϕ, Λ_+, and Λ_- as above, define the Lagrangian exactly as in Equation (3) by

$$\mathrm{L}_{\text{Toda-Weyl}} = \frac{1}{2}(\partial_\mu \phi, \partial^\mu \phi) - (\exp(\mathrm{ad}\ \phi)(\Lambda_+), \Lambda_-)$$

We call this the Toda-Weyl theory.

Suppose σ is a Coxeter element and Λ_+ has eigenvalue $e^{2\pi i/h}$. The Kac coordinates are known to be $(1, \cdots, 1)$ and hence $\mathfrak{g}_0 = \mathfrak{h}_{\mathrm{Kac}}$. If the Cartan algebra $\mathfrak{h}$ in Section 1 is identified as $\mathfrak{h}_{\mathrm{Kac}}$, the Toda-Weyl theory recovers affine Toda theory.

2 Mass calculations

For $1 \leq i \leq s$ consider complex-valued scalar fields $\phi_i : \mathbb{R}^2 \to \mathbb{C}$. Let $\phi = (\phi_1, \cdots, \phi_s)^{\mathrm{T}}$ and consider a Lagrangian density

$$\mathrm{L} = \frac{1}{2} \cdot \partial_\mu \overline{\phi}^{\mathrm{T}} A \partial_\mu \phi + B\phi + \overline{\phi}^{\mathrm{T}} C \phi + \cdots$$

where A and C are $s \times s$ matrices and B is of the form $(b_1, \cdots, b_s)$. Under a field redefinition $\phi \mapsto X\phi$ (with X invertible) one obtains

$$\mathrm{L} \mapsto \mathrm{L}_X = \frac{1}{2} \cdot \partial_\mu \overline{\phi}^{\mathrm{T}} (\overline{X}^{\mathrm{T}} A X) \partial_\mu \phi + BX\phi + \overline{\phi}^{\mathrm{T}} (\overline{X}^{\mathrm{T}} C X) \phi + \cdots$$

Suppose that for some X

$$\mathrm{L}_X = \frac{1}{2} \cdot \partial_\mu \overline{\phi}^{\mathrm{T}}\, \partial_\mu \phi - \frac{1}{2} \cdot \overline{\phi}^{\mathrm{T}} \begin{bmatrix} m_1^2 & & & \\ & m_2^2 & & \\ & & \ddots & \\ & & & m_s^2 \end{bmatrix} \phi + \cdots \tag{4}$$

where the m_i's are non-negative real numbers. The masses of the Lagrangian are then by definition $m_1, \cdots, m_s$. If the masses exist, they are well defined: If Y is another field redefinition such that L_Y is as in Equation (4), then $Y = XU$ for a unitary matrix U. Hence the spectrum of $\overline{X}^{\mathrm{T}} CX$ agrees with the spectrum of $\overline{Y}^{\mathrm{T}} CY$. Furthermore, the linear ϕ term in L_X vanishes if and only if $B = 0$ and hence vanishes in L_Y as well.

In the remainder of this work we calculate the masses of the Toda-Weyl Lagrangians. The key idea is to construct a basis of $\mathfrak{g}_0$ in terms of the root space decomposition with respect to $\mathfrak{h}^{\mathrm{Weyl}}$ (for Coxeter elements such a description goes back to work of Kostant [18]). This allows a description of the masses in terms of pairings $\Lambda_+ \cdot \alpha := \alpha(\Lambda_+)$ for suitable roots α, viewed as functionals on $\mathfrak{h}^{\mathrm{Weyl}}$.

Theorem 1. *Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra with a Cartan subalgebra $\mathfrak{h}^{\mathrm{Weyl}}$. Let σ be an element in the Weyl group of $\mathfrak{g}$ such that:*

(i) *1 is not an eigenvalue of σ acting on $\mathfrak{h}^{\mathrm{Weyl}}$.*

(ii) *There is an inner automorphism* $\tilde\sigma$ *of* $\mathfrak{g}$ *such that* $\tilde\sigma|_{\mathfrak{h}^{\mathrm{Weyl}}} = \sigma$ *and* $\operatorname{ord}\tilde\sigma = \operatorname{ord}\sigma$.

Let $\Lambda_+ \in \mathfrak{h}^{\mathrm{Weyl}}$ *be an eigenvector of* σ *and let* $\Lambda_- = *(\Lambda_+)$. *Let* $\gamma_1, \cdots, \gamma_s$ *denote orbit representatives for the action of the cyclic group* $\langle\sigma\rangle$ *on the set of roots. Then the masses of*

$$\mathrm{L} = \frac{1}{2}(\partial_\mu\phi, \partial^\mu\phi) - (\exp(\mathrm{ad}\ \phi)(\Lambda_+), \Lambda_-)$$

are given by

$$m_i = |\Lambda_+ \cdot \gamma_i|$$

Proof. Let us show that the absolute values $|\Lambda_+ \cdot \gamma_i|$ are independent of the choice of orbit representatives: Let $\tilde\Lambda_+ = (\Lambda_+, -)$ be the element in root space associated to Λ_+. Recall that Λ_+ is an eigenvector of σ, denote the corresponding eigenvalue μ, a suitable root of unity. It follows that for all j

$$\begin{aligned} |\Lambda_+ \cdot \sigma^j\gamma_i| &= |(\sigma^j\gamma_i)(\Lambda_+)| \\ &= |(\sigma^j\gamma_i, \tilde\Lambda_+)| \\ &= |(\gamma_i, \sigma^{-j}\tilde\Lambda_+)| \\ &= |\mu^{-j}(\gamma_i, \tilde\Lambda_+)| \\ &= |\Lambda_+ \cdot \gamma_i| \end{aligned}$$

where we have used that the inner product in root space is invariant under the Weyl group action.

Let $\tilde\sigma$ be a finite-order inner automorphism lifting σ. For each root α let e_α be a generator of the root space with respect to $\mathfrak{h}_{\mathrm{Weyl}}$. Then $\tilde\sigma e_\alpha$ is a generator of the root space of $\sigma(\alpha)$. By the assumptions of the theorem, we can assume that $\tilde\sigma$ is of the same order as σ. Hence, one can choose the generators of the root spaces such that for each root α

$$\tilde\sigma e_\alpha = e_{\sigma(\alpha)} \tag{5}$$

For each orbit $\mathcal{O}_i$ pick a representative γ_i and define

$$A_i := \frac{1}{\sqrt{|\mathcal{O}_i|}} \cdot \sum_{j=0}^{|\mathcal{O}_i|-1} e_{\sigma^j(\gamma_i)}$$

From Equation (5) it follows that A_i is fixed by $\tilde\sigma$ and hence lies in $\mathfrak{g}_0$. Since the root space generators are linearly independent, so are the A_i's. We have seen that $\tilde\sigma$ permutes the various root spaces, and since $\tilde\sigma|_{\mathfrak{h}^{\mathrm{Weyl}}} = \sigma$ it follows that $\mathfrak{g}_0$ has a basis given by the A_i's together with a basis of $(\mathfrak{h}^{\mathrm{Weyl}})^{\sigma=1}$. By our assumption on σ, $(\mathfrak{h}^{\mathrm{Weyl}})^{\sigma=1} = 0$ and therefore the A_i's are in fact a basis of $\mathfrak{g}_0$. We now show that in this basis one can read off the masses of the Toda-Weyl Lagrangian. Write

$$\phi = \sum_{i=1}^{s} \phi_i A_i$$

Since $(e_\alpha, e_\beta) = 0$ unless $\alpha + \beta = 0$, one can deduce that

$$(A_i, A_j) = \delta_{i,\pi(j)} \tag{6}$$

where π is a permutation of the indices such that $-\gamma_j$ is in $A_{\pi(j)}$ for all j. By [4] (Theorem 2.4)

$$[[A_j, \Lambda_+], \Lambda_-] \quad = \quad (\Lambda_+ \cdot \gamma_j) \cdot (\Lambda_- \cdot \gamma_j) \cdot A_j$$

Note that by [18] (Equation 6.1.1), for every complex scalar c

$$(c \cdot e_\alpha)^* = \overline{c} \cdot e_{-\alpha} \tag{7}$$

It follows from the construction of $*$ that for all x, y in $\mathfrak{g}$

$$(*(x), *(y)) = \overline{(x, y)}$$

In particular:

$$\begin{aligned}\Lambda_- \cdot \gamma_j &= *(\Lambda_+) \cdot *(\gamma_j) \\ &= \overline{\Lambda_+ \cdot \gamma_j}\end{aligned}$$

Therefore

$$\begin{aligned}([A_i, [A_j, \Lambda_+]], \Lambda_-) &= (A_i, [[A_j, \Lambda_+], \Lambda_-]) \\ &= (\Lambda_+ \cdot \gamma_j) \cdot (\Lambda_- \cdot \gamma_j)(A_i, A_j) \\ &= |\Lambda_+ \cdot \gamma_i|^2 \cdot \delta_{i,\pi(j)}\end{aligned}$$

Together with Equation (6), this implies that the Toda-Weyl Lagrangian density is given by

$$\begin{aligned}\mathrm{L} &= \frac{1}{2} \cdot \sum_{i=1}^{s} \partial_\mu \phi_i \partial^\mu \phi_{\pi(i)} - (\Lambda_+, \Lambda_-) - \sum_{i=1}^{s} \phi_i([A_i, \Lambda_+], \Lambda_-) - \frac{1}{2!} \sum_{i,j} \phi_i \phi_j ([A_i, [A_j, \Lambda_+]], \Lambda_-) + \text{higher order terms} \\ &= \frac{1}{2} \cdot \sum_{i} \partial_\mu \phi_i \partial^\mu \phi_{\pi(i)} - (\Lambda_+, \Lambda_-) - \sum_{i=1}^{s} \phi_i(A_i, [\Lambda_+, \Lambda_-]) - \frac{1}{2!} \sum_{i,j} \phi_i \phi_j |\Lambda_+ \cdot \gamma_i|^2 \cdot \delta_{i,\pi(j)} + \text{higher order terms}\end{aligned}$$

To simplify further, note that by construction the image of ϕ is contained in the space fixed by $*$. Hence

$$\begin{aligned}\sum_{i=1}^{s} \phi_i A_i &= * \left(\sum_{i=1}^{s} \phi_i A_i \right) \\ &= \sum_{i=1}^{s} \overline{\phi_i} A_{\pi(i)}\end{aligned}$$

and therefore

$$\phi_{\pi(i)} = \overline{\phi_i}$$

for all i. Furthermore $[\Lambda_+, \Lambda_-] = 0$. Therefore, up to the constant (Λ_+, Λ_-), the Lagrangian density is given by

$$\mathrm{L} = \frac{1}{2} \cdot \sum_{i} \partial_\mu \phi_i \partial^\mu \overline{\phi}_i - \frac{1}{2} \sum_{i} |\phi_i|^2 |\Lambda_+ \cdot \gamma_i|^2 + \text{ higher order terms}$$

This implies the theorem. □

Remark 1. The number of Toda-Weyl masses often has a very simple expression. Suppose for example σ is a regular Weyl group element such that 1 is not an eigenvalue. As discussed by Reeder in [19] (Proposition 2.2), since σ is regular there indeed exists a lift $\tilde{\sigma}$ of the same order as σ. Hence the conditions of Theorem 1 are satisfied. Since σ is regular, it follows from the work of Springer [21] (Proposition 4.1) that every orbit has exactly $n = \text{ord } \sigma$ elements. The total number of roots is known to be hr where h is the Coxeter number and r is the rank of $\mathfrak{g}$. It follows that the number s of orbits of σ satisfies

$$\text{ord } \sigma \cdot s = h \cdot \text{rank } \mathfrak{g}$$

In particular, the number of Toda-Weyl masses is given by

$$s = \frac{h \cdot \text{rank } \mathfrak{g}}{\text{ord } \sigma}$$

In Section 2.1 and Section 2.2 we give two illustrative examples of how to use Theorem 1 to obtain the precise mass spectrum.

2.1 Example I

Let $\mathfrak{g} = \mathfrak{e}_6$, with simple roots $\alpha_1, \cdots, \alpha_6$ indexed as in [1]. Consider the Weyl group element

$$\sigma = r_{\alpha_1} r_{\alpha_2} r_{\alpha_5} r_{\alpha_6} r_{\alpha_2+\alpha_4} r_{\alpha_3+\alpha_4}$$

Since the 6 roots involved in the definition of σ are a basis of root space, it follows from work of Carter [5] (Lemma 3) that 1 is not an eigenvalue of σ. Furthermore, one can calculate that the eigenvalues are distinct: For $\zeta_9 = e^{2\pi i/9}$, they are $\zeta_9, \zeta_9^2, \zeta_9^4, \zeta_9^5, \zeta_9^7, \zeta_9^8$, see for example [2] (Table 1). Hence, by work of Springer [21] (Lemma 4.11), the element σ is regular. By Remark 1 it follows that the conditions of Theorem 1 are satisfied. We apply the theorem with Λ_+ an eigenvector with eigenvalue ζ_9. Since ζ_9 has multiplicity 1, up to an overall scaling, the masses are independent of Λ_+.

Let $\zeta = e^{2\pi i/36}$ be a primitive 36'th root of unity. It has minimal polynomial over $\mathbb{Q}$ given by $x^{12} - x^6 + 1$. We claim that one can take Λ_+ as

$$\zeta\alpha_1 + (-\zeta^7 + \zeta^5 + \zeta)\alpha_2 + (-\zeta^9 + \zeta^3 + \zeta)\alpha_3 + (-\zeta^{11} - \zeta^7 + \zeta^5 + \zeta^3 + \zeta)\alpha_4 + (-\zeta^{11} + \zeta^5 + \zeta^3)\alpha_5 + (-\zeta^{11} + \zeta^5)\alpha_6 \tag{8}$$

To show this, note that the 8 orbits $\mathcal{O}_1, \cdots, \mathcal{O}_8$ of the cyclic group $\langle\sigma\rangle$ acting on the set of roots can be calculated easily:

$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$
$-\alpha_6$	$-\alpha_2-\alpha_4$	$\alpha_3-\alpha_4$	α_5
$\alpha_5+\alpha_6$	$\alpha_4+\alpha_5$	$\alpha_1+\cdots+\alpha_5$	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$
$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5$	$\alpha_2+\alpha_4+\alpha_5+\alpha_6$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$
$\alpha_2+\cdots+\alpha_6$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$
$\alpha_1+\alpha_3+\alpha_4$	α_3	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6$	$\alpha_3+\alpha_4$
$-\alpha_1$	$-\alpha_2$	$-\alpha_5$	$-\alpha_1-\cdots-\alpha_5$
$-\alpha_2-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_1-\alpha_3$	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6$	$-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-2\alpha_5-\alpha_6$
$-\alpha_1-\alpha_3-\alpha_4-\alpha_5-\alpha_6$	$-\alpha_3-\alpha_4-\alpha_5$	$-\alpha_1-2\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6$	$-\alpha_1-\alpha_2-2\alpha_3-3\alpha_4-2\alpha_5-\alpha_6$
$-\alpha_2-\alpha_3-2\alpha_4-\alpha_5$	$-\alpha_4-\alpha_5-\alpha_6$	$-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5-\alpha_6$	$-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6$

$\mathcal{O}_5$	$\mathcal{O}_6$	$\mathcal{O}_7$	$\mathcal{O}_8$
α_2	α_1	α_4	$\alpha_1+\alpha_3+\alpha_4+\alpha_5$
$\alpha_1+\alpha_3$	$\alpha_2+\cdots+\alpha_5$	$-\alpha_1-\alpha_3-\alpha_4-\alpha_5$	$\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$
$\alpha_3+\alpha_4+\alpha_5$	$\alpha_1+\alpha_3+\cdots+\alpha_6$	$-\alpha_2-\alpha_3-2\alpha_4-2\alpha_5-\alpha_6$	$\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6$
$\alpha_4+\alpha_5+\alpha_6$	$\alpha_2+\alpha_3+2\alpha_4+\alpha_5$	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5-\alpha_6$	$\alpha_2+\alpha_3+\alpha_4$
$\alpha_2+\alpha_4$	α_6	$-\alpha_2-\alpha_3-\alpha_4$	$-\alpha_2-\alpha_4-\alpha_5$
$-\alpha_4-\alpha_5$	$-\alpha_5-\alpha_6$	$\alpha_2+\alpha_4+\alpha_5$	$-\alpha_1-\cdots-\alpha_6$
$-\alpha_2-\alpha_4-\alpha_5-\alpha_6$	$-\alpha_1-\alpha_2-\alpha_3-2\alpha_4-\alpha_5$	$\alpha_1+\cdots+\alpha_6$	$-\alpha_1-\alpha_2-2\alpha_3-2\alpha_4-\alpha_5$
$-\alpha_1-\alpha_2-\alpha_3-\alpha_4$	$-\alpha_2-\cdots-\alpha_6$	$\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5$	$-\alpha_3-\alpha_4-\alpha_5-\alpha_6$
$-\alpha_3$	$-\alpha_1-\alpha_3-\alpha_4$	$\alpha_3+\alpha_4+\alpha_5+\alpha_6$	$-\alpha_4$

In particular

$$\begin{aligned}
\sigma(\alpha_1) &= \alpha_2+\alpha_3+\alpha_4+\alpha_5\\
\sigma(\alpha_2) &= \alpha_1+\alpha_3\\
\sigma(\alpha_3) &= -\alpha_2\\
\sigma(\alpha_4) &= -\alpha_1-\alpha_3-\alpha_4-\alpha_5\\
\sigma(\alpha_5) &= \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\\
\sigma(\alpha_6) &= -\alpha_5-\alpha_6
\end{aligned}$$

It follows that

$$\sigma(\Lambda_+) = \zeta^4\cdot\Lambda_+ = \zeta_9\cdot\Lambda_+$$

and Λ_+ is indeed the desired eigenvector. Orbit representatives for the σ action on the set of roots are given by

$$\alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ \alpha_4\ ,\ \alpha_5\ ,\ -\alpha_1\ ,\ -\alpha_4\ ,\ -\alpha_5$$

By Theorem 1 one obtains the following 8 masses:

$$\begin{aligned}
|\Lambda_+\cdot\pm\alpha_1| &= |\zeta^9-\zeta^3+\zeta| \approx 0.684\\
|\Lambda_+\cdot\alpha_2| &= |\zeta^{11}-\zeta^7+\zeta^5-\zeta^3+\zeta| \approx 0.446\\
|\Lambda_+\cdot\alpha_3| &= |\zeta^{11}-2\zeta^9+\zeta^7-\zeta^5+\zeta^3| \approx 0.446\\
|\Lambda_+\cdot\pm\alpha_4| &= |-\zeta^{11}+\zeta^9-\zeta^7| \approx 0.879\\
|\Lambda_+\cdot\pm\alpha_5| &= |\zeta^7+\zeta^3-\zeta| \approx 1.286
\end{aligned}$$

Let us normalize the masses so that the lowest mass equals 1 and let us index these normalized masses as

$$m_1\le\cdots\le m_8$$

In the following calculations we repeatedly exploit that $\zeta^{12}-\zeta^6+1=0$. One has

$$\left|\frac{\zeta^{11}-\zeta^7+\zeta^5-\zeta^3+\zeta}{\zeta^{11}-2\zeta^9+\zeta^7-\zeta^5+\zeta^3}\right| = \left|\zeta^8-\zeta^2\right|$$
$$= \left|\zeta^{13}\cdot(\zeta^8-\zeta^2)\right|$$
$$= \left|-\zeta^9\right|$$
$$= 1$$

Hence $m_1 = m_2 = 1$. Furthermore

$$m_3 = m_4 = \left|\frac{\zeta^9-\zeta^3+\zeta}{\zeta^{11}-\zeta^7+\zeta^5-\zeta^3+\zeta}\right|$$
$$= \left|\zeta^4+\zeta^{-4}\right|$$
$$= 2\cos\left(\frac{2\pi}{9}\right)$$

$$m_5 = m_6 = \left|\frac{-\zeta^{11}+\zeta^9-\zeta^7}{\zeta^{11}-\zeta^7+\zeta^5-\zeta^3+\zeta}\right|$$
$$= \left|-\zeta^2-1\right|$$
$$= \left|-\zeta^{17}\cdot(\zeta^2+1)\right|$$
$$= \left|\zeta^1+\zeta^{-1}\right|$$
$$= 2\cos\left(\frac{\pi}{18}\right)$$

$$m_7 = m_8 = \left|\frac{\zeta^7+\zeta^3-\zeta}{\zeta^{11}-\zeta^7+\zeta^5-\zeta^3+\zeta}\right|$$
$$= \left|-\zeta^{11}+\zeta^5+\zeta^3+\zeta\right|$$
$$= \left|\zeta\cdot(\zeta^2+\zeta^{-2})\cdot(\zeta^4+\zeta^{-4})\right|$$
$$= 4\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)$$

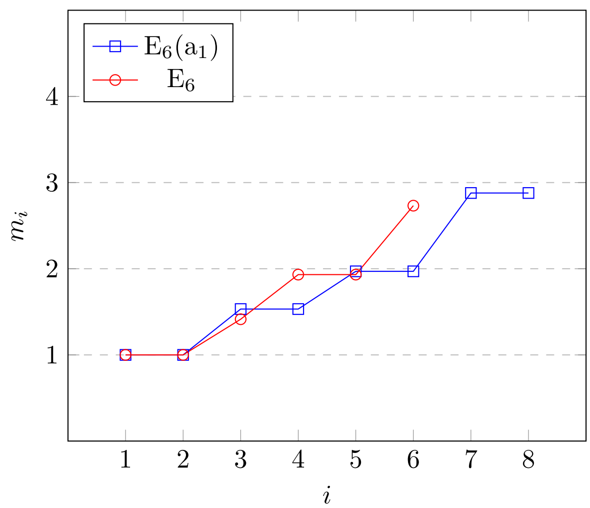
Recall that if σ_{Coxeter} is a Coxeter element, the mass spectrum corresponds to the Perron-Frobenius eigenvector of the Cartan matrix. In particular, the mass spectrum has elegant trigonometric expressions. We have shown that similar formulas hold for our choice of σ:

	σ_{Coxeter}	σ
order	12	9
m_1	1	1
m_2	1	1
m_3	$2\cos\left(\frac{3\pi}{12}\right)$	$2\cos(\frac{2\pi}{9})$
m_4	$4\cos(\frac{\pi}{12})\cos(\frac{4\pi}{12})$	$2\cos(\frac{2\pi}{9})$
m_5	$4\cos(\frac{\pi}{12})\cos(\frac{4\pi}{12})$	$2\cos(\frac{\pi}{18})$
m_6	$4\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{3\pi}{12}\right)$	$2\cos(\frac{\pi}{18})$
m_7		$4\cos(\frac{\pi}{9})\cos(\frac{2\pi}{9})$
m_8		$4\cos(\frac{\pi}{9})\cos(\frac{2\pi}{9})$

In Table 1 we plot the normalized masses for the Coxeter case, denoted by E_6, as well as for σ, denoted by $\mathrm{E}_6(\mathrm{a}_1)$ (since this is the conjugacy class of σ in the notation of [5]).

Table 1:

Mass ratios



E_6	$\mathrm{E}_6(a_1)$
1	1
1	1
1.414...	1.532...
1.932...	1.532...
1.932...	1.970...
2.732...	1.970...
	2.879...
	2.879...

2.2 Example II

Let $\mathfrak{g} = \mathfrak{f}_4$ and let $\alpha_1, \cdots, \alpha_4$ be simple roots, indexed as in [1]. Consider the Weyl group element

$$\sigma = r_{\alpha_1} r_{\alpha_3+\alpha_4} r_{\alpha_2} r_{\alpha_1+\alpha_2+\alpha_3}$$

It is of order 6 and by the same argument as in Example I one sees that 1 is not an eigenvalue of σ. For $\zeta_6 = e^{2\pi i/6}$ the eigenvalues are

$$\zeta_6, \zeta_6, \zeta_6^5, \zeta_6^5$$

See [2] (Table 1). The check for regularity is different than in Example I since both eigenvalues occur with multiplicity bigger than 1. One approach is to calculate explicitly an eigenvector that is not orthogonal to any root, along the lines of the calculations in the current section. Instead, we show in Section 2.3 that σ lies in the conjugacy class $\mathrm{F}_4(a_1)$, which is known to be regular. Either way, the conditions of Theorem 1 are satisfied.

Let $\zeta = \zeta_{24} = e^{2\pi i/24}$ be a primitive 24'th root of unity, its minimal polynomial over $\mathbb{Q}$ is $x^8 - x^4 + 1$. We claim that

$$\Lambda_+ = \zeta\alpha_1 + (\zeta + \zeta^{-3})\alpha_2 + 2\zeta\alpha_3 + 2\zeta\alpha_4 \tag{9}$$

is an eigenvector of σ with eigenvalue ζ_6. The 8 orbits $\mathcal{O}_1, \cdots, \mathcal{O}_8$ of the action of the cyclic group $\langle\sigma\rangle$ on the set of roots are as follows:

$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$
α_1	α_2	α_3	α_4
$-\alpha_2 - 2\alpha_3$	$-\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4$	$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$	$\alpha_2 + \alpha_3 + \alpha_4$
$-\alpha_1 - \alpha_2 - 2\alpha_3$	$-\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\alpha_2 + \alpha_3$
$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$
$\alpha_2 + 2\alpha_3$	$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$	$-\alpha_1 - \alpha_2 - 2\alpha_3 - \alpha_4$	$-\alpha_2 - \alpha_3 - \alpha_4$
$\alpha_1 + \alpha_2 + 2\alpha_3$	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$	$-\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$	$-\alpha_2 - \alpha_3$

$\mathcal{O}_5$	$\mathcal{O}_6$	$\mathcal{O}_7$	$\mathcal{O}_8$
$\alpha_3 + \alpha_4$	$\alpha_1 + \alpha_2 + \alpha_3$	$\alpha_1 + \alpha_2$	$\alpha_1 + 2\alpha_2 + 2\alpha_3$
$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$	$-\alpha_2 - 2\alpha_3 - \alpha_4$	$-\alpha_1 - 2\alpha_2 - 4\alpha_3 - 2\alpha_4$	$-\alpha_2 - 2\alpha_3 - 2\alpha_4$
$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$	$-\alpha_1 - 2\alpha_2 - 3\alpha_1 - \alpha_4$	$-2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$	$-\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$
$-\alpha_3 - \alpha_4$	$-\alpha_1 - \alpha_2 - \alpha_3$	$-\alpha_1 - \alpha_2$	$-\alpha_1 - 2\alpha_2 - 2\alpha_3$
$-\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4$	$\alpha_2 + 2\alpha_3 + \alpha_4$	$\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_2 + 2\alpha_3 + 2\alpha_4$
$-\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	$\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

In particular

$$\begin{aligned}
\sigma(\alpha_1) &= -\alpha_2 - 2\alpha_3 \\
\sigma(\alpha_2) &= -\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 \\
\sigma(\alpha_3) &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \\
\sigma(\alpha_4) &= \alpha_2 + \alpha_3 + \alpha_4
\end{aligned}$$

It follows that

$$\sigma(\Lambda_+) = \zeta^4 \cdot \Lambda_+ = \zeta_6 \cdot \Lambda_+$$

as desired. Orbit representatives for the σ action on the set of roots are given by

$$\alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ \alpha_4\ ,\ \alpha_3 + \alpha_4\ ,\ \alpha_1 + \alpha_2 + \alpha_3\ ,\ \alpha_1 + \alpha_2\ ,\ \alpha_1 + 2\alpha_2 + 2\alpha_3$$

Using Theorem 1 one obtains the following 8 masses, repeatedly using $\zeta^8 - \zeta^4 + 1 = 0$:

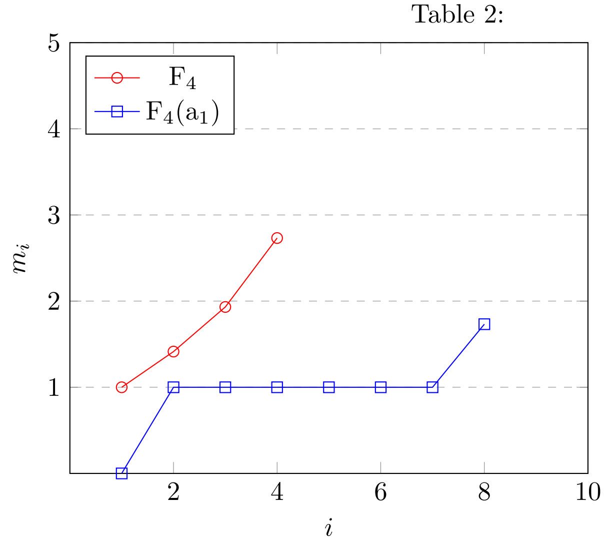
$$
\begin{aligned}
|\Lambda_+ \cdot \alpha_1| &= |\zeta - \zeta^{-3}| = |\zeta^5| = 1 \\
|\Lambda_+ \cdot \alpha_2| &= |-\zeta + 2\zeta^{-3}| = |\zeta^5 \cdot (-\zeta + 2\zeta^{-3})| = |\zeta^2 + \zeta^{-2}| = 2\cos\left(\frac{\pi}{6}\right) \\
|\Lambda_+ \cdot \alpha_3| &= |-\zeta^{-3}| = 1 \\
|\Lambda_+ \cdot \alpha_4| &= |\zeta| = 1 \\
|\Lambda_+ \cdot (\alpha_3 + \alpha_4)| &= |\zeta - \zeta^{-3}| = 1 \\
|\Lambda_+ \cdot (\alpha_1 + \alpha_2 + \alpha_3)| &= 0 \\
|\Lambda_+ \cdot (\alpha_1 + \alpha_2)| &= |\zeta^{-3}| = 1 \\
|\Lambda_+ \cdot (\alpha_1 + 2\alpha_2 + 2\alpha_3)| &= |-\zeta + \zeta^{-3}| = 1
\end{aligned}
$$

As in Example I, we have shown that the Toda-Weyl mass spectrum for σ has trigonometric expressions analogous to those for affine Toda theory:

	σ_{Coxeter}	σ
order	12	6
m_1	1	0
m_2	$2\cos\left(\frac{3\pi}{12}\right)$	1
m_3	$2\cos(\frac{\pi}{12})$	1
m_4	$4\cos(\frac{\pi}{12})\cos(\frac{3\pi}{12})$	1
m_5		1
m_6		1
m_7		1
m_8		$2\cos\left(\frac{\pi}{6}\right)$

The masses are plotted in Table 2, normalized so that the first non-zero mass equals 1.

Table 2:



F_4	$F_4(a_1)$
1	0
1.414...	1
1.932...	1
2.732...	1
	1
	1
	1
	1.73205...

2.3 A general theory

As demonstrated in Example I and II, Theorem 1 allows the effective calculation of the Toda-Weyl mass spectrum. Nonetheless, the meaning of the spectrum might still be open. To this end, recall that in the Coxeter case a crucial insight is the relation to the Perron-Frobenius eigenvector of the Cartan matrix. With some mathematical effort, a corresponding theory can be developed for the Toda-Weyl Lagrangians. The full details will be presented elsewhere, but we describe the approach for the two previously considered examples.

In Example I, the Weyl group element of $\mathfrak{e}_6$ is given by

$$\sigma = r_{\alpha_1} r_{\alpha_2} r_{\alpha_5} r_{\alpha_6} r_{\alpha_2+\alpha_4} r_{\alpha_3+\alpha_4}$$

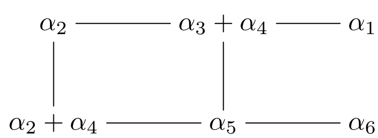
For simple roots, the relative geometry is captured by the Cartan matrix and the Dynkin diagram. Consider now analogous constructions that capture the geometry of the six roots

$$\alpha_1, \alpha_2, \alpha_5, \alpha_6, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4$$

involved in the definition of σ. Order them arbitrarily as $\gamma_1, \cdots, \gamma_6$ and define the Carter matrix K in complete analogy with the Cartan matrix via $K_{i,j} = 2 \cdot (\gamma_i, \gamma_j)/(\gamma_j, \gamma_j)$. Now define a graph with vertices corresponding to the γ_i's and the i'th and j'th vertices are joined by $N_{i,j}$ lines where

$$N_{i,j} = K_{i,j} \cdot K_{j,i}$$

This graph generalizes the notion of Dynkin diagram and was introduced in seminal work by Carter in [5], classifying conjugacy classes of Weyl groups. In the present situation one obtains



In the notation of [5], this means that σ lies in the conjugacy class $\mathrm{E}_6(a_1)$. If we let $\gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \gamma_3 = \alpha_5, \gamma_4 = \alpha_6, \gamma_5 = \alpha_2 + \alpha_4, \gamma_6 = \alpha_3 + \alpha_4$ one obtains the corresponding Carter matrix

$$K = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Let $\zeta = e^{2\pi i/36}$. A simple calculation yields an eigenvector $(x_1, \cdots, x_6)^{\mathrm{T}}$ of K with eigenvalue $\lambda = 2 - (\zeta^2 + \zeta^{-2})$:

$$x_1 = 1 = x_4$$

$$x_2 = \zeta^{10} - \zeta^8 + 1 = 1 - 2\cos\left(\frac{4\pi}{9}\right) = x_5$$

$$x_3 = -\zeta^{10} + \zeta^4 + \zeta^2 = 2\cos\left(\frac{\pi}{9}\right) = x_6$$

It turns out that the relative geometry of the roots γ_i allows to write down the desired eigenvector Λ_+ of σ. The definition of Λ_+ in Equation (8) is simply

$$\Lambda_+ = \zeta \cdot (x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3) + \zeta^{-1} \cdot (x_4\gamma_4 + x_5\gamma_5 + x_6\gamma_6)$$

Therefore, the eigenvector of the Carter matrix determines the pairings of Λ_+ with a basis of root space, and hence via Theorem 1 determines the mass spectrum.

To make this even more clear, re-scale our choice of eigenvector $(x_1, \cdots, x_6)^{\mathrm{T}}$ of K such that the smallest entry is 1, in other words we divide the original entries by $1 - 2\cos\left(\frac{4\pi}{9}\right)$. Ordered by size, the re-scaled entries are then

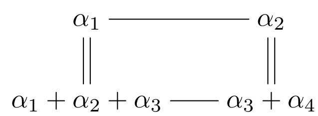
$$1, 1, 2\cos\left(\frac{2\pi}{9}\right), 2\cos\left(\frac{2\pi}{9}\right), 4\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right), 4\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)$$

So (rank $\mathfrak{g}$) = 6 out of the 8 masses of the Toda-Weyl theory associated to σ come from the eigenvector of the Carter matrix. In this manner, the celebrated relation between the affine Toda mass spectrum and the Perron-Frobenius eigenvector of the Cartan matrix is generalized to the element σ in the conjugacy class $\mathrm{E}_6(a_1)$.

The same phenomenon persists in Example II where we consider the element of the Weyl group of $\mathfrak{f}_4$ given by

$$\sigma = r_{\alpha_1} r_{\alpha_3+\alpha_4} r_{\alpha_2} r_{\alpha_1+\alpha_2+\alpha_3}$$

The corresponding graph equals



Let $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_3 + \alpha_4$, $\gamma_3 = \alpha_2$, $\gamma_4 = \alpha_1 + \alpha_2 + \alpha_3$. The corresponding Carter matrix is then

$$K = \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 2 & -1 & -1 \\ -1 & -2 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

Note that it is non-symmetric and we now choose a left eigenvector! Let $\zeta = e^{2\pi i/24}$ and let $\lambda = 2 - (\zeta^2 + \zeta^{-2})$. One possible corresponding left eigenvector of K is given by $(x_1, x_2, x_3, x_4)^{\mathrm{T}}$ with

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 2\cos\left(\frac{\pi}{6}\right)$$

$$x_4 = 0$$

As in Example I, the desired eigenvector Λ_+ of σ can be expressed in terms of this data: The choice of Λ_+ in Equation (9) is simply

$$\Lambda_+ = \zeta \cdot (x_1\gamma_1 + x_2\gamma_2) + \zeta^{-1} \cdot (x_3\gamma_3 + x_4\gamma_4)$$

To realize part of the mass spectrum in terms of the eigenvector entries $x_1, \cdots, x_4$ requires more care than in example I: In non-simply laced cases an interesting duality occurs, the σ eigenvector is expressed in term of a left eigenvector of K, whereas the inner products of Λ_+ with the roots γ_i are expressed in terms of a right eigenvector. This duality is already present in the classical Coxeter case, see [12] (Equation 24). To make this explicit, let us normalize the root lengths so that $\alpha_1^2 = \alpha_2^2 = 2$ and $\alpha_3^2 = \alpha_4^2 = 1$. Then $\gamma_1^2 = \gamma_3^2 = 2$ and $\gamma_2^2 = \gamma_4^2 = 1$ and the right eigenvector corresponding to the left eigenvector $(x_1, x_2, x_3, x_4)^{\mathrm{T}}$ has entries $\gamma_i^2 \cdot x_i$. Its entries are therefore

$$2, 2, 4\cos\left(\frac{\pi}{6}\right), 0$$

Hence, after scaling to make the lowest entry 1, one sees that (rank $\mathfrak{g}$) = 4 of the Toda-Weyl masses calculated in Section 2.2 can be read off from a suitable eigenvector of the Carter matrix.

This relation between the relative geometry of special sets of roots (the "Carter roots" γ_i) and eigenvectors of Weyl group elements allows to generalize the mass description of affine Toda theories in terms of the Perron-Frobenius eigenvector of the Cartan matrix. We will describe the full mathematical details elsewhere.

3 Conclusions

Starting with the formulation of affine Toda theory in Equation (3), we generalized this set-up by considering Lagrangians involving eigenvectors Λ_+ of other Weyl group elements σ.

Note that the resulting Toda-Weyl theory does not usually have a simple description of the form as in Equation (1): Typically, when the Weyl group element eigenvector is described in terms of root spaces, it involves two roots ξ and ν such that $\xi - \nu$ is again a root. This is one of the reasons we simply took the formulation of affine Toda theory in Equation (3) as our starting point.

Under some technical conditions on σ that are frequently satisfied, we obtained a description of the classical mass spectrum in terms of the pairings of Λ_+ with orbit representatives of the action of $\langle\sigma\rangle$ on the set of roots. After calculating the masses in some illustrative examples, we sketched in Section 2.3 how one can construct the desired eigenvectors Λ_+ in terms of eigenvectors of generalized Cartan matrices. We will describe the full mathematical details elsewhere. This relation between the linear algebra of Weyl group elements and matrices describing the relative geometry of special sets of roots generalizes the celebrated description of the affine Toda mass spectrum in terms of a Perron-Frobenius eigenvector of the relevant Cartan matrix.

There are many open questions regarding these Toda-Weyl theories. For example, one should calculate the three point couplings, as is done in usual Toda theory in [12]. Another open question concerns the integrability or failure thereof. We do not address this issue here at all but hope to return to it on a future occasion.

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The author declares to have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability:

No data was used for the research described in the article.

References

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4, 5 et 6. Hermann, Paris, 1968

[2] P. Bouwknegt, Lie algebra automorphisms, the Weyl group, and tables of shift vectors, J. Math. Phys. **30**, 571 (1989)

[3] H. W. Braden, E. Corrigan, P. Dorey, Affine Toda field theory and exact S-matrices, Nucl. Phys. B **338** (1990), 689-746

[4] L. Brillon, V. Schechtman, Coxeter element and particle masses, Selecta Math. **22** (2016), 2591-2609

[5] R. W. Carter, Conjugacy classes in the Weyl group, Compositio Math. **25** (1972), 1-59

[6] R. Coldea, D. A. Tennant, E. M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smeibidl, K. Kiefer, Quantum criticality in an Ising Chain: Experimental evidence for emergent E_8 symmetry, Science **327** (2010), 177-180

[7] G. W. Delius, M.T. Grisaru, S. Penati, D. Zanon, Exact S-matrix and perturbative calculations in affine Toda theories based on Lie superalgebras, Nuclear Phys. B **359** (1991), 125-167

[8] P. Dorey, Root systems and purely elastic S-matrices, Nucl. Phys. B **351** (1991), 654-676

[9] M. D. Freeman, On the mass spectrum of affine Toda field theory, Phys. Lett. B **261**, 57-61

[10] A. Fring, C. Korff: Colour valued scattering matrices, Phys. Lett. B **477** (2000), 380-386

[11] A. Fring, C. Korff, Affine Toda field theories related to Coxeter groups of noncrystallographic type, Nuclear Phys. B **729** (2005), 361-386

[12] A. Fring, H. C. Liao, D. I. Olive, The mass spectrum and coupling in affine Toda theories, Phys. Lett. B **266** (1991), 82-86

[13] A. Fring, S. Whittington, Lorentzian Toda field theories, Reviews in Mathematical Physics **33** (2021): 2150017

[14] R. Gebert, S. Mizoguchi, I. Takeo, Toda field theories associated with hyperbolic Kac-Moody algebra - Painlevé properties and W algebras, International Journal of Modern Physics A (1996), 5479-5493

[15] C. Heid, H. Weitzel, P. Burlet, M. Bonnet, W. Gonschorek, T. Vogt, J. Norwig, H. Fuess, Magnetic phase diagram of $CoNb_2O_6$: a neutron diffraction study, J. Magn. Magn. Mater. **151** (1995), 123-131

[16] T. J. Hollowood, P. Mansfield, Rational conformal field theories at, and away from, criticality as Toda field theories, Phys. Lett. B **226** (1989), 73-79

[17] V. Kac, Infinite-Dimensional Lie Algebras, 3rd ed., Cambridge University Press, 1990

[18] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. (1959), 973-1032

[19] M. Reeder, Torsion automorphisms of simple Lie algebras, L'Enseignement Math. **56** (2010), 3-47

[20] W. Scharf et al., Magnetic structures of $CoNb_2O_6$, J. Magn. Magn. Mater. **13** (1979), 121-124

[21] T. A. Springer, Regular elements of finite reflection groups, Inventiones Math. **25** (1974), 159-198

[22] A. B. Zamolodchikov, Integrable field theory from conformal field theory, Adv. Stud. Pure Math (1989), 641-674