Crystalline Cohomology of Superschemes

Martin T. Luu

Abstract

We introduce a notion of crystalline cohomology for superschemes and show that it is isomorphic to the usual crystalline cohomology of the underlying commutative scheme if 2 is invertible on the base.

Wess-Zumino terms play a crucial role in the matching of theoretical predictions in the framework of Quantum Chromodynamics of the decay of the neutral pion π^0 into two photons

$$\pi^0 \longrightarrow \gamma + \gamma$$

and the experimentally observed frequency. Wess-Zumino terms are often known to correspond to suitable cohomology groups related to de Rham cohomology of the target space of a suitable Σ-model. There are situations of physical interest where the target space is not a commutative manifold but rather a superspace. For example, consider the Green-Schwarz Wess-Zumino term in the superstring action as discussed in [10]. This concerns a Σ-model quantum field theory with maps

$$\phi : \Sigma \longrightarrow M^{\text{super}}$$

where Σ is a surface and M^{super} is a certain supermanifold whose associated commutative manifold is simply $\mathbb{R}^m$ for a suitable m. The Wess-Zumino term is given by the integral

$$\text{WZ}[\phi] := \int_T \Omega_3$$

where T is a subspace of M^{super} with boundary $\phi(\Sigma)$ and Ω_3 is the restriction to T of a suitable 3-form on M^{super}. If one wants a cohomological interpretation of this Wess-Zumino terms it is useful to have comparison theorems between the cohomology of superspaces and the cohomology of the associated commutative spaces. For the Green-Schwarz superstring the work of Kostant [6] is sufficient, it shows that the suitably defined de Rham cohomology of a supermanifold agrees with the usual de Rham cohomology of the underlying commutative, or "bosonic", manifold. In this work, a more arithmetic version of such a cohomological comparison theorem will be shown. One motivation for this is the following:

One can conjecture that developing tools for quantum field theories over more general ground rings than the real or complex numbers could be useful. An example of this is the work [7] of Kontsevich-Schwarz-Vologodsky that approaches the integrality question of instanton numbers via a p-adic version of the B-model topological string. This is a motivation for us to study cohomology theories of superspaces in the algebro-geometric context. The action of Frobenius on de Rham cohomology plays a central role in [7]. This action comes from positive characteristic via a comparison with crystalline cohomology and this is one reason we do not want to restrict to variants of de Rham theory that are more useful in characteristic 0 than in positive characteristic. Since the work of Grothendieck it has been known that the crystalline cohomology of usual commutative schemes is a variant of de Rham cohomology that can be more useful in positive characteristic.

This leads to the results of the current work: The relevant algebro-geometric notion of superspace, so called superscheme, was introduced in 1974 by Leites [9]. This followed the work of Berezin in the early 1970's, who started developing what is now called supermathematics while working on the method of second quantization. We refer to [14] for a clear historical description of the events and a description of the relation to the development of supersymmetric theories in physics. In this work we define and calculate the crystalline cohomology of superschemes. Our work is partially motivated by the connection between supergeometry and crystalline cohomology developed in [12]. We show that a suitable generalization of crystalline cohomology to superschemes, denoted by $\mathrm{H}^\bullet_{\mathrm{s-cris}}(-)$, agrees with the usual crystalline cohomology $\mathrm{H}^\bullet_{\mathrm{cris}}(-)$ of the underlying even scheme under some hypothesis.

Theorem 1. *Let S be a scheme over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2$, let X/S be a superscheme and let X_{even} denote the commutative scheme associated to X. Then there exists an isomorphism*

$$\mathrm{H}^i_{\mathrm{s-cris}}(X) \cong \mathrm{H}^i_{\mathrm{cris}}(X_{\mathrm{even}})$$

for all $i \geq 0$.

The idea of the proof is to show that the super-crystalline cohomology of a superscheme only depends on the underlying even scheme and then to show that the super-crystalline cohomology of a commutative scheme is in fact isomorphic to the usual crystalline cohomology. This is achieved by writing down functors between suitable sites and showing that these in fact give rise to morphisms of corresponding topoi. More precisely, if X is a superscheme over a commutative scheme S, with underlying even scheme X_{even}, then there will be a functor α from the super-crystalline site of X/S to the super-crystalline site of X_{even}/S and there will be the inclusion functor β of the crystalline site of X_{even}/S to the corresponding super-crystalline site:

$$\mathrm{SCris}(X/S) \xrightarrow{\alpha} \mathrm{SCris}(X_{\mathrm{even}}/S) \xleftarrow{\beta} \mathrm{Cris}(X_{\mathrm{even}}/S)$$

Studying these functors and the associated Leray spectral sequences will show the theorem.

Note that there is related work by Cortiñas: In [5] Kapranov defined NC-schemes and superschemes are special cases of these non-commutative schemes. In [3] it is shown under suitable hypotheses that if X is an NC-scheme then the suitably defined de Rham cohomology agrees with the usual algebraic de Rham cohomology of the underlying commutative scheme and a similar result is shown for analogues of infinitesimal cohomology in the context of NC-schemes.

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1 Superschemes

We briefly recall basic notions concerning superschemes. Convenient references are [2] and [8]. Throughout this work, all rings are assumed to have an identity. Moreover we will assume that $1 + 1$ is a unit for all rings that we work with.

A super-commutative ring R is a $\mathbb{Z}/2\mathbb{Z}$-graded ring $R = R_0 \oplus R_1$ such that

$$a \cdot b = (-1)^{d(a)d(b)} b \cdot a$$

for all a, b which are either in R_0 or R_1 and where $d(x) = 0$ if and only if $x \in R_0$ and $d(x) = 1$ if and only if $x \in R_1$. A homogeneous one-sided ideal in R will automatically be two-sided and hence in the following we will call such ideals simply homogeneous ideals. A super-commutative ring is called local if and only if it has a unique maximal homogeneous ideal. The category of super-commutative rings has as objects super-commutative rings and as morphisms grading preserving ring homomorphisms.

Let $R = R_0 \oplus R_1$ be a super-commutative ring. By a super locally ringed space we mean a ringed space such that the sheaf of rings is a sheaf of super-commutative rings whose stalks are local rings. The affine superscheme $\mathrm{Spec}(R)$ associated to a super-commutative ring R is then defined to be the super locally ringed space

$$\mathrm{Spec}(R) := (\mathrm{Spec}(R_0), \tilde{R})$$

where $\tilde{R}$ is the quasi-coherent sheaf on $\mathrm{Spec}(R_0)$ associated to the R_0-module R. Note that the stalk of $\tilde{R}$ at a prime ideal $\mathfrak{p}$ of R_0 has an obvious structure of super-commutative ring.

In analogy with usual commutative algebraic geometry, a superscheme is defined to be a a super locally ringed space locally isomorphic to an affine superscheme. More precisely: A superscheme X is a super locally ringed space $(|X|, \mathcal{O}_X)$, with $\mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1}$ a sheaf such that $(X, \mathcal{O}_{X,0})$ is a scheme and $\mathcal{O}_{X,1}$ is a quasi-coherent sheaf of $\mathcal{O}_{X,0}$-modules. This is equivalent to describing a superscheme to locally be an affine superscheme, see [2] (Proposition 10.1.3).

If $f : X \to Y$ is a continuous map between locally ringed spaces and $\mathcal{F}$ is a sheaf on X, then $f_*(\mathcal{F})$ will denote the push-forward. A morphism $f : X \to Y$ of superschemes is defined in [2] (Section 10.1) to be a pair $(|f|, f^\sharp)$ consisting of a continuous map of topological spaces $|f| : |X| \to |Y|$ as well as a morphisms of sheaves

$$f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$$

which gives rise to local maps on stalks and which on open sets gives rise to morphisms of superalgebras, where the $\mathbb{Z}/2\mathbb{Z}$-grading on $f_*(\mathcal{O}_X)$ is given by

$$(f_*\mathcal{O}_X)_i = f_*(\mathcal{O}_{X,i})$$

We denote by **S-Schemes** the category whose objects are superschemes and whose morphisms are as defined above. Note that in particular usual commutative schemes are objects in this category.

We will make the following definitions: A morphism $f : X \to Y$ is called a closed immersion if $|f|$ is a homeomorphism from its source to the image and $f^\sharp$ is an epimorphism in the category of sheaves of sets on Y. Note that as for commutative schemes, closed immersions can be described via sheaves of ideals. For a superscheme X, by a Zariski sheaf on X we will mean a sheaf with respect to the topology on the locally ringed space X.

We assume for the remainder of this section that X is a superscheme over $\mathbb{Z}[\frac{1}{2}]$ in the sense that there is a morphism of superschemes $X \to \operatorname{Spec}(\mathbb{Z}[\frac{1}{2}])$.

Definition 1. Given a superscheme X, let $\mathcal{O}_{X,0}/(\mathcal{O}_{X,1})^2$ denote the sheafification of the presheaf given by $U \mapsto \mathcal{O}_{X,0}(U)/(\mathcal{O}_{X,1})^2(U)$. The underlying even scheme X_{even} of X is defined to be

$$X_{\text{even}} := (|X|, \mathcal{O}_{X,0}/(\mathcal{O}_{X,1})^2)$$

Let

$$i_X : X_{\text{even}} \longrightarrow X$$

denote the closed immersion of superschemes given by the identity map on underlying topological spaces and $\mathcal{O}_X \to i_{X*}\mathcal{O}_{X_{\text{even}}}$ is defined such that for an open set U of $|X|$ it is the composition of the quotient map $\mathcal{O}_X(U) \to \mathcal{O}_{X,0}(U)/\mathcal{O}_{X,1}(U)^2$ with the sheafification.

We now define the evenization functor that maps a superscheme to its associated even scheme:

Definition 2. Let **Schemes** denote the category of schemes. Define a functor

$$\textbf{S-Schemes} \longrightarrow \textbf{Schemes}$$

in the following manner: An object X of **S-Schemes** is taken to the even scheme X_{even}. If $f : X \to Y$ is a morphism of superschemes, then one maps it to the morphism $f_{\text{even}} : X_{\text{even}} \longrightarrow Y_{\text{even}}$ of underlying schemes which is defined in the following way: Since $|X| = |X_{\text{even}}|$ and $|Y| = |Y_{\text{even}}|$, one obtains a map $|X_{\text{even}}| \to |Y_{\text{even}}|$. By definition of a morphism of superschemes, for $i = 0, 1$ there exists a morphism

$$\mathcal{O}_{Y,i} \longrightarrow (f_*\mathcal{O}_X)_i$$

Hence one obtains a morphism

$$\mathcal{O}_{Y,0}/(\mathcal{O}_{Y,1})^2 \longrightarrow f_*(\mathcal{O}_{X,0})/f_*(\mathcal{O}_{X,1}^2) \longrightarrow f_*(\mathcal{O}_{X,0}/\mathcal{O}_{X,1}^2)$$

and hence one has obtained a morphism f_{even}.

We will use in subsequent arguments the following properties of the above constructions:

(i) If $f : X \to Y$ is a closed immersion of superschemes then f_{even} is a closed immersion of commutative schemes.

(ii) If Z is an even scheme, X is a superscheme and $f : Z \to X$ is a morphism of superschemes then the following diagram commutes

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, equal] & X \\ Z_{\text{even}} \arrow[r, "f_{\text{even}}"] & X_{\text{even}} \arrow[u, "i_x"'] \end{tikzcd}$$

2 Super-crystalline site and the associated topos

In this section we define an analogue of the crystalline site in the context of superschemes. It will be the third step in a three step process away from classical de Rham theory:

(i) Infinitesimal site

(ii) Crystalline site

(iii) Super-crystalline site

The usual crystalline site of a commutative scheme is a variant, better suited to positive characteristic, of the infinitesimal site. The latter site was used in the 60's by Grothendieck to show that one can often calculate de Rham cohomology without ever talking about differential forms. More precisely, for a scheme X which is smooth over a scheme S of characteristic 0 Grothendieck showed in [4] (Theorem 4.1) that the algebraic de Rham cohomology of X is isomorphic to the cohomology of the infinitesimal site. In positive characteristic situations the infinitesimal cohomology can not easily be related to the de Rham cohomology since the analogue of the Poincaré lemma fails in general. Instead one introduces the crystalline site which is a variant of the infinitesimal site that also requires the existence of certain divided power structures:

If A be a commutative ring and I an ideal then a divided power structure, or pd-structure, $\gamma = (\gamma_n)_{n\geq 0}$ on I is collection of maps

$$\gamma_n : I \longrightarrow A$$

for each $n \geq 0$, satisfying certain conditions, see [1] (Chapter 3), which include that for all $x, y \in I$

$$\gamma_2(x + y) = \gamma_2(x) + xy + \gamma_2(y)$$

It follows that for general ideals in super-commutative rings the above conditions do not hold for the maps

$$\gamma_n : x \longmapsto \frac{x^n}{n!}$$

assuming for example that $n!$ is invertible for all n. Hence one should modify the notion of divided power structure when working with superschemes. For our purpose, namely to develop a notion of crystalline cohomology for superschemes, this is fortunately not necessary. It turns out that it suffices to work with divided power structures on underlying even schemes. We now give precise definitions.

For the remainder of this work S will always denote a scheme over $\mathbb{F}_q$ with $\text{char}(\mathbb{F}_q) \neq 2$ and $I \subseteq \mathcal{O}_S$ will be a sheaf of ideals with pd-structure γ. By a superscheme X/S we mean a superscheme X together with a morphism of superschemes $X \to S$ such that the pd-structure γ extends to X_{even}. See [1] (Chapter 3) for the definitions of compatibility and extension of pd-structures as well as S-pd morphisms.

We now define the analogue of the crystalline site in the setting of superschemes. Note that in the context of infinitesimal or crystalline cohomology of commutative schemes there are at least two conventions for defining the relevant sites: In Grothendieck's work [4] one looks at suitable closed immersions $U \to T$ defined by nilpotent ideals.

In other references, for example [1], one looks at the more general notion of a thickening $U \to T$, which means a closed immersion where U and T have the same underlying topological space. We will adapt the latter definition to the super-setting:

Definition 3. Let X/S be a superscheme such that the pd-structure γ extends to X_{even}. We now define the category $\mathrm{SCris}(X/S)$:

(i) An object in $\mathrm{SCris}(X/S)$ is a triple $(U, f : U \to T, \delta)$ with

- U a Zariski open subset of X
- a closed S-immersion $f : U \to T$ which is a thickening, in the sense that U and T have the same topological space, such that the sheaf of ideals $\mathcal{J}$ defining the closed immersion $f_{\text{even}} : U_{\text{even}} \to T_{\text{even}}$ has a pd-structure δ which is compatible with γ.

Sometimes we will denote such a triple simply by (U, T, δ).

(ii) A morphism

$$(U', T', \delta') \longrightarrow (U, T, \delta)$$

between two objects of $\mathrm{SCris}(X/S)$ is a pair (ρ, g) where $\rho : T' \to T$ is a morphism of superschemes such that ρ^{even} is a S-pd morphism and $g : U' \to U$ is an inclusion in the Zariski topology such that the following diagram commutes:

$$\begin{tikzcd} T' \arrow[r, "\rho"] & T \\ U' \arrow[u] \arrow[r, "g"] & U \arrow[u] \end{tikzcd}$$

The following is an example of an object in $\mathrm{SCris}(X/S)$: Let us take $S = X = U = \operatorname{Spec} \mathbb{F}_q$ where as before $\mathrm{char}(\mathbb{F}_2) \neq 2$ and let $T = \operatorname{Spec} \mathbb{F}_q[x, \xi_1, \xi_2]$ where $x^3 = \xi_1^2 = \xi_2^2 = 0$ and x is an even variable and ξ_1, ξ_2 are odd variables. Furthermore, define the pd-structure on $\delta = (x)$ via $\gamma_0(y) = 1$, $\gamma_1(y) = y$, $\gamma_2(y) = y^2 \cdot 2^{-1}$ and $\gamma_i(y) = 0$ for $i \geq 3$ for all y in δ.

Lemma 2.1. *Let X/S be a superscheme, then the category* $\mathrm{SCris}(X/S)$ *has fiber products.*

Proof. As discussed for example in [8] (Proposition 1.1.3), the category of superschemes has fiber products, which as in the commutative case, is given on affine superschemes by taking the spectrum of the relevant tensor product. If A, B, C are super-commutative rings and given maps $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ and $\operatorname{Spec}(C) \to \operatorname{Spec}(A)$ the corresponding fiber product is $\operatorname{Spec}(B \otimes_A C)$ where the grading on $B \otimes_A C$ is the following: For $i = 0, 1$ let $b^{(i)}$ denote an element of B_i and $c^{(i)}$ an element of C_i. Then $(B \otimes_A C)_0$ is spanned by elements of the form $b^{(1)} \otimes c^{(1)}$ and $b^{(0)} \otimes c^{(0)}$ and furthermore $(B \otimes_A C)_1$ is spanned by elements of the form $b^{(0)} \otimes c^{(1)}$ and $b^{(1)} \otimes c^{(0)}$. The multiplication comes from defining for k, l, m, n being even or odd elements

$$(k \otimes l) \cdot (m \otimes n) = (-1)^{d(l)d(m)}(km \otimes ln)$$

where $d(-)$ denotes the parity. Consider now two morphisms f_α and f_β in $\mathrm{SCris}(X/S)$ of the form:

$$\begin{tikzcd} & (U_\beta, T_\beta, \delta_\beta) \arrow[d, "f_\beta"] \\ (U_\alpha, T_\alpha, \delta_\alpha) \arrow[r, "f_\alpha"] & (U, T, \delta) \end{tikzcd}$$

Then $U_\alpha \times_U U_\beta$ can be viewed as an open sub-superscheme of X and there is a closed immersion $f_{\alpha,\beta} : U_\alpha \times_U U_\beta \longrightarrow T_\alpha \times_T T_\beta$ with corresponding closed immersion

$$f_{\alpha,\beta}^{\text{even}} : (U_\alpha \times_U U_\beta)^{\text{even}} \longrightarrow (T_\alpha \times_T T_\beta)^{\text{even}}$$

One obtains a pd-structure, which we denote by $\delta_\alpha \times \delta_\beta$, on the sheaf of ideals defining this closed immersion by noticing that taking the fiber product commutes with passing to underlying even schemes: For example, for affine superschemes this follows from the fact that if $A \to B$ and $A \to C$ are morphisms of supercommutative rings and if $b^{(1)} \in B_1$ and $c^{(1)} \in C_1$ one has that $b^{(1)} \otimes c^{(1)} \in (B \otimes_A C)_0$ is in fact in $(B \otimes_A C)_1^2$. Hence the fiber product in $\mathrm{SCris}(X/S)$ is given by

$$(U_\alpha, T_\alpha, \delta_\alpha) \times_{(U,T,\delta)} (U_\beta, T_\beta, \delta_\beta) = (U_\alpha \times_U U_\beta, T_\alpha \times_T T_\beta, \delta_\alpha \times \delta_\beta)$$

□

Definition 4. A covering family of $(U, T, \delta) \in \mathrm{SCris}(X/S)$ will be defined to be a collection of morphisms

$$f_\alpha : (U_\alpha, T_\alpha, \delta_\alpha) \longrightarrow (U, T, \delta)$$

such that the maps $T_\alpha \to T$ are open immersions and such that $\bigcup_\alpha f_\alpha(T_\alpha) = T$.

The category $\mathrm{SCris}(X/S)$ together with the above defined covering families defines a pretopology as defined in [11] (II.1.3): We need to verify the properties PT 0) - PT 3) of loc. cit. and by the definition of fiber products in $\mathrm{SCris}(X/S)$ this follows from the same arguments as in the case of the usual crystalline site of a scheme. For example, if $\{(U_\alpha, T_\alpha, \delta_\alpha) \to (U, T, \delta)\}_\alpha$ is a covering family of (U, T, δ) and if $(U_Y, Y, \delta_Y) \to (U, T, \delta)$ is a morphism in $\mathrm{SCris}(X/S)$, then

$$\{(U_\alpha, T_\alpha, \delta_\alpha) \times_{(U,T,\delta)} (U_Y, Y, \delta_Y) \to (U_Y, Y, \delta_Y)\}$$

is again a covering family. We call the site corresponding to the above pretopology, see loc. cit., the super-crystalline site.

Following loc. cit. (IV.1.1), by a topos we mean a category which is equivalent to the category of sheaves of sets of a site.

Definition 5. Let X/S be a superscheme. Denote by $(X/S)_{\mathrm{s-cris}}$ the topos of the super-crystalline site $\mathrm{SCris}(X/S)$.

In complete analogy with the case of sheaves on the usual crystalline site of commutative schemes, see [1] (Proposition 5.1), one has a Zariski interpretation of objects of $(X/S)_{\mathrm{s-cris}}$:

Let $(U, T, \delta) \in \mathrm{SCris}(X/S)$ and let $\mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_1$ be the sheaf of ideals defining the closed immersion $U \to T$. Let T' be an open sub-superscheme of T. Then there is a closed immersion $U \cap T' \to T'$ defined by $\mathcal{I}|_{T'}$ with grading $(\mathcal{I}|_{T'})_i := \mathcal{I}_i|_{T'}$ for $i = 0, 1$. Recall that δ is a pd-structure on $\mathcal{I}_0/\mathcal{I}_1^2$. Let $\delta|_{T'}$ denote its restriction to T'. Since sheafification commutes with restriction this gives a pd-structure on the sheaf of ideals defining the closed immersion $(U \cap T')_{\mathrm{even}} \to T'_{\mathrm{even}}$ and therefore

$$(U \cap T', T', \delta|_{T'}) \in \mathrm{SCris}(X/S)$$

Hence the assignment

$$T' \mapsto \mathcal{F}((U \cap T', T', \delta|_{T'}))$$

defines a sheaf on the Zariski topology of T.

To give a sheaf $\mathcal{F}$ on $\mathrm{SCris}(X/S)$ is then seen to be the same as to give for every object (U, T, δ) of $\mathrm{SCris}(X/S)$ a Zariski sheaf $\mathcal{F}_{(U,T,\delta)}$ on T and for every morphism $u : (U', T', \delta') \to (U, T, \delta)$ in $\mathrm{SCris}(X/S)$ a morphism of sheaves

$$\rho_u : u^{-1}(\mathcal{F}_{(U,T,\delta)}) \to \mathcal{F}_{(U',T',\delta')}$$

satisfying certain conditions, see [1] (Proposition 5.1).

Definition 6. Let X/S be a superscheme. The structure sheaf $\mathcal{O}_{X/S}^{\mathrm{super}}$ is the object of $(X/S)_{\mathrm{s-cris}}$ such that

$$(\mathcal{O}_{X/S}^{\mathrm{super}})_{(U,T,\delta)} = \mathcal{O}_T$$

for all $(U, T, \delta) \in \mathrm{SCris}(X/S)$.

Let e be a final object of $(X/S)_{\mathrm{s-cris}}$. It exists and is unique up to unique isomorphism. Let **Set** denote the category of sets. The global section functor $\Gamma_{\mathrm{s-cris}}$ is defined to be the functor

$$\Gamma_{\mathrm{s-cris}} : (X/S)_{\mathrm{s-cris}} \longrightarrow \mathbf{Set}$$

given by $\mathcal{F} \mapsto \mathrm{Hom}_{(X/S)_{\mathrm{s-cris}}}(e, \mathcal{F})$. It is a left exact functor and since the category of sheaves of abelian groups on a site has enough injectives we can define the corresponding right derived functors:

Definition 7. Let X/S be a superscheme and define $\mathrm{H}^i_{\mathrm{s-cris}}(X, -)$ to be the i'th right derived functor of $\Gamma_{\mathrm{s-cris}}$. We will write

$$\mathrm{H}^i_{\mathrm{s-cris}}(X) := \mathrm{H}^i_{\mathrm{s-cris}}(X, \mathcal{O}^{\mathrm{super}}_{X/S})$$

3 Calculation of cohomology

To prove Theorem 1 we use various morphisms of topoi and associated spectral sequences. Let X/S be a superscheme. As mentioned earlier, the key are suitable functors between sites:

$$\begin{tikzcd}
\mathrm{Cris}(X_{\mathrm{even}}/S) \arrow[dr, "\alpha"] & & \mathrm{SCris}(X/S) \arrow[dl, "\beta"'] \\
& \mathrm{SCris}(X_{\mathrm{even}}/S) &
\end{tikzcd}$$

Here the functor α could be called the super-refinement functor and the functor β could be called the bosonization functor. Via suitable Leray spectral sequences these functors will give rise to isomorphisms:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{cris}}(X_{\mathrm{even}}) \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "\sim"] & & \mathrm{H}^i_{\mathrm{s-cris}}(X) \arrow[dl, "\sim"'] \\
& \mathrm{H}^i_{\mathrm{s-cris}}(X_{\mathrm{even}}) &
\end{tikzcd}$$

Before we carry out this strategy we now briefly recall relevant definitions concerning topoi:

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topoi. A morphism of topoi

$$g = (g_*, g^*) : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$$

is defined to be a functor $g_* : \mathcal{T}_1 \to \mathcal{T}_2$ which has a left adjoint $g^* : \mathcal{T}_2 \to \mathcal{T}_1$ which commutes with finite inverse limits. By [11] (V.5.3), for a morphism of topoi as above and a sheaf $\mathcal{F}$ in $\mathcal{T}_1$ which is a sheaf of abelian groups there exists a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathcal{T}_2, \mathrm{R}^q g_* \mathcal{F}) \Longrightarrow \mathrm{H}^{p+q}(\mathcal{T}_1, \mathcal{F})$$

We refer to loc. cit. (V.2.1) for the definition of the cohomology of a sheaf on a site. The above spectral sequence is called the Leray spectral sequence associated to the morphism $g = (g_*, g^*)$.

The morphisms of topoi that we will use will come from functors between the underlying sites in the following manner: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sites, or more precisely pretopologies, with corresponding topoi $\mathcal{T}_1$ and $\mathcal{T}_2$ and consider a continuous functor $i : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$. Continuity means that i takes coverings to coverings and if $Y \to X$ is a morphism in $\mathcal{S}_1$ and $\{X_\alpha \to X\}_\alpha$ is a covering in $\mathcal{S}_1$ then the morphism $i(Y \times_X X_\alpha) \longrightarrow i(Y) \times_{i(X)} i(X_\alpha)$ is an isomorphism. Let $i_* : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ be given by mapping $\mathcal{F} \in \mathcal{T}_2$ to $\mathcal{F} \circ i$. If i is a continuous functor and i_* has an exact left adjoint i^* then (i_*, i^*) is a morphism of topoi.

Lemma 3.1. *Let S be a scheme over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2$ and let X/S be a scheme. Consider the inclusion*

$$i : \mathrm{Cris}(X/S) \longrightarrow \mathrm{SCris}(X/S)$$

It gives rise to a map $i_*: (X/S)_{\mathrm{s-cris}} \longrightarrow (X/S)_{\mathrm{cris}}$ defined by $i_*\mathcal{F} = \mathcal{F} \circ i$. Then i is a continuous functor and i_* has an exact left adjoint i^* and therefore (i_*, i^*) is a morphism of topoi.

Proof. Clearly i is a continuous functor. We now check that i_* has an exact left adjoint i^*. First note that via the Zariski interpretation of sheaves, the topoi $(X/S)_{\mathrm{s-cris}}$ and $(X/S)_{\mathrm{cris}}$ have enough points and so exactness of morphism of sheaves can be checked for stalks: The sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact if and only if for every (U, T, δ) an object of the site and every $x \in |T| = |U|$ the sequence

$$\mathcal{F}_{T,x} \longrightarrow \mathcal{G}_{T,x} \longrightarrow \mathcal{H}_{T,x}$$

is exact. We now describe the left adjoint functor i^*:

Let $(U, T, \delta) \in \mathrm{SCris}(X/S)$ and let $\mathcal{I}_{(U,T,\delta)}$ denote the category whose objects are pairs $((U', T', \delta'), \phi)$ with

- $(U', T', \delta') \in \mathrm{Cris}(X/S)$
- $\phi \in \mathrm{Hom}_{\mathrm{SCris}(X/S)}((U, T, \delta), (U', T', \delta'))$

The morphisms between $((U_1', T_1', \delta_1'), \phi_1)$ and $((U_2', T_2', \delta_2'), \phi_2)$ are the morphisms

$$\Big\{\lambda \in \mathrm{Hom}_{\mathrm{Cris}(X/S)}((U_1', T_1', \delta_1'), (U_2', T_2', \delta_2')) \big| \phi_2 = \lambda \circ \phi_1\Big\}$$

For $\mathcal{F}$ an object of $(X/S)_{\mathrm{cris}}$ define a functor

$$\mathcal{F} - (U, T, \delta) : \mathcal{I}^{\mathrm{opp}}_{(U,T,\delta)} \longrightarrow \mathbf{Set}$$

which on objects is given by

$$((U', T', \delta'), \phi) \mapsto \mathcal{F}(U', T', \delta')$$

Consider the presheaf $\tilde{i}\mathcal{F}$ on $\mathrm{SCris}(X/S)$ given by

$$(U, T, \delta) \mapsto \mathrm{colim}_{\mathcal{I}^{\mathrm{opp}}_{(U,T,\delta)}} \mathcal{F} - (U, T, \delta)(((U', T', \delta'), \phi)) = \mathrm{colim}_{\mathcal{I}^{\mathrm{opp}}_{(U,T,\delta)}} \mathcal{F}(U', T', \delta')$$

By [13] (TAG 00VC) the functor i_* has a left adjoint i^* such that $i^*\mathcal{F}$ is given by the sheafification of $\tilde{i}\mathcal{F}$.

Assume now that

$$\mathcal{F} \xrightarrow{\ f\ } \mathcal{G} \xrightarrow{\ g\ } \mathcal{H}$$

is an exact sequence of objects of $(X/S)_{\mathrm{cris}}$. By the same argument as in the case of sheaves on the site $\mathrm{SCris}(X/S)$ one has an interpretation of presehaves on that site in terms of a collection of presheaves on the Zariski topology of T where (U, T, δ) is an arbitrary object of the super-crystalline site. We will use the analogous notation as in the case for sheaves to denote these Zariski presheaves. Since sheafification is an exact functor, see for example [13] (Lemma 7.10.14), it follows that in order to show that i^* is exact it suffices to show that the sequence of stalks

$$(\tilde{i}\mathcal{F})_{(U,T,\delta),x} \longrightarrow (\tilde{i}\mathcal{G})_{(U,T,\delta),x} \longrightarrow (\tilde{i}\mathcal{H})_{(U,T,\delta),x}$$

is exact for all $(U, T, \delta) \in \mathrm{SCris}(X/S)$ and all $x \in |T|$. For the following arguments note that if $\mathcal{M}$ is an object of $(X/S)_{\mathrm{cris}}$ then the stalk of $\tilde{i}\mathcal{M}$ is given by

$$(\tilde{i}\mathcal{M})_{(U,T,\delta),x} = \mathrm{colim}_{(\tilde{V} \cap U, \tilde{V}, \tilde{\delta}) \in \mathrm{SCris}(X/S), x \in |\tilde{V}|, \tilde{V} \subseteq T} \Big(\mathrm{colim}_{\mathcal{I}^{\mathrm{opp}}_{(\tilde{V} \cap U, \tilde{V}, \tilde{\delta})}} \mathcal{M}(U', T', \delta')\Big)$$

where $\tilde{V} \subseteq T$ means that $\tilde{V}$ is an open subscheme of T. Let us fix some notation: For $t \in \mathcal{M}(U', T', \delta')$ we denote by $[t]$ the corresponding element of the stalk $(\tilde{i}\mathcal{M})_{(U,T,\delta),x}$.

Suppose now that

$$s \in \mathrm{Ker}\Big((\tilde{i}\mathcal{G})_{(U,T,\delta),x} \longrightarrow (\tilde{i}\mathcal{H})_{(U,T,\delta),x}\Big)$$

Write $s = [t]$ where $t \in \mathcal{G}(A, B, \mu)$ for some $(A, B, \mu) \in \mathrm{Cris}(X/S)$ such that there there is $(\tilde{V} \cap U, \tilde{V}, \delta|_{\tilde{V}})$ in $\mathrm{SCris}(X/S)$ with $x \in |\tilde{V}|$ and $\tilde{V} \subseteq T$ and some ϕ such that

$$((A, B, \mu), \phi) \in \mathcal{I}_{(\tilde{V} \cap U, \tilde{V}, \delta|_{\tilde{V}})}$$

Then by possibly replacing $(\tilde{V} \cap U, \tilde{V}, \delta|_{\tilde{V}})$ we can assume that $g(t) = 0$. Note that $x \in |B|$. By exactness of the sequence of stalks

$$\mathcal{F}_{(A,B,\mu),x} \longrightarrow \mathcal{G}_{(A,B,\mu),x} \longrightarrow \mathcal{H}_{(A,B,\mu),x}$$

there exists $(F \cap A, F, \mu|_F) \in \mathrm{Cris}(X/S)$ with $x \in |F|$ and $F \subseteq B$ such that restriction $\mathrm{Res}(t)$ of t to it is in the image of $\mathcal{F}((F \cap A, F, \mu|_F))$ under the map f. Let r denote the inclusion morphism

$$r : (F \cap A, F, \mu|_F) \longrightarrow (A, B, \mu)$$

The reason that the desired exactness does not immediately follow is that one does not know if there is a ξ such that

$$((F \cap A, F, \mu|_F), \xi) \in \mathcal{I}_{(\tilde{V} \cap U, \tilde{V}, \tilde{\delta})} \text{ and } r \circ \xi = \phi$$

The map ϕ gives rise to a morphism of schemes $\tilde{V} \longrightarrow B$ which we will also denote by ϕ. Let $W = \phi^{-1}(F)$ and note that $x \in |W|$ since $x \in |F|$ and $x \in |\tilde{V}|$. Then the following diagram commutes:

$$\begin{tikzcd}
& (A, B, \mu) \\
(W \cap U, W, \delta|_W) \arrow[ur, "\phi|_W"] \arrow[dr, "\phi|_W"'] & \\
& (F \cap A, F, \mu|_F) \arrow[uu, "r"']
\end{tikzcd}$$

In particular $((F \cap A, F, \mu|_F), \phi|_W) \in \mathcal{I}_{(W \cap U, W, \delta|_W)}$ and since

$$s = [t] = [\mathrm{Res}(t)]$$

it follows that the sequence $(\tilde{i}\mathcal{F})_{(U,T,\delta),x} \longrightarrow (\tilde{i}\mathcal{G})_{(U,T,\delta),x} \longrightarrow (\tilde{i}\mathcal{H})_{(U,T,\delta),x}$ is exact, as desired. ☐

The previous lemma can be used to show:

Corollary 3.2. *Let S be a scheme over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2$ and let X/S be a scheme. Then for all $i \geq 0$ there is an isomorphism*

$$\mathrm{H}^i_{\text{cris}}(X) \cong \mathrm{H}^i_{\text{s-cris}}(X)$$

Proof. Let

$$(i_*, i^*) : (X/S)_{\text{s}-\text{cris}} \longrightarrow (X/S)_{\text{cris}}$$

be the morphism of topoi constructed in the previous lemma. Then

$$i_*(\mathcal{F})_{(U,T,\delta)} = \mathcal{F}_{(U,T,\delta)}$$

Via the Zariski interpretation of sheaves on the crystalline and super-crystalline site of X/S it follows that i_* is an exact functor since this can be checked on stalks for all the Zariski sheafs. Moreover,

$$\begin{aligned} i_*(\mathcal{O}_{X/S}^{\text{super}})((U,T,\delta)) &= \mathcal{O}_{X/S}^{\text{super}}((U,T,\delta)) \\ &= \mathcal{O}_T(T) \\ &= \mathcal{O}_{X/S}((U,T,\delta)) \end{aligned}$$

and therefore

$$i_*(\mathcal{O}_{X/S}^{\text{super}}) = \mathcal{O}_{X/S}$$

Define $\mathrm{R}^q i_*$ as the q'th derived functor of i_*. Consider the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}_{\text{cris}}^p(X, \mathrm{R}^q i_* \mathcal{O}_{X/S}^{\text{super}}) \Longrightarrow \mathrm{H}_{\text{s}-\text{cris}}^{p+q}(X, \mathcal{O}_{X/S}^{\text{super}})$$

By the exactness of i_* the spectral sequence degenerates and it follows that

$$\mathrm{H}_{\text{cris}}^i(X) \cong \mathrm{H}_{\text{s}-\text{cris}}^i(X)$$

for all $i \geq 0$. □

To prove the main theorem we now compare the super-crystalline cohomology of a superscheme X to the super-crystalline cohomology of the underlying even scheme.

Lemma 3.3. *Let S be a scheme over $\mathbb{F}_q$ with $\operatorname{char}(\mathbb{F}_q) \neq 2$ and let X/S be a superscheme. Consider the functor*

$$i : \mathrm{SCris}(X/S) \longrightarrow \mathrm{SCris}(X_{\text{even}}/S)$$

given by $(U,T,\delta) \mapsto (U_{\text{even}}, T, \delta)$. It gives rise to a map $i_* : (X_{\text{even}}/S)_{\text{s}-\text{cris}} \longrightarrow (X/S)_{\text{s}-\text{cris}}$ given by $i_*\mathcal{F} = \mathcal{F} \circ i$. Then i is a continuous functor and i_* has an exact left adjoint i^* and therefore (i_*, i^*) is a morphism of topoi.

Proof. The proof is very similar to the proof of Lemma 3.1 and hence we will be more succinct here. If A, B, C are superschemes and there are morphisms of superschemes $f : A \to C$ and $g : B \to C$ then there is an isomorphism for the corresponding fiber product $(A \times_C B)_{\text{even}} \cong A_{\text{even}} \times_{C_{\text{even}}} B_{\text{even}}$. Using this one sees that i is a continuous functor.

We now show that i_* has an exact left adjoint. Let $(U,T,\delta) \in \mathrm{SCris}(X_{\text{even}}/S)$ and let $\mathcal{I}_{(U,T,\delta)}$ denote the category whose objects are pairs $((U',T',\delta'),\phi)$ with

- $(U',T',\delta') \in \mathrm{SCris}(X/S)$
- $\phi \in \mathrm{Hom}_{\mathrm{SCris}(X_{\text{even}}/S)}((U,T,\delta),(U'_{\text{even}},T',\delta'))$

The morphisms are defined similar to what is done in the proof of Lemma 3.1. For $\mathcal{F}$ an object of $(X/S)_{\text{cris}}$ define a functor

$$\mathcal{F} - (U,T,\delta) : \mathcal{I}_{(U,T,\delta)}^{\text{opp}} \longrightarrow \mathbf{Set}$$

in an analogous manner to Lemma 3.1. Define the presheaf $\tilde{i}\mathcal{F}$ on $\mathrm{SCris}(X_{\text{even}}/S)$ given by

$$(U,T,\delta) \mapsto \operatorname{colim}_{\mathcal{I}_{(U,T,\delta)}^{\text{opp}}} \mathcal{F} - (U,T,\delta)(((U',T',\delta'),\phi))$$

As before, the functor i_* has a left adjoint i^* such that $i^*\mathcal{F}$ is given by the sheafification of $\tilde{i}\mathcal{F}$. Assume now that

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is an exact sequence of objects of $(X/S)_{\text{s}-\text{cris}}$. Let

$$s \in \operatorname{Ker}\Big((\tilde{i}\mathcal{G})_{(U,T,\delta),x} \longrightarrow (\tilde{i}\mathcal{H})_{(U,T,\delta),x}\Big)$$

Write $s = [t]$ where $t \in \mathcal{G}(A, B, \mu)$ for some $(A, B, \mu) \in \mathrm{SCris}(X/S)$ such that there there is $(\tilde{V} \cap U, \tilde{V}, \delta|_{\tilde{V}})$ in $\mathrm{SCris}(X/S)$ with $x \in |\tilde{V}|$ and $\tilde{V} \subseteq T$ and some ϕ such that

$$((A, B, \mu), \phi) \in \mathcal{I}_{(\tilde{V} \cap U, \tilde{V}, \delta|_{\tilde{V}})}$$

Then we can assume that $g(t) = 0$. By exactness of

$$\mathcal{F}_{(A,B,\mu),x} \longrightarrow \mathcal{G}_{(A,B,\mu),x} \longrightarrow \mathcal{H}_{(A,B,\mu),x}$$

there exists $(F \cap A, F, \mu|_F) \in \mathrm{SCris}(X/S)$ with $x \in |F|$ and $F \subseteq B$ and the inclusion morphism

$$r : (F \cap A, F, \mu|_F) \longrightarrow (A, B, \mu)$$

such that restriction $\mathrm{Res}(t)$ of t to it is in the image of $\mathcal{F}(F \cap A, F, \mu|_F)$ under the map f. The map ϕ gives rise to a morphism of schemes $\tilde{V} \longrightarrow B$ which we will also denote by ϕ. Let $W = \phi^{-1}(F)$ and note that $x \in |W|$ since $x \in |F|$ and $x \in |\tilde{V}|$. Then the following diagram commutes:

$$\begin{tikzcd}
 & (A, B, \mu) \\
(W \cap U, W, \delta|_W) \arrow[ur, "\phi|_W"] \arrow[dr, "\phi|_W"'] & \\
 & (F \cap A, F, \mu|_F) \arrow[uu, "r"']
\end{tikzcd}$$

and we conclude as in the proof of Lemma 3.1. □

The following lemma completes the proof of Theorem 1.

Lemma 3.4. *Let S be a scheme over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2$ and let X/S be a superscheme. Then for all $i \geq 0$ there is an isomorphism*

$$\mathrm{H}^i_{\mathrm{s-cris}}(X/S) \cong \mathrm{H}^i_{\mathrm{s-cris}}(X_{\mathrm{even}}/S)$$

Proof. Let

$$(i_*, i^*) : (X_{\mathrm{even}}/S)_{\mathrm{s-cris}} \longrightarrow (X/S)_{\mathrm{s-cris}}$$

be the morphism of topoi constructed in the previous lemma. Note that

$$i_*(\mathcal{F})_{(U,T,\delta)} = \mathcal{F}_{(U_{\mathrm{even}},T,\delta)}$$

and note that i_* is an exact functor. Also:

$$\begin{aligned}
i_*(\mathcal{O}^{\mathrm{super}}_{X_{\mathrm{even}}/S})((U, T, \delta)) &= \mathcal{O}^{\mathrm{super}}_{X_{\mathrm{even}}/S}((U_{\mathrm{even}}, T, \delta)) \\
&= \mathcal{O}_T(T) \\
&= \mathcal{O}^{\mathrm{super}}_{X/S}((U, T, \delta))
\end{aligned}$$

and therefore $i_*(\mathcal{O}^{\mathrm{super}}_{X_{\mathrm{even}}/S}) = \mathcal{O}^{\mathrm{super}}_{X/S}$. By using the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\mathrm{s-cris}}(X, \mathrm{R}^q i_* \mathcal{O}^{\mathrm{super}}_{X_{\mathrm{even}}/S}) \Longrightarrow \mathrm{H}^{p+q}_{\mathrm{s-cris}}(X_{\mathrm{even}}, \mathcal{O}^{\mathrm{super}}_{X_{\mathrm{even}}/S})$$

and the exactness of i_* it follows that $\mathrm{H}^i_{\mathrm{s-cris}}(X) \cong \mathrm{H}^i_{\mathrm{s-cris}}(X_{\mathrm{even}})$. □

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M. Luu, Department of Mathematics, University of California, Davis
E-mail address: mluu@math.ucdavis.edu