Deformation Theory and Local-Global Compatibility of Langlands Correspondences

Martin T. Luu

Abstract

The deformation theory of automorphic representations is used to study local properties of Galois representations associated to automorphic representations of general linear groups and symplectic groups. In some cases this allows to identify the local Galois representations with representations predicted by a local Langlands correspondence.

Contents

1 Preface

Parts of this memoir are based on my Ph.D. thesis written at Princeton University under the direction of professor Christopher Skinner. I am deeply indebted to professor Skinner for sharing numerous mathematical ideas with me, many of which are used in this work. I also thank him for many discussions of the mathematics treated in this memoir. It is also my great pleasure to thank B. Conrad, S. Dasgupta, A. Jorza, C. Sorensen for conversations and correspondences that have been very helpful. Moreover, I thank the referee for remarks concerning the improvement of the exposition.

2 Introduction

Let τ be in the complex upper half-plane and let $q = \exp(2\pi i\tau)$. Fix a prime number ℓ. In the year 1954 Martin Eichler associated to the modular form

$$f = \sum_{n\geq 1} a_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots$$

of weight 2 and level $\Gamma_0(11)$ a very different type of object: A group representation

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that for all primes $p \notin \{11, \ell\}$ the representation $\rho_{f,\ell}$ is unramified at p and

$$\mathrm{Trace}\ \rho_{f,\ell}(\mathrm{Fr}_p) = a_p$$

and

$$\det \rho_{f,\ell}(\mathrm{Fr}_p) = p$$

where Fr_p denotes a lift of Frobenius at p. A priori, this association makes no connection between the local properties at $p = 11$ of the modular form and the group representation. It is then natural to attempt to strengthen Eichler's correspondence between the analytic object f and the algebraic object $\rho_{f,\ell}$, and many results for generalizations of Eichler's correspondence to more general settings have been obtained in the last half a century in this direction. Such questions are now often called the problem of local-global compatibility of Langlands correspondences. The current memoir is meant to contribute to these ongoing efforts.

Traditionally, the main tool in this area is the detailed study of the geometry of Shimura varieties. This should not be too surprising: The very construction by Eichler of $\rho_{f,\ell}$ uses modular curves. For modular forms a reference for the traditional approach to local-global compatibility is [LAN], for Hilbert modular forms there is for example [CAR], and a more recent work in this tradition is the work [HT] by Harris-Taylor.

The methods of our work are different, we use the deformation theory of automorphic forms and Galois representations instead: The description of f above as an element of $\mathbb{Z}[[q]]$ lends itself to an obvious notion of congruences to other such formal power series in q. In many situations one knows the existence of an abundance of such congruences to other modular forms. We will exploit this to develop an approach to local-global compatibility questions that largely avoids the study of bad reduction of Shimura varieties. Phrased in terms of Eichler's modular form f we would like to advertise the following maxim: The local behavior at 11 of the associated Galois representation is governed in a direct way by unramified local behavior of modular forms, not necessarily of the individual modular form f itself but of families of modular forms deforming f. This approach is very different than the standard approach to such problems via studying singularities of Shimura varieties and therefore in particular is useful in the following situations:

(i) there is no variety to work with in the first place

(ii) the singularities of the relevant Shimura varieties are not sufficiently well understood

Examples for these two situations that we treat in this memoir are:

(i) Hilbert modular forms of partial weight one

(ii) Automorphic forms on symplectic groups

Let us come back to Eichler's modular form f. To study the behavior of $\rho_{f,\ell}$ at $p = 11$ we look at the corresponding Weil-Deligne representation (r, N). See for example [TAT] for a definition. We refer to r as the semi-simple part and to N as the monodromy operator. We use the following methods to study r and N as well as their analogues for more general automorphic Galois representations:

To study r we use the vertical deformation theory of automorphic forms, namely eigenvarieties and associated families of Galois representations. Our approach to calculating the local monodromy operator N is given by the following principle:

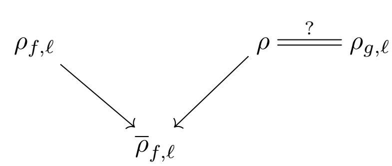
$$\left.\begin{array}{l}\text{automorphic congruences}\\ \quad + \\ \text{modularity lifting theorems}\end{array}\right\}\ \text{automorphic monodromy operators}$$

We now illustrate these approaches to local-global compatibility in some examples. The examples hopefully demonstrate the usefulness of the deformation theoretic approach to local-global compatibility questions concerning Galois representations whose local properties are not easily detectable by standard approaches via singularities of Shimura varieties. On a more philosophical level they might illustrate that ramified local-global compatibility can sometimes be deduced from unramified local-global compatibility in pleasing accordance with the way unramified properties often abstractly determine the Galois representation via the Chebotarev density theorem.

Calculating N for Eichler's modular form:

Let us explain our approach to monodromy operators in the case of Eichler's modular form f. The space $S_2(\mathrm{SL}_2(\mathbb{Z}))$ sitting inside $S_2(\Gamma_0(11))$ is actually trivial. A reflection of this fact for the Galois representation $\rho_{f,\ell}$ is that one expects the inertia action at 11 to be non-trivial and unipotent and this has been known for many decades. However, analogous questions in higher dimensions can be much more difficult, hence we now sketch in this simple situation of Eichler's modular form a deformation theoretic approach to studying the ramification:

The key are the revolutionary techniques developed by Wiles [WIL] in his proof of Fermat's last theorem, namely modularity lifting results. The idea of modularity lifting theorems can be described schematically as:



By this we mean that ρ is a representation whose reduction is isomorphic to the reduction $\overline{\rho}_{f,\ell}$ of $\rho_{f,\ell}$ and we want to study if ρ is equal to $\rho_{g,\ell}$ for a suitable modular form g. With this in mind our idea to prove the above described form of the inertia action of $\rho_{f,\ell}$ at 11 is an approach that can schematically be described by the following diagram:

$$\begin{array}{ccc} R & \xrightarrow{\sim} & \mathbb{T}(11) \\ \downarrow & & \downarrow \\ R(11\text{-unr}) & \xrightarrow{\sim} & \mathbb{T}(1) \end{array}$$

Here R parametrizes suitable deformations of the residual representation $\overline{\rho}_{f,\ell}$ and $R(11\text{-unr})$ parametrizes the subclass of deformations unramified at 11, $\mathbb{T}(11)$ denotes a Hecke algebra corresponding to certain modular forms whose level divides 11 and $\mathbb{T}(1)$ corresponds to the subclass of modular forms of level 1. However, since $S_2(\mathrm{SL}_2(\mathbb{Z}))$ is trivial, the above diagram can not be used directly. This is similar to a problem Wiles faced in the proof of Fermat's last theorem: When working with so called minimal deformation problems it can be non-trivial to show that the relevant Hecke algebras are non-empty. In the work of Wiles this is solved by using level-lowering theorems. In our case, we can use the potential level-lowering results of Skinner and Wiles to bypass the above described problem: In many situations one can find congruences to less ramified modular forms after a suitable base change. An added advantage of the Skinner-Wiles results for our aims is that it avoids the study of singularities of Shimura varieties that is employed to prove more classical level-lowering results.

Since modularity lifting theorems are known in many situations, variants of the above described arguments can also be used to study monodromy operators in situations not easily accessible via more standard geometric methods. Hilbert modular forms of partial weight one are one such example which we treat in the current memoir. Since the Galois representations for such modular forms are constructed by using families of modular forms one loses control of the monodromy operators at places of Steinberg ramification. We demonstrate how the deformation theory can be used to calculate the monodromy operators nonetheless in certain situations.

Calculating r (and consequences for N) in a symplectic example:

We now discuss an example of our approach to the semi-simple parts of Weil-Deligne representations associated to automorphic Galois representations. In the case of modular forms we have indicated earlier for the example of Eichler's modular form how the horizontal deformation theory, by which we mean the theory of modularity lifting, can be used to study N. To study r we use the theory of eigenvarieties: Under suitable assumptions many automorphic representations can be put into p-adic families and there are corresponding families of Galois representations. It follows from the important work of Kisin in [KIS] that for such families one often has a variation of crystalline periods. We use this to obtain the matching of r with what is predicted by the local Langlands correspondence in some cases. For this method we study p-adic families of Galois representations and we study the local properties of the Galois representations at p. If independence of ℓ results are known then one can safely restrict to this case of critical characteristic. The control of the variation of only one crystalline period translates the study of r into a combinatorial problem involving exterior powers of Galois representations and different ways of injecting a suitable local component of π into a principal series representation. We discuss some of the resulting combinatorics, in particular in low dimensions, in this memoir. Note that very recent results on strengthening of Kisin's work towards existence of global triangulations allow to discard the combinatorics in many situations and we briefly discuss this later on.

The motivation for choosing the specific example we are about to discuss stems from the work of Skinner-Urban [SU] on the main conjecture of modular forms. There, it would be useful to know certain local-global compatibility results for automorphic representations of GSp_4. Since the existence of a Langlands transfer to GL_4 is not always known one cannot simply appeal to the well established results for automorphic representations of general linear and unitary groups. Moreover, contrary to the detailed study of the bad reduction of certain unitary group Shimura varieties carried out by Harris-Taylor in [HT], the corresponding understanding of the bad reduction of symplectic Shimura varieties is much less developed. Hence this is a situation where the methods developed in this memoir come into play.

Let F be a totally real number field and let π be a suitable automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ which is in particular cuspidal, algebraic and globally generic. Let $\mathrm{rec}_{\mathrm{GSp}_4}$ denote the local Langlands correspondence for GSp_4 constructed by Gan and Takeda. In the following statement we in fact use it only for unramified representations where it is a classical result.

For ℓ a prime and ι an isomorphism from $\overline{\mathbb{Q}}_\ell$ to $\mathbb{C}$ there exists a unique continuous semi-simple representation

$$\rho_{\pi,\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$$

such that for $v \nmid \ell$ with π_v an unramified principal series representation one has

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-3/2})$$

where c denotes the symplectic similitude character. An analogue of the behavior of Eichler's modular form f at $p = 11$ is to assume that $v \nmid \ell$ is a finite place of F such that

$$\pi_v \hookrightarrow \mathrm{Ind}(\chi_1|\cdot|^{1/2}, \chi_1|\cdot|^{-1/2}; \chi_2)$$

where $|\cdot|$ is normalized absolute value of F_v and χ_1 and χ_2 are unramified characters of $F_v^\times$ such that $\chi_1^2 \notin \{|\cdot|^{\pm 1}, |\cdot|^{\pm 3}\}$ and the right hand side is the normalized induction of the character of the diagonal torus of $\mathrm{GSp}_4(F_v)$ given by

$$t = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & c(t)t_2^{-1} & \\ & & & c(t)t_1^{-1} \end{bmatrix} \rightarrow \chi_1(t_1)|t_1|^{1/2}\chi_1(t_2)|t_2|^{-1/2}\chi_2(c(t))$$

Let (r, N) denote the Weil-Deligne representation associated to $\rho_{\pi,\ell,\iota}$ at the place v. Let ϖ be a uniformizer of F_v, let q be the size of residue field of F_v, let Fr_v be a lift of geometric Frobenius and let S be the multiset of eigenvalues of $r(\mathrm{Fr}_v)$ multiplied by $q^{-3/2}$. Local-global compatibility predicts that

$$S = \{\alpha\beta q^{1/2}, \alpha\beta q^{-1/2}, \alpha^2\beta, \beta\} \text{ where } \alpha := \chi_1(\varpi) \text{ and } \beta := \chi_2(\varpi)$$

By using the vertical deformation theory, by which we mean the theory of eigenvarieties, we show under some assumptions, including an independence of ℓ assumption, the following: Let $1 \leq j \leq 4$ and let

$$\mu = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \mu_3 & \\ & & & \mu_4 \end{bmatrix}$$

be an unramified character of the diagonal torus of $\mathrm{GL}_4(F_v)$ such that

$$\mathrm{St}_2(\chi_1\chi_2) \boxplus \chi_1^2\chi_2 \boxplus \chi_2 \hookrightarrow \mathrm{Ind}_{B_{\mathrm{GL}_4(F_v)}}^{\mathrm{GL}_4(F_v)}(\mu)$$

Then there exists a multiset $\{x_1, \cdots, x_j\}$ contained in S such that

$$\prod_{k=1}^{j} x_k = \prod_{k=1}^{j} \mu_k(\varpi)$$

Remark 1. Assuming an independence of ℓ hypothesis deserves some discussion since it might sound rather strong. However, this type of result has already proven to be useful, for example in the work of Jorza in [JOR]. There, via a multiplicity one result, symplectic local-global compatibility is related to GL_4 local-global compatibility but only up to a quadratic twist. Via the methods developed in the current memoir this twist can be removed, since the independence of ℓ for GL_4 is indeed known.

We now use the above described result for various pairs (μ, j) and the combination of the information obtained from these applications will yield that S is as predicted by the local Langlands correspondence.

- Pair 1:

Here we take

$$(\mu, j) = (\begin{bmatrix} \chi_1\chi_2|\cdot|^{1/2} & & & \\ & \chi_1\chi_2|\cdot|^{-1/2} & & \\ & & \chi_1^2\chi_2 & \\ & & & \chi_2 \end{bmatrix}, 1)$$

It follows that $\alpha\beta q^{-1/2} \in S$

- Pair 2:

Here we take

$$(\mu, j) = (\begin{bmatrix} \chi_2 & & & \\ & \chi_1^2\chi_2 & & \\ & & \chi_1\chi_2|\cdot|^{1/2} & \\ & & & \chi_1\chi_2|\cdot|^{-1/2} \end{bmatrix}, 3)$$

It follows that there is $\{x_1, x_2, x_3\} \subset S$ such that

$$x_1x_2x_3 = \alpha^3\beta^3q^{-1/2}$$

- Pair 3:

Here we take

$$(\mu, j) = (\begin{bmatrix} \chi_2 & & & \\ & \chi_1^2\chi_2 & & \\ & & \chi_1\chi_2|\cdot|^{1/2} & \\ & & & \chi_1\chi_2|\cdot|^{-1/2} \end{bmatrix}, 4)$$

It follows that there is $\{y_1, y_2, y_3, y_4\} = S$ such that

$$y_1y_2y_3y_4 = \alpha^4\beta^4$$

Hence, without loss of generality,

$$y_4 = \alpha\beta q^{1/2} \in S$$

- Pair 4:

Here we take

$$(\mu, j) = (\begin{bmatrix} \chi_2 & & & \\ & \chi_1^2\chi_2 & & \\ & & \chi_1\chi_2|\cdot|^{1/2} & \\ & & & \chi_1\chi_2|\cdot|^{-1/2} \end{bmatrix}, 1)$$

It follows that $\beta \in S$ and since $\alpha^2 \notin \{q^{\pm 1}, q^{\pm 3}\}$ one has $\beta \neq \alpha\beta q^{\pm 1/2}$ and therefore

$$S = \{\alpha\beta q^{-1/2}, \alpha\beta q^{1/2}, \beta, z\}$$

for some number z. Now using the pair $(\mu, 4)$ implies $\alpha^2\beta^3 z = \alpha^4\beta^4$ and hence

$$z = \alpha^2\beta$$

and S is as desired.

Note that this local-global compatibility result concerning r has consequences for what N can be. The analogue of the conjecture of Skinner-Urban in our current situation predicts that the rank of N equals one. In fact, for their applications it is only necessary to show that the rank is at most one and this can be deduced from results of the above type concerning r.

Remark 2. The approach to local-global compatibility that we outlined above actually allows to go beyond the results based on singularities of Shimura varieties. By working directly with symplectic group eigenvarieties one can attack the non-globally generic case as well. Note that the conjecture of Skinner-Urban is now essentially a theorem of Jorza. He proves this by showing a certain strong multiplicity one result for automorphic representations of GSp_4 and using the already known local-global compatibility results for automorphic representations of GL_4. To prove the full compatibility, there was originally a problem with quadratic-twists. After learning of these results we suggested to solve this problem via the methods outlined above and this is now included in [JOR2]. The general deformation theoretic approach that we wish to develop gives another, more self-contained, approach to showing such local-global compatibility results. Since it does not appeal to GL_4 results it should be easier to apply it in more general situations where the understanding of bad reduction of Shimura varieties is less developed.

A more detailed outline of this memoir:

We focus throughout this work on automorphic representations of GL_n and GSp_{2n} and while we usually focus on Iwahori-spherical ramification one can obtain more general results by using base change methods. One should also note that several of the results we prove in this work are not new. Namely, some results are special cases of the local-global compatibility results of [HT] and [TY] and subsequent variations of the methods of these two references. The results in these references are proved by a detailed study of singularities of Shimura varieties and related algebraic varieties, which our approach avoids. Some new results that we prove via the deformation theoretic methods include: Suppose the automorphic Galois representation is not known to be realized in the cohomology of an algebraic variety. To show how deformation theory can deal with this situation, in this case we prove non-triviality results for monodromy operators for Hilbert modular forms of partial weight one.

We also obtain lower bounds on the rank of monodromy operators associated to symplectic automorphic representations. In this symplectic example we show how to bypass the possible lack of strong multiplicity one results by the use of the γ-factors coming from the doubling method. These methods should work more generally. Another difficult case for the standard approach to local-global compatibility is if the automorphic Galois representation is known to be realized in the cohomology of an algebraic variety but a detailed understanding of the singularities of the variety is not known: To show how deformation theory can deal with this situation we prove local-global compatibility results for symplectic automorphic representations. These results not withstanding, the aim of this memoir is not only to present new results: In particular for results concerning automorphic representations of GL_n, the modest aim of much of what we do in the present work is to illustrate the usefulness of the deformation theoretic approach which then can be applied to more difficult situations in the future. We now give a brief outline of the structure of this memoir.

In Chapter 2 we recall some results about Iwahori-spherical representations and p-adic Hodge theory. These results will be used extensively in the applications of families of automorphic forms. In Chapter 3 we give examples for Hilbert modular forms of the way eigenvarieties and modularity lifting theorems can be used to obtain local-global compatibility results for automorphic Galois representations. In Chapter 4 the deformation theoretic approach to local-global compatibility questions is developed in the setting of unitary groups and general linear groups. In Chapter 5 the results of the previous chapter are used to obtain local global compatibility results for automorphic Galois representations of general linear groups. In Chapter 6 the results of the Chapter 4 are used to obtain local global compatibility results fro automorphic Galois representations of symplectic groups. In Chapter 7 we prove potential level-lowering results for general linear groups, unitary groups, and symplectic groups. In Chapter 8 we prove non-triviality results for monodromy operators associated to automorphic Galois representations of general linear groups. In Chapter 9 we develop, via the example of symplectic groups, a variant of the modularity lifting approach to monodromy operators that is developed in Chapter 8 that is based on γ-factors from the doubling method rather than strong multiplicity one results.

3 Preliminaries

Since such results will be used on many occasions throughout this memoir, in this chapter we collect some preliminary results about Iwahori-spherical representations as well as crystalline periods of Galois representations.

3.1 Notation

Fix throughout an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ for each prime p. We will denote by $\mathrm{v}_p(-)$ the valuation on $\overline{\mathbb{Q}}_p$ such that $\mathrm{v}_p(p) = 1$ and let $|\cdot|_p$ denote the corresponding absolute value such that $|p|_p = 1/p$. Throughout this work, by a p-adic field we will mean a finite extension of $\mathbb{Q}_p$ for some rational prime p. For a p-adic field K let $|\cdot|_K$ denote the absolute value normalized such that for a uniformizer ϖ one has $|\varpi|_K = 1/q$ where q denotes the size of the residue field of K. If the context is clear then $|\cdot|_K$ will sometimes simply be denoted by $|\cdot|$.

For a p-adic field K normalize local class field theory so that uniformizers correspond to lifts of geometric Frobenius. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ and let $W_K \subset G_K$ denote the Weil group and for $g \in W_K$ let $\nu(g) \in \mathbb{Z}$ be such that g is a lift of the $\nu(g)$'th power of geometric Frobenius. For a character χ of $K^\times$ we will denote by $\tilde{\chi}$ the character of W_K corresponding to it by local class field theory. Let the maximal absolutely unramified subfield of K be denoted by K_0 and let K_0^{ur} denote its maximal unramified extension.

Suppose F is a number field and $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is a continuous representation. Consider the isomorphism class of the semi-simplification of the residual representation of the representation on a Galois stable lattice coming from ρ. Let $\overline{\rho}$ denote the scalar extension of this representation to the algebraic closure.

For an extension B/A of number fields let $\mathrm{Spl}_{B/A}$ denote the set of finite places of A which split completely in B.

3.2 Weil-Deligne representations

We briefly recall some standard results on Weil-Deligne representations and refer to [BH, Chapter 7] for more details.

Let K be a p-adic field and fix an algebraic closure $\overline{K}$. Let W_K be the Weil group which is a topological group such that the inertia group I_K is open and the topology of I_K agrees with the topology on I_K viewed as a subset of $\mathrm{Gal}(\overline{K}/K)$ equipped with the Krull topology. With this topology the Weil group is a locally profinite topological group. As for any locally profinite topological group, one defines a smooth representation of W_K as follows: Let E be a field of characteristic zero and V an E-vector space. Then a smooth representation is defined to be a homomorphism from W_K to the group of E-linear automorphisms of V such that every $v \in V$ has an open stabilizer.

Let $v_K : K \longrightarrow \mathbb{Z}$ be defined by taking geometric Frobenius elements to 1, let q denote the size of the residue field of K and let $||\sigma|| := q^{-v_K(\sigma)}$ for $\sigma \in W_K$. A Weil-Deligne representation (ρ, N) of W_K over a field E of characteristic zero consists of a smooth representation ρ of W_K on a finite-dimensional E-vector space V and a nilpotent E-linear endomorphism N of V such that

$$\rho(\sigma)N\rho(\sigma)^{-1} = ||\sigma||N$$

for all $\sigma \in W_K$. A morphism between Weil-Deligne representations over E is defined to be an E-linear map between underlying E-vector spaces which commutes with the relevant smooth representations and nilpotent endomorphisms. Let $\text{WD-Rep}_E(W_K)$ denote the category of Weil-Deligne representations of W_K on E-vector spaces. Suppose $\iota : E \longrightarrow E'$ is a field isomorphism of fields of characteristic zero. Consider the corresponding functor from $\text{WD-Rep}_E(W_K)$ to $\text{WD-Rep}_{E'}(W_K)$ which on objects takes a Weil-Deligne representation (ρ, N) with underlying E-vector space V to the Weil-Deligne representation (ρ', N') over E' whose underlying E'-vector space is $V \otimes_E E'$ with

$$\rho'(\sigma)(v \otimes 1) = \rho(\sigma)(v) \otimes 1 \text{ and } N'(v \otimes 1) = N(v) \otimes 1$$

for $v \in V$. This yields an equivalence of categories. In particular, for any choice ι of isomorphism between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ there is an equivalence of categories between $\text{WD-Rep}_{\overline{\mathbb{Q}}_\ell}(W_K)$ and $\text{WD-Rep}_{\mathbb{C}}(W_K)$.

Let p denote the residue characteristic of K and let $\ell \neq p$ be a rational prime. View $\overline{\mathbb{Q}}_\ell$ as a topological space via the ℓ-adic topology and hence consider $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ as a topological space. Let

$$\begin{array}{c}\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}}_\ell}(W_K)\\ \Big\| \\ \text{category of f.d. continuous representations of } W_K \text{ on } \overline{\mathbb{Q}}_\ell\text{-vector spaces}\end{array}$$

Fix a lift $\mathrm{Fr} \in W_K$ of Frobenius and fix a surjection $t : I_K \longrightarrow \mathbb{Z}_\ell$. It is well known, see for example [BH, p.204], that if ρ is a continuous finite-dimensional representation of W_K on a $\overline{\mathbb{Q}}_\ell$-vector space V then there is a unique nilpotent $\overline{\mathbb{Q}}_\ell$-linear endomorphism N_ρ of V such that for all σ in some open subgroup of I_K one has

$$\rho(\sigma) = \exp(t(\sigma)N_\rho)$$

For ρ an object in $\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}}_\ell}(W_K)$ define the homomorphism

$$\rho_{\mathrm{Fr}} : W_K \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

by

$$\rho_{\mathrm{Fr}}(\mathrm{Fr}^i\sigma) := \rho(\mathrm{Fr}^i\sigma)\exp(-t(\sigma)N_\rho)$$

for all $i \in \mathbb{Z}$ and all $\sigma \in I_K$. This is a smooth representation of W_K. It is well known, see for example [BH, p.206], that there is an equivalence of categories between $\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}}_\ell}(W_K)$ and $\mathrm{WD}\text{-}\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_K)$ which takes an object ρ of $\mathrm{Rep}^{\mathrm{f}}_{\overline{\mathbb{Q}}_\ell}(W_K)$ to $(\rho_{\mathrm{Fr}}, N_\rho)$ and that the isomorphism class of $(\rho_{\mathrm{Fr}}, N_\rho)$ is in fact independent of the choice of surjection $t : I_K \longrightarrow \mathbb{Z}_\ell$ and choice of lift of Frobenius Fr.

The typical situation encountered in this work is that F is a number field, ι is an isomorphism from $\overline{\mathbb{Q}}_\ell$ to $\mathbb{C}$ and

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

a continuous homomorphism and v a finite place of F. Hence, via the above described equivalence of categories, one obtains a Weil-Deligne representation of W_{F_v} over $\overline{\mathbb{Q}}_\ell$ and via ι a Weil-Deligne representation of W_{F_v} over $\mathbb{C}$. If $W = (r, N)$ is a Weil-Deligne representation over $\overline{\mathbb{Q}}_\ell$ we will denote by W_ι the corresponding Weil-Deligne representation over $\mathbb{C}$ via ι. We let $W^{\text{F-ss}}$ denote the Frobenius semi-simplification of W and let W^{ss} denote the Frobenius semi-simplification of r.

Let $\rho_1 = (r_1, N_1)$ and $\rho_2 = (r_2, N_2)$ be two Weil-Deligne representations of W_K, with underlying vector spaces V_1 and V_2. The direct sum $\rho_1 \oplus \rho_2 = (r', N')$ of ρ_1 and ρ_2 is defined to have underlying vector-space $V_1 \oplus V_2$ and $r' = r_1 \oplus r_2$ and N' is defined via

$$N'((v_1, v_2)) = (N_1(v_1), N_2(v_2)$$

for $v_1 \in V_1$ and $v_2 \in V_2$. The tensor product $(r, N) := \rho_1 \otimes \rho_2$ of ρ_1 and ρ_2 is defined to be the Weil-Deligne representation with underlying vector space $V_1 \otimes V_2$ and for every $\sigma \in W_K$ one has

$$r(\sigma)(v_1 \otimes v_2) = r_1(\sigma)v_1 \otimes r_2(\sigma)v_2 \text{ and } N(v_1 \otimes v_2) = N_1v_1 \otimes v_2 + v_1 \otimes N_2v_2$$

The i'th exterior product $\wedge^i\rho_1$ of ρ_1 is defined by viewing $\wedge^iV_1$ as a sub-space of the i-fold tensor product $V_1^{\otimes i}$.

3.3 Iwahori-spherical representations

The local components of the automorphic representations to which we will later apply arguments involving families of automorphic representations are of a specific type: They are Iwahori-spherical representations. Hence we now describe now for later use some known results about Iwahori-spherical and closely related types of representations. Let us first fix some notation:

Let K be a p-adic field, let $\mathcal{O}$ be its valuation ring and let k be its residue field. Let $G = \mathrm{GL}_n(K)$ for some $n \geq 2$.

Let B be the upper triangular Borel subgroup of $\mathrm{GL}_n(K)$. Let I denote the Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O})$ associated to B and let I_1 denote the subgroup of I corresponding to unipotent matrices in the reduction modulo the maximal ideal of $\mathcal{O}$. Let T be the diagonal torus of GL_n. Let $\delta_B : B \to \mathbb{C}^\times$ be the modulus character, which takes $b \in B$ to $|\det_b|_K$ where $\det_b$ denotes the determinant of the conjugation action of b on the set $\mathfrak{n}$ of strictly upper triangular matrices in $M_n(K)$.

Definition 1. Let π be an irreducible admissible representation of $\mathrm{GL}_n(K)$.

- Let $\pi^I := \{v \in \pi \big| i \cdot v = v \text{ for all } i \in I\}$
- For any character $\rho : I \to \mathbb{C}^\times$ that is trivial on I_1 define

$$\pi^\rho := \{v \in \pi \;\big|\; i \cdot v = \rho(i)v \;\text{ for all } i \in I\}$$

The representation π is called Iwahori-spherical if $\pi^I \neq (0)$.

Note that automorphic representations whose local components at some finite place satisfy $\pi^\rho \neq (0)$ for some non-trivial character ρ as above will be used in the potential level-lowering results that we prove in Section 8.1.

3.3.1 Hecke action

For a character $\rho : I \to \mathbb{C}^\times$ as in the above definition, define

$$\mathcal{H}(G,\rho) := \{f : G \longrightarrow \mathbb{C} \big| f(xgy) = \rho^{-1}(x)f(g)\rho^{-1}(y) \text{ for all } x,y \in I \text{ and } g \in G\}$$

where all functions are required to be locally constant and with compact support. Fix a Haar measure on G such that I has measure 1. The product of $f_1, f_2 \in \mathcal{H}(G,\rho)$ is defined by

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1})f_2(y)dy$$

The algebras $\mathcal{H}(G,\rho)$ are in general not abelian and we will sometimes work with certain subalgebras which we now describe. Fix a uniformizer ϖ of K and let

$$T^+ := \{ \begin{bmatrix} \varpi^{a_1} & & & \\ & \varpi^{a_2} & & \\ & & \ddots & \\ & & & \varpi^{a_n} \end{bmatrix} \big| a_1, \cdots, a_n \in \mathbb{Z} \text{ and } a_1 \geq a_2 \cdots \geq a_n\}$$

Definition 2. For $t \in T^+$ let

$$\phi_t^\rho \in \mathcal{H}(G,\rho)$$

be the element which has support ItI and which satisfies $\phi_t^\rho(t) = 1$. Let $\mathcal{H}_\rho^+$ denote the subalgebra of $\mathcal{H}(G,\rho)$ generated by ϕ_t^ρ for $t \in T^+$. Note that the elements ϕ_t^ρ as above are invertible as recalled in [HAI, Cor. 5.2.2].

For any irreducible admissible representation π of $\mathrm{GL}_n(K)$ there is an action of $\mathcal{H}(G,\rho)$ on π^ρ given by

$$f \cdot v = \int_G f(y)(y \cdot v)dy$$

for $f \in \mathcal{H}(G,\rho)$ and $v \in \pi^\rho$. The vector space π^ρ with its $\mathcal{H}_\rho^+$-action has an alternative description in terms of the Jacquet module of π which we describe in Theorem 1. First let

$$\xi : T(\mathcal{O}) \longrightarrow \mathbb{C}^\times$$

be the character given by the composition of the natural map $T(\mathcal{O}) \to T(k)$ with the character of $T(k)$ determined by ρ via the isomorphism $I/I_1 \cong T(k)$. Let $J(\pi)$ denote the Jacquet module of π with respect to B and for ξ as above let $J(\pi)^\xi$ denote the subspace of $J(\pi)$ on which $T(\mathcal{O})$ acts through ξ. The following theorem in the case $\xi = 1$ is due to Borel and Casselman and a proof in the case where ξ is non-trivial can be found in [HAI, Prop. 6.0.1]:

Theorem 1. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(K)$. There is a $T(\mathcal{O})$-equivariant isomorphism of $\mathbb{C}$-vector spaces*

$$\pi^\rho \cong J(\pi)^\xi \otimes \delta_B^{-1}$$

Furthermore, this isomorphism is T^+-equivariant where $t \in T^+$ acts on π^ρ via ϕ_t^ρ and on $J(\pi)^\xi \otimes \delta_B^{-1}$ via the $T(K)$-action.

Note that the T^+-equivariance follows from [HAI, Lem. 6.0.3, Lem. 6.0.4] and [CAS, Lem. 1.5.1]. This equivariance, both for the case where ρ is trivial and where it is non-trivial, will be used in Section 8.1 to obtain certain residual local-global compatibility results for automorphic Galois representations.

3.3.2 Refinements

We now describe some known results about realizing Iwahori-spherical representations as sub-representations of principal series representations. This leads to the notion of accessible refinement of an Iwahori-spherical representation and the notion of accessible refinement is crucial for our intended application to local-global compatibility questions: Different accessible refinements of relevant local components of an automorphic representation, if they exist, correspond in general to different families deforming the automorphic representations. In the applications to Hilbert modular forms in Chapter 4 the phenomenon of multiple accessible refinements does not yet play a role but it becomes crucial in the higher dimensional cases treated later on.

Let χ be a smooth character of $T(K)$. For $1 \leq i \leq n$ define $\chi_i : K^\times \to \mathbb{C}^\times$ by

$$x \mapsto \chi(\mathrm{diag}(1, \cdots, 1, x, 1, \cdots, 1))$$

where x is at the i'th entry. For the characters χ_i as above we will write

$$\chi = \prod_{i=1}^{n} \chi_i$$

Throughout this work we identify the principal series representation associated to χ as

$$\mathrm{Ind}_B^G(\chi) =$$

$$\{f : G \to \mathbb{C} \big| f(bg) = \delta_B(b)^{1/2}\chi(b)f(g) \text{ for all } b \in B \text{ and } g \in G \text{ and } f \text{ smooth } \}$$

This representation will also be denoted by $\mathrm{Ind}(\chi)$ as well as $\mathrm{Ind}(\chi_1, \cdots, \chi_n)$. As in [CHE] we make the following definition:

Definition 3. Let π be an irreducible admissible representation of $\mathrm{GL}_n(K)$ which is Iwahori-spherical. A smooth unramified character $\chi : T(K) \to \mathbb{C}^\times$ such that

$$\pi \hookrightarrow \mathrm{Ind}(\chi)$$

is called an accessible refinement of π.

One can deduce from Theorem 1 that if π is as in the above definition, then it has an accessible refinement. In Theorem 2 we will describe in more detail such accessible refinements in the case where π is generic.

Definition 4. Let $\chi : K^\times \to \mathbb{C}^\times$ be a smooth character. By [ZEL, Thm. 6.1 (a)] the representation

$$\mathrm{Ind}(\chi|\cdot|^{\frac{n-1}{2}}, \chi|\cdot|^{\frac{n-3}{2}}, \cdots, \chi|\cdot|^{\frac{1-n}{2}})$$

of $\mathrm{GL}_n(K)$ has a unique irreducible subrepresentation which we will denote by $\mathrm{St}_n(\chi)$.

For a partition $n = \sum_{i=1}^r n_i$ let $\mathrm{P}_{n_1,\cdots,n_r}$ be the corresponding standard parabolic subgroup of $\mathrm{GL}_n(K)$ given by matrices of the form

$$A = \begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & * & \vdots \\ \vdots & 0 & \ddots & * \\ 0 & \cdots & 0 & A_r \end{bmatrix}$$

for some $A_j \in \mathrm{GL}_{n_j}(K)$. Suppose now that π is an irreducible admissible representation of $\mathrm{GL}_n(K)$ which is Iwahori-spherical and generic. It then follows from [ZEL, Thm. 9.7(b)] that there exist numbers $n_1, \cdots, n_r$ in $\mathbb{Z}^{\geq 1}$ and smooth characters $\chi_1, \cdots, \chi_r$ of the form $K^\times \to \mathbb{C}^\times$ such that $n = \sum_{i=1}^r n_i$ and

$$\pi \cong \mathrm{Ind}_{\mathrm{P}_{n_1,\cdots,n_r}}^{\mathrm{GL}_n}(\mathrm{St}_{n_1}(\chi_1), \cdots, \mathrm{St}_{n_r}(\chi_r))$$

and such that the corresponding segments are unlinked in the sense of [ZEL, Sect. 4.1]. Here the induction is normalized induction.We will denote the above representation by

$$\mathrm{St}_{n_1}(\chi_1) \boxplus \cdots \boxplus \mathrm{St}_{n_r}(\chi_r)$$

For such representations one has:

Theorem 2 (Bernstein-Zelevinsky)**.** *Assume π is an irreducible admissible Iwahori-spherical generic representation of $\mathrm{GL}_n(K)$ which is isomorphic to a representation of the form $\mathrm{St}_{n_1}(\chi_1) \boxplus \cdots \boxplus \mathrm{St}_{n_r}(\chi_r)$ where $\sum_{i=1}^r n_i = n$. Then*

$$\pi \hookrightarrow \mathrm{Ind}_{P_{1,\cdots,1}}^{\mathrm{GL}_n}(\mu_1, \cdots, \mu_n)$$

if there is a permutation w in the symmetric group S_r such that there is an equality of ordered n-tuples

$$(\mu_1, \cdots, \mu_n) =$$

$$(\chi_{w(1)}|\cdot|^{\frac{n_{w(1)}-1}{2}}, \cdots, \chi_{w(1)}|\cdot|^{\frac{1-n_{w(1)}}{2}}, \cdots, \chi_{w(r)}|\cdot|^{\frac{n_{w(r)}-1}{2}}, \cdots, \chi_{w(r)}|\cdot|^{\frac{1-n_{w(r)}}{2}})$$

The theorem is a special case of [ZEL, Thm. 1.2] as we will now explain. Let us fix the following notation:

Definition 5. For a sub-partition $n = m_1 + \cdots + m_s$ of a partition

$$n = n_1 + \cdots + n_r$$

and for a collection of irreducible admissible representations π_i of $\mathrm{GL}_{n_i}(K)$ let $\otimes_{i=1}^r \pi_i$ denote the corresponding representation of $\prod_{i=1}^r \mathrm{GL}_{n_i}(K)$ and let

$$\mathrm{J}_{(n_1,\cdots,n_r)}^{(m_1,\cdots,m_s)}(\otimes_{i=1}^r \pi_i)$$

denote the Jacquet module of the representation with respect to $\mathrm{P}_{m_1,\cdots,m_s}$. Denote by

$$\mathrm{J}_{(n_1,\cdots,n_r)}^{(m_1,\cdots,m_s)}(-)_{\mathrm{BZ}}$$

the corresponding Jacquet module with the action normalized as in [BZ, Sect. 1.8]. For ease of notation let $J(-) := \mathrm{J}_{(n)}^{(1,\cdots,1)}(-)$ and note that

$$J(-) \cong \delta_B^{1/2} \otimes J(-)_{\mathrm{BZ}}$$

Fix π as in the statement of Theorem 2 and for such a representation we say that two numbers $1 \leq i, j \leq n$ belong to the same block if there is $k \in \mathbb{Z}$ such that

$$n_1 + \cdots + n_k < i, j \leq n_1 + \cdots + n_{k+1}$$

Let $W = S_n$ be the symmetric group and consider it as a subgroup of $\mathrm{GL}_n(K)$ by taking $w \in W$ to the matrix, also denoted by w, whose (i,j)'th entry is $\delta_{i,w(j)}$. For a character λ of $T(K)$ let $w \circ \lambda$ be the character of $T(K)$ defined by

$$x \to \lambda(w^{-1}xw)$$

for all $x \in T(K)$. Let

$$W' = \{w \in W \big| w(i) < w(j) \text{ whenever } i < j \text{ and } i,j \text{ belong to the same block}\}$$

By applying [ZEL, Thm. 1.2, Prop. 1.5, Sect. 9.1] one obtains

$$\begin{aligned} J(\pi)^{ss}_{\mathrm{BZ}} &\cong \bigoplus_{w \in W'} [w \circ \mathrm{J}^{(1,\cdots,1)}_{(n_1,\cdots,n_r)}(\otimes_i \mathrm{St}_{n_i}(\chi_i))_{\mathrm{BZ}}] \\ &\cong \bigoplus_{w \in W'} [w \circ (\otimes_i \mathrm{J}^{(1,\cdots,1)}_{(n_i)}(\mathrm{St}_{n_i}(\chi_i))_{\mathrm{BZ}}] \\ &\cong \bigoplus_{w \in W'} [w \circ (\otimes_i (\otimes_{k=0}^{n_i-1} \chi_i | \cdot |^{\frac{n_i-1}{2}-k}))] \end{aligned}$$

And it follows from [ZEL, Cor. 1.3] that $J(\pi)_{\mathrm{BZ}}$ has $\otimes_i(\otimes_{k=0}^{n_i-1}\chi_i|\cdot|^{\frac{n_i-1}{2}-k}))$ as a quotient. Furthermore, since π is generic it follows that for any permutation $w \in S_r$ one has

$$\pi \cong \mathrm{St}_{n_{w(1)}}(\chi_{w(1)}) \boxplus \cdots \boxplus \mathrm{St}_{n_{w(r)}}(\chi_{w(r)})$$

By [CAS, Thm. 3.2.4] one has

$$\mathrm{Hom}_{\mathrm{GL}_n(K)}(\pi, \mathrm{Ind}(\chi)) = \mathrm{Hom}_{T(K)}(J(\pi), \chi \otimes \delta_B^{1/2})$$

Hence Theorem 2 follows.

3.4 Crystalline periods

Proving the existence of certain crystalline periods of automorphic Galois representations and their exterior powers plays a key role in our deformation theoretic approach to local-global compatibility. In this section we recall relevant notions from p-adic Hodge theory and collect useful results concerning crystalline periods for later use.

3.4.1 Preliminaries

Let $\mathrm{B}_{\mathrm{cris}}$, $\mathrm{B}^+_{\mathrm{cris}}$, B_{st}, B_{dR} and $\mathrm{B}^+_{\mathrm{dR}}$ be the period rings of p-adic Hodge theory as defined in [FON1]. Let $t \in \mathrm{B}^+_{\mathrm{dR}}$ be a generator of the maximal ideal of $\mathrm{B}^+_{\mathrm{dR}}$ that is a period for the cyclotomic character. Let φ denote the crystalline Frobenius as defined in [FON1, Sect. 2] and let $N : \mathrm{B}_{\mathrm{st}} \to \mathrm{B}_{\mathrm{st}}$ be as defined in [FON1, Sect. 3]. Fix a finite extension $K/\mathbb{Q}_p$, let $q = p^f$ denote the size of the residue field of K and let $\varphi_K := \varphi^f$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on the period rings mentioned above and this action commutes with φ and N. For a finite dimensional $\overline{\mathbb{Q}}_p$-vector space V with continuous G_K-action define

- $$\mathrm{D}_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}})^{\mathrm{Gal}(\overline{K}/K)}$$
- $$\mathrm{D}_{\mathrm{st}}(V) = (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}})^{\mathrm{Gal}(\overline{K}/K)}$$
- $$\mathrm{D}_{\mathrm{pst}}(V) = \cup_{L/K \text{ s.t. } L \subset \overline{K} \text{ and } [L:K]<\infty} (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}})^{\mathrm{Gal}(\overline{K}/L)}$$

-

$$\mathrm{D}_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}})^{\mathrm{Gal}(\overline{K}/K)}$$

The representation V is called crystalline if $\mathrm{D}_{\mathrm{cris}}(V)$ is a free $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_0$-module of rank $\dim_{\overline{\mathbb{Q}}_p}(V)$. One similarly defines what it means for V to be semi-stable, potentially semi-stable or de Rham.

The action of W_K on $\mathrm{D}_{\mathrm{pst}}(V)$ is $\overline{\mathbb{Q}}_p$-linear and K_0^{ur}-semi linear, where K_0^{ur} is as defined in the beginning of the current chapter. A $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_0^{\mathrm{ur}}$-linear action of W_K on $\mathrm{D}_{\mathrm{pst}}(V)$ is defined in [FON3] with $g \in W_K$ acting on $v \in \mathrm{D}_{\mathrm{pst}}(V)$ by

$$v \to g \cdot \varphi^{\nu(g)} \cdot v$$

For any embedding $K_0^{\mathrm{ur}} \hookrightarrow \overline{\mathbb{Q}}_p$ the previous construction together with the action of N defines a Weil-Deligne representation over $\overline{\mathbb{Q}}_p$. The isomorphism class of this is independent of the embedding and we will denote it by $\mathrm{WD}(\rho)$.

If ρ is a Hodge-Tate representation of $\mathrm{Gal}(\overline{K}/K)$ then for any embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ let $\mathrm{HT}_\tau(\rho)$ denote the corresponding multiset of Hodge-Tate weights. If ρ is de Rham these are the opposites of the jumps in the filtration on $\mathrm{D}_{\mathrm{dR}}(\rho)$ coming from the filtration of B_{dR}. For use in later chapters let us also fix the following notation. Suppose all elements of $\mathrm{HT}_\tau(\rho)$ are in $\mathbb{Z}$. Order the elements of $\mathrm{HT}_\tau(\rho)$ in increasing order and for $i \in \mathbb{Z}^{\geq 1}$ let $\mathrm{HT}_\tau^{(i)}(\rho)$ denote the i'th element of this ordered set in the sense that for example $\mathrm{HT}_\tau^{(1)}(\rho)$ denotes a smallest element of $\mathrm{HT}_\tau(\rho)$.

Let us recall some definitions from [FON2]. Let σ be the absolute Frobenius of K_0. A (φ, N)-module is a K_0-vector space D with a σ semi-linear injective map $\varphi : D \longrightarrow D$ and a K_0-linear endomorphism N of D such that

$$N\varphi = p\varphi N$$

The dimension of a (φ, N)-module is the K_0-dimension of D. Let $\underline{\mathrm{Mod}}(\varphi, N)$ denote the category whose objects are finite dimensional (φ, N)-modules. Let D_1 and D_2 be two objects of $\underline{\mathrm{Mod}}(\varphi, N)$ whose underlying K_0-vector spaces we denote by V_1 and V_2. Then the tensor product $D_1 \otimes D_2$ is defined to be the object in $\underline{\mathrm{Mod}}(\varphi, N)$ whose underlying K_0-vector space is $V_1 \otimes_{K_0} V_2$ and $\varphi(d_1 \otimes d_2) := \varphi d_1 \otimes \varphi d_2$ and

$$N(d_1 \otimes d_2) := N d_1 \otimes d_2 + d_1 \otimes N d_2$$

for $d_1 \in V_1$ and $d_2 \in V_2$.

A filtered (φ, N)-module is (φ, N)-module D with a decreasing, exhaustive and separated $\mathbb{Z}$-indexed filtration on $D_K := D \otimes_{K_0} K$. Let $\mathrm{MF}_K(\varphi, N)$ denote the category whose objects are filtered (φ, N)-modules whose underlying K_0-vector space is finite dimensional and a morphism between two such objects D_1 and D_2 is a morphism η of underlying (φ, N)-modules whose extension of scalars η_K to a map $D_{1,K} \longrightarrow D_{2,K}$ satisfies $\eta_K(\mathrm{Fil}^i(D_{1,K})) \subseteq \mathrm{Fil}^i(D_{2,K})$ for all $i \in \mathbb{Z}$. Let D_1 and D_2 be two objects in $\mathrm{MF}_K(\varphi, N)$. Their tensor product $D_1 \otimes D_2$ is defined to be the object in $\mathrm{MF}_K(\varphi, N)$ whose underlying (φ, N)-module is the tensor product of the underlying (φ, N)-modules of D_1 and D_2 and whose filtration is given by

$$\mathrm{Fil}^i(D_1 \otimes D_2)_K := \sum_{j+k=i} \mathrm{Fil}^j D_{1,K} \otimes \mathrm{Fil}^k D_{2,K}$$

Hence one can define exterior products: Let D be an object in $\mathrm{MF}_K(\varphi, N)$ of K_0-dimension equal to r and with underlying K_0-vector space V. For $1 \leq i \leq r$ the exterior product $\wedge^i V$ is a sub-object of the i-fold tensor product of $V \otimes_{K_0} \cdots \otimes_{K_0} V$. One hence can view it as an object in $\mathrm{MF}_K(\varphi, N)$ which will be denoted by $\wedge^i D$.

Let D be an object of $\mathrm{MF}_K(\varphi, N)$ and let r be $\dim_{K_0} D$ and consider the object $\wedge^r D \in \underline{\mathrm{Mod}}(\varphi, N)$ whose K_0-dimension is 1. Let $d \in \wedge^r D$ be such that $\varphi d = \lambda d$ and define $t_N(D) := v_{\mathrm{p}}(\lambda)$ which is well defined. Define $t_H(D)$ to be the largest integer i such that $\mathrm{Fil}^i((\wedge^r D)_K) \neq (0)$. An object D in $\mathrm{MF}_K(\varphi, N)$ is called weakly admissible if

$$t_N(D) = t_H(D) \text{ and } t_N(D') \geq t_H(D')$$

for all sub-objects, see [FON2] (Section 4.3.3), D' of D.

3.4.2 Rigid geometry

We recall some important notions from rigid geometry which will be used in describing families of automorphic representations as well as associated families of Galois representations. For more details we refer to [BGR].

Let K be a field with a non-trivial complete non-archimedean valuation. Then a K-algebra with a complete K-algebra norm is said to be a K-Banach algebra. For $n = 0$ let $T_0 = K$ and for $n \in \mathbb{Z}^{\geq 1}$ and $X_1, \cdots, X_n$ indeterminates consider the K-algebra

$$T_n := K\langle X_1, \cdots, X_n\rangle = \{\sum c_{i_1,\cdots,i_n} X_1^{i_1} \cdots X_n^{i_n} \big| c_{i_1,\cdots,i_n} \in K \text{ and } \lim c_{i_1,\cdots,i_n} = 0\}$$

An affinoid K-algebra $\mathcal{R}$ is a K-Banach algebra such that there exists a continuous epimorphism $T_n \longrightarrow \mathcal{R}$ for some $n \geq 0$. For an affinoid K-algebra let $\mathrm{Sp}(\mathcal{R})$ denote the set of maximal ideals of $\mathcal{R}$. A locally G-ringed space over K consists of a G-topological space X and an associated sheaf $\mathcal{O}_X$ of rings which are K-algebras and such that the stalk of $\mathcal{O}_X$ at every $x \in X$ is a local ring. A rigid space X over K is a locally G-ringed space over K that locally looks like $\mathrm{Sp}(\mathcal{R})$ for an affinoid K-algebra $\mathcal{R}$ in a sense made precise in [BGR] (Section 9.3). Define $\mathcal{O}_X$.

Suppose $\mathcal{R}$ is a Tate algebra and $U \subseteq \mathrm{Sp}(\mathcal{R})$. Then U is called Zariski dense if U equals W^{an} for a Zariski open and scheme-theoretically dense $W \subseteq \mathrm{Spec}(\mathcal{R})$. Here W^{an} denotes the analytification of W as defined in [BGR, Section 9.3.4].

3.4.3 Variation of crystalline periods

In later chapters, to study an automorphic Galois representation we will work with all its exterior powers and the following lemma will turn out to be useful:

Lemma 3.5. *Let K be a p-adic field and let V be an n-dimensional $\overline{\mathbb{Q}}_p$-vector space which is a potentially semi-stable representation of $\mathrm{Gal}(\overline{K}/K)$. Suppose $\alpha \in \overline{\mathbb{Q}}_p$ is such that*

$$\mathrm{D}_{\mathrm{cris}}(\wedge^i V)^{\varphi_K = \alpha} \neq (0)$$

for some $1 \leq i \leq n$. For $\sigma \in W_K$ let S_σ denote the set of eigenvalues of σ acting on $\mathrm{WD}(\rho)$. Then if $\nu(\sigma) = [K_0 : \mathbb{Q}_p]$ there exists a subset $\{\alpha_1, \cdots, \alpha_i\} \subseteq S_\sigma$ such that

$$\prod_{j=1}^{i} \alpha_j = \alpha$$

Proof. If $\mathrm{D}_{\mathrm{cris}}(\wedge^i V)^{\varphi_K = \alpha} \neq (0)$ then also

$$\mathrm{D}_{\mathrm{pst}}(\wedge^i V)^{G_K = 1, N = 0, \varphi_K = \alpha} \neq (0)$$

Hence, for any $\sigma \in W_K$ with $\nu(\sigma) = [K_0 : \mathbb{Q}_p]$ the action on $\mathrm{WD}(\wedge^i V)$ has α as an eigenvalue. Since V is potentially semi-stable so is $\wedge^i V$ and

$$\mathrm{WD}(\wedge^i V) \cong \wedge^i \mathrm{WD}(V)$$

Hence there is a subset $\{\alpha_1, \cdots, \alpha_i\} \subseteq S_\sigma$ such that $\prod_{j=1}^{i} \alpha_j = \alpha$. ☐

When we apply this lemma in later chapters, the existence of the relevant crystalline periods will be obtained via families of automorphic Galois representations and Kisin's results on variation of crystalline periods in such families. We now describe the work of Kisin in more detail.

Let E be a finite extension of $\mathbb{Q}_p$ contained in $\overline{K}$ and containing the Galois closure of K in $\overline{K}$. Let $\mathcal{R}$ be an affinoid E-algebra and M a finite free $\mathcal{R}$-module with a continuous $\mathcal{R}$-linear G_K-action. As discussed for example in [NAK, Sect. 3.2] let

$$P_M(T) \in (K \otimes_{\mathbb{Q}_p} \mathcal{R})[T]$$

denote the Sen polynomial and note that via the isomorphism $K \otimes_{\mathbb{Q}_p} E \cong \oplus_{\sigma: K \hookrightarrow \overline{K}} E$ it factors as

$$(P_M(T)_\sigma)_\sigma \in \oplus_\sigma \mathcal{R}[T]$$

In case that $P_M(T)_\sigma(0)$ vanishes for all σ we will write that $P_M(T) = T \cdot Q(T)$ for some $Q(T) \in (K \otimes_{\mathbb{Q}_p} \mathcal{R})[T]$. For the statement of the next theorem note that if X and X' are rigid analytic spaces over E and $Y \in \mathcal{O}(X)^\times$ then in [KIS, Sect. 5.2] it is defined what it means for a map of analytic spaces $f : X' \to X$ to be Y-small: Namely, there exists a finite extension E'/E and $\lambda \in \mathcal{O}_{X' \otimes_E E'}(X' \otimes_E E')^\times$ such that $E'(\lambda)$ is a product of finite field extensions of E and $Y\lambda^{-1} - 1$ is topologically nilpotent on $X' \otimes_E E'$. For the following theorem see [KIS, Cor. 5.15] and [NAK, Prop. 3.14]:

Theorem 3 (Kisin-Nakamura)**.** *Let K and E be as above. Let $\mathcal{R}$ be an affinoid E-algebra and M a finite-free $\mathcal{R}$-module with a continuous G_K-action. Let $Y \in \mathcal{R}^\times$ and assume that the identity map $\mathrm{Sp}(\mathcal{R}) \to \mathrm{Sp}(\mathcal{R})$ is Y-small. Assume that*

$$P_M(T) = T \cdot Q(T) \in (K \otimes_{\mathbb{Q}_p} \mathcal{R})[T]$$

for some

$$Q(T) \in (K \otimes_{\mathbb{Q}_p} \mathcal{R})[T]$$

Let $\{\mathcal{R}_i\}_{i \in I}$ be a collection of affinoid E-algebras which are $\mathcal{R}$-algebras and let Y_i denote the image of Y in $\mathcal{R}_i$. Assume for all $k \in \mathbb{Z}^{\geq 1}$ there exists $I_k \subset I$ such that

- *For every $i \in I_k$ the natural map gives rise to an isomorphism*

$$(\mathrm{B}^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} (M \otimes_{\mathcal{R}} \mathcal{R}_i))^{G_K, \varphi_K = Y_i} \otimes_{K_0} K \xrightarrow{\sim} (\mathrm{B}^+_{\mathrm{dR}}/t^k \mathrm{B}^+_{\mathrm{dR}} \widehat{\otimes}_{\mathbb{Q}_p} (M \otimes_{\mathcal{R}} \mathcal{R}_i))^{G_K}$$

- *For every $i \in I_k$ the image of $\prod_{j=0}^{k-1} Q(-j)$ in $\mathcal{R}_i$ is a unit*
- *The map*

$$\mathcal{R} \to \prod_{i \in I_k} \mathcal{R}_i$$

is injective

Let $E \subset \mathbb{C}_p$ be a closed subfield and $f : \mathcal{R} \to E$ a continuous map. Then

$$(\mathrm{B}^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} (M \otimes_{\mathcal{R}, f} E))^{G_K, \varphi_K = f(Y)} \neq (0)$$

To verify the first assumption of Theorem 3 we will later use the following standard lemma:

Lemma 3.6. *Let $k \in \mathbb{Z}^{\geq 1}$ be an integer. Suppose M is an n-dimensional $\overline{\mathbb{Q}}_p$-vector space with a continuous G_K-action such that*

$$\mathrm{D}_{\mathrm{cris}}(M)^{\varphi_K = \alpha} \neq (0)$$

for some $\alpha \in \overline{\mathbb{Q}}_p^\times$ and suppose that M is Hodge-Tate such that for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ one has $\mathrm{HT}_\tau(M) = \{0, k_{\tau,2}, \cdots, k_{\tau,n}\}$ with

- $0 > k_{\tau,2} \geq \cdots \geq k_{\tau,n}$
- $|k_{\tau,2}| > \max(k, [K : \mathbb{Q}_p] \mathrm{v}_p(\alpha))$

Then the natural map gives rise to an isomorphism

$$(\mathrm{B}^+_{\mathrm{cris}} \widehat{\otimes}_{\mathbb{Q}_p} M)^{G_K, \varphi_K = \alpha} \otimes_{K_0} K \xrightarrow{\sim} (\mathrm{B}^+_{\mathrm{dR}}/t^k \mathrm{B}^+_{\mathrm{dR}} \widehat{\otimes}_{\mathbb{Q}_p} M)^{G_K}$$

of $K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-modules.

Proof. Since for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ one has

$$\#\{j \in \mathrm{HT}_\tau(M)\big||j| \leq k\} = 1$$

it follows from [KIS, Cor. 2.6] that $(\mathrm{B}_{\mathrm{dR}}^+/t^k\mathrm{B}_{\mathrm{dR}}^+\widehat{\otimes}_{\mathbb{Q}_p}M)^{G_K}$ is a finite flat $K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$ module of rank 1. Since $\mathrm{HT}_\tau(M) \subset \mathbb{Z}^{\leq 0}$ for all τ it follows from [NAK, Lem. 3.8] that

$$(\mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} M)^{G_K} = (\mathrm{B}_{\mathrm{cris}}^+ \otimes_{\mathbb{Q}_p} M)^{G_K}$$

and hence it follows from the assumptions of the lemma that

$$(\mathrm{B}_{\mathrm{cris}}^+ \otimes_{\mathbb{Q}_p} M)^{G_K,\varphi_K=\alpha} \neq (0)$$

and this is a free $K_0 \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-module of rank at least 1. Consider the natural map

$$f : (\mathrm{B}_{\mathrm{cris}}^+\widehat{\otimes}_{\mathbb{Q}_p}M)^{G_K,\varphi_K=\alpha} \otimes_{K_0} K \to (\mathrm{B}_{\mathrm{dR}}^+\widehat{\otimes}_{\mathbb{Q}_p}M)^{G_K}$$

Suppose now for contradiction that the image of f has non-trivial intersection with

$$(t^k\mathrm{B}_{\mathrm{dR}}^+\widehat{\otimes}_{\mathbb{Q}_p}M)^{G_K}$$

where $k \in \mathbb{Z}^{\geq 1}$ is as in the statement of the lemma. Since $\mathrm{D}_{\mathrm{cris}}(M)$ is weakly admissible and since for all $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ one has

$$j \in \{i \in \mathrm{HT}_\tau(M)\big||i| \geq k\} \Longrightarrow |j| \geq |k_{\tau,2}|$$

and since by assumption one has $|k_{\tau,2}| > [K : \mathbb{Q}_p]\mathrm{v}_p(\alpha)$, it follows [careful, see [Kisin] (Lem. 6.7) where somehow the inequalities are in opposite if expected direction!] that

$$\mathrm{v}_p(\alpha)/[K_0 : \mathbb{Q}_p] > \mathrm{v}_p(\alpha)$$

This is a contradiction. Let

$$g : (\mathrm{B}_{\mathrm{cris}}^+\widehat{\otimes}_{\mathbb{Q}_p}M)^{G_K,\varphi_K=\alpha} \otimes_{K_0} K \longrightarrow (\mathrm{B}_{\mathrm{dR}}^+/t^k\mathrm{B}_{\mathrm{dR}}^+\widehat{\otimes}_{\mathbb{Q}_p}M)^{G_K}$$

be the natural map. Then the image of g is a free $K \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-submodule of rank at least 1 and the lemma follows. ☐

4 Local-global compatibility for Hilbert Modular Forms

In this chapter we illustrate the main ideas of this memoir at the example of Hilbert modular forms. First we will deal with Hilbert modular forms of cohomological weight where we simply reprove some classical results via our deformation theoretic methods. Afterwards we prove new results about the non-triviality of monodromy operators associated to Hilbert modular forms of partial weight one. The relevant Galois representations were constructed via congruences by Jarvis in [JAR]. A drawback to this method is that one looses control over local monodromy operators at places of Steinberg ramification. To generalize our methods from cohomological to non-cohomological weight we develop a slightly different version of the approach to monodromy operators via monodromy lifting theorems that we use in the case of cohomological weight. While the methods should work more generally, we focus here on the case of Hilbert modular forms for simplicity. Roughly speaking the idea is to replace identification of small deformation rings and small Hecke algebras with the corresponding identification of big deformation rings and big Hecke algebras. For simplicity of exposition we restrict to the case of ordinary forms but the methods can be adapted to deal with the general case as well and we hope to address this matter in the future.

4.1 Cohomological weight

Note that we will not prove the most general results possible in this chapter in order to present the main ideas in a simplified setting. In particular, this allows us to appeal directly to the families of Hilbert modular forms used in [SKI] and the potential level-lowering results of [SW], and this shortens the arguments.

Let F be totally real field and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. It is said to be of weight

$$\kappa = ((k_\tau)_{\tau\in \mathrm{Hom}(F,\mathbb{R})}; w) \in (\mathbb{Z})^{\mathrm{Hom}(F,\mathbb{R})} \times \mathbb{Z}$$

if for the infinite place v_τ corresponding to τ the representation π_{v_τ} is an essentially discrete series representation of $\mathrm{GL}_2(F_{v_\tau})$ of Blattner parameter k_τ and with central character

$$x \mapsto \mathrm{sign}(x)^{k_i}|x|^{-w}$$

The weight is called cohomological if $k_\tau \geq 2$ and $w \equiv k_\tau \mod 2$ for all τ.

Let $\mathrm{rec}_{\mathrm{GL}_2}(-)$ be a local Langlands correspondence for GL_2 normalized such that in particular the following holds. Suppose K is a p-adic field and χ_1 and χ_2 are characters of $K^\times$ such that the normalized induction $\mathrm{Ind}(\chi_1, \chi_2)$ is irreducible. Then

$$\mathrm{rec}_{\mathrm{GL}_2}(\mathrm{Ind}(\chi_1, \chi_2)) \cong \tilde{\chi}_1 \oplus \tilde{\chi}_2$$

Here $\tilde{\chi}_1$ and $\tilde{\chi}_2$ are as defined at the beginning of Chapter 3.

The following local-global compatibility hypothesis will be combined with the deformation theory of automorphic forms to deduce more general local-global compatibility results. The compatibility at all places where the automorphic representation is unramified is crucial, as is the independence of ℓ hypothesis at places of Iwahori-spherical ramification.

Hypothesis 4.1.1. Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of cohomological weight

$$\kappa = ((k_\tau)_{\tau\in \mathrm{Hom}(F,\mathbb{R})}; w)$$

For any rational prime ℓ and choice of $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ there exists a continuous semi-simple representation

$$\rho_{\pi,\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that

(i) for all finite places $v \nmid \ell$ of F such that π_v is a principal series representation one has

$$\mathrm{WD}_\iota\big(\rho_{\pi,\ell,\iota}|_{W_{F_v}}\big)^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_2}(\pi_v \otimes |\det|^{-\frac{1}{2}})$$

(ii) if $v|\ell$ then

- if π_v is unramified then $\rho_{\pi,\ell,\iota}|_{G_{F_v}}$ is crystalline and

$$\mathrm{WD}_\iota\big(\rho_{\pi,\ell,\iota}|_{G_{F_v}}\big)^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_2}(\pi_v \otimes |\det|^{-\frac{1}{2}})$$

- $\rho_{\pi,\ell,\iota}$ is Hodge-Tate at v and if $\tau \in \mathrm{Hom}(F, \mathbb{R})$ corresponds to v via ι then the Hodge-Tate weights with respect to τ are given by

$$(-\frac{w-k_\tau}{2}, -\frac{w+k_\tau-2}{2})$$

- if π_v is Iwahori-spherical then $\rho_{\pi,\ell,\iota}$ is potentially semi-stable at v

(iii) let v be a finite place of F such that π_v is Iwahori-spherical. Then for all rational primes ℓ_1, ℓ_2 and isomorphisms

$\iota_{\ell_1} : \overline{\mathbb{Q}}_{\ell_1} \xrightarrow{\sim} \mathbb{C}$ and $\iota_{\ell_2} : \overline{\mathbb{Q}}_{\ell_2} \xrightarrow{\sim} \mathbb{C}$ one has

$$\mathrm{WD}_{\iota_{\ell_1}}\big(\rho_{\pi,\ell_1,\iota_{\ell_1}}|_{W_{F_v}}\big)^{\mathrm{ss}} \cong \mathrm{WD}_{\iota_{\ell_2}}\big(\rho_{\pi,\ell_2,\iota_{\ell_2}}|_{W_{F_v}}\big)^{\mathrm{ss}}$$

and these representations are unramified

This hypothesis will be assumed throughout the current chapter.

Remark 3. Concerning part (iii) of the hypothesis, see [SAI2] for independence of ℓ results for ℓ-adic Galois representations associated to Hilbert modular forms. We do not assume the results of [SAI2] since one aim of this work is to obtain local-global compatibility results without studying the singularities of Shimura varieties. In [SAI3] and [OCH] independence of ℓ results for Galois representations on the ℓ-adic cohomology of quite general algebraic varieties are obtained which however do not imply the full independence of ℓ assumption that we make in Hypothesis 4.1.1: The case where, in the notation of Hypothesis 4.1.1 (iii), the characteristic of the residue field of F_v equals ℓ_1 does not follow directly from the above mentioned references.

4.1.2 Semi-simplification

The next proposition is an example of how Hypothesis 4.1.1 can be used to deduce local-global compatibility results for the local semi-simplifications of automorphic Galois representations at places where the local Langlands correspondence predicts ramification.

Note that while we focus throughout this work on the case of Iwahori-spherical ramification, the particularly restrictive assumption used in the next proposition is included only to be able to appeal directly to the families of Hilbert modular forms used in [SKI]. The assumption on the weight is also included only to make the argument shorter. See corollaries 6.3 and 9.9 where more general results for Hilbert modular forms are deduced as special cases of results for automorphic forms of GL_n over CM-fields.

Proposition 4.2. *Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $(2, \cdots, 2; 2)$. Assume v is a finite place of F of residue characteristic p such that π_w is Iwahori-spherical for all places $w|p$ of F. Then*

$$\mathrm{WD}_{\iota}\big(\rho_{\pi,\ell,\iota}|_{W_{F_v}}\big)^{\mathrm{ss}} \cong \big(\mathrm{rec}_{\mathrm{GL}_2}(\pi_v \otimes |\det|^{-\frac{1}{2}})\big)^{\mathrm{ss}}$$

Proof. Let v, p and ℓ be as in the statement of the proposition. By Hypothesis 4.1.1 (iii) one can assume that $p = \ell$. After making a base change to a suitable solvable totally real base extension in which v splits completely one can assume that $[F : \mathbb{Q}]$ is even. In the notation of Section 3.3 write $\pi_v \cong \mathrm{St}_2(\chi)$ for some unramified character χ of $F_v^{\times}$. For each place $w|p$ of F choose a uniformizer ϖ_w and let Iw_w denote the Iwahori subgroup corresponding to the upper triangular Borel subgroup of $\mathrm{GL}_2(F_w)$. Let

$$U_w = \mathrm{char}(\mathrm{Iw}_w \begin{bmatrix} \varpi_w & \\ & 1 \end{bmatrix} \mathrm{Iw}_w) \in \mathcal{H}(\mathrm{GL}_2(F_w), 1)$$

where $\mathrm{char}(-)$ denotes the characteristic function and $\mathcal{H}(\mathrm{GL}_2(F_w), 1)$ is the Iwahori-Hecke algebra as defined in Chapter 3. Let $\mathcal{A}_w$ be the subalgebra of $\mathcal{H}(\mathrm{GL}_2(F_w), 1)$ generated by U_w. Let S be the union of the set of infinite places of F, the set of places $w|p$ of F and the set of finite places of F where π is ramified. For a place $w \notin S$ let $\mathcal{H}_w$ denote the spherical Hecke algebra at w. This is the algebra consisting of functions on $\mathrm{GL}_2(F_w)$ which are locally constant, have compact support and are $\mathrm{GL}_2(\mathcal{O}_{F_w})$ bi-invariant. Let

$$\mathcal{H} = \otimes'_{w \notin S} \mathcal{H}_w \otimes_{w|p} \mathcal{A}_w$$

and let

$$K^{(S)} = \prod_{w \notin S} \mathrm{GL}_2(\mathcal{O}_{F_w})$$

Let $f \in \pi^{K^{(S)}\prod_{w|p}\mathrm{Iw}_w} \neq (0)$ be an eigenvector for $\mathcal{H}$. For each $w|p$ there exists an unramified character χ_w such that $\pi_w \hookrightarrow \mathrm{Ind}(\chi_w)$ and one has

$$U_w \cdot f = \chi_w(\mathrm{diag}(\varpi_w, 1)) \cdot q^{1/2} \cdot f$$

We now recall the existence of certain families of Hilbert modular forms as described in [SKI]. The only difference is that there the local components of π at places dividing p are all unramified principal series representations.

Let $K/\mathbb{Q}_p$ be a p-adic field contained in $\overline{\mathbb{Q}}_p$ and containing the image of all embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$. Let $r \in |K^\times|_p$ and let $\mathrm{Sp}(\mathcal{A}_r)$ be the closed rigid ball over K of radius r. Then $\mathcal{A}_r$ is a $\mathcal{A}_1 \cong K\langle T\rangle$-module where T is an indeterminate and

$$K\langle T\rangle := \{\sum_{n\geq 0} a_n T^n \Big| a_n \in K \text{ and } |a_n|_K \to 0 \text{ as } n \to 0\}$$

is a Tate algebra. Let $\mathbf{p} := p$ if p is odd and $\mathbf{p} := 4$ otherwise. As explained in [SKI], there exists $r \in |K^\times|_p$ and a reduced finite torsion-free $\mathcal{A}_r$-algebra $\mathcal{R}$ such that

(i) there is a morphism

$$\phi : \mathcal{H} \longrightarrow \mathcal{R}$$

such that for each $\kappa \in \mathrm{Hom}_K(\mathcal{R}, \overline{\mathbb{Q}}_p)$ such that $\kappa(1+T) = (1+\mathbf{p})^{n_\kappa}$ for some $n_\kappa \in p(p-1)\mathbb{Z}^{\geq 1}$ there is an automorphic representation π_κ of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $(2, \cdots, 2; 2) + n_\kappa(1, \cdots, 1; 1)$ and $f_\kappa \in \pi_\kappa^{K^{(S)}\prod_{w|p}\mathrm{Iw}_w}$ such that

$$S \cdot f_\kappa = (\kappa \circ \phi)(S) \cdot f_\kappa$$

for all $S \in \mathcal{H}$. Here $\mathrm{Hom}_K(\mathcal{R}, \overline{\mathbb{Q}}_p)$ denotes the set of continuous homomorphisms

(ii) there is a constant C such that if $n_\kappa > C$ and π_κ are as above, then $\pi_{\kappa,v}$ is an unramified principal series representation

(iii) there exists κ_0 with $\kappa_0(1+T) = 1$ such that $\kappa_0 \circ \phi$ gives the Hecke eigenvalues of f_0

(iv) for v as in the statement of the proposition the slope $\mathrm{v}_p((\kappa\circ\phi)(U_v))$ for κ as in (i) is constant and will be denoted by $\mathrm{v}_p(\phi(U_v))$

(v) there exists a free $\mathcal{R}$-module $V_{\mathcal{R}}$ of rank 2 and a continuous Galois representation

$$\rho_{\mathcal{R}} : G_F \longrightarrow \mathrm{GL}(V_{\mathcal{R}})$$

such that for κ_0 as above the semi-simplification $\rho^{\mathrm{ss}}_{\mathcal{R},\kappa_0}$ of the representation on $V_{\mathcal{R}} \otimes_{\mathcal{R},\kappa_0} \overline{\mathbb{Q}}_p$ is isomorphic to $\rho_{\pi,\ell,\iota}$ and for all $\kappa \in \mathrm{Hom}_K(\mathcal{R}, \overline{\mathbb{Q}}_p)$ as in (i) the semi-simplification $\rho^{\mathrm{ss}}_{\mathcal{R},\kappa}$ of the representation on $V_{\mathcal{R}} \otimes_{\mathcal{R},\kappa} \overline{\mathbb{Q}}_p$ is isomorphic to $\rho_{\pi_\kappa,\ell,\iota}$

Fix $k \in \mathbb{Z}^{\geq 1}$ and let

$$\mathrm{I}_k :=$$
$$\{\kappa \in \mathrm{Hom}_K(\mathcal{R}, \overline{\mathbb{Q}}_p) \big| \ \kappa \text{ is as in (i) with } n_\kappa > \max(C, k-1, [F_v : \mathbb{Q}_p]\mathrm{v}_p(\phi(U_v)) - 1)\}$$

Moreover, for $\kappa \in \mathrm{I}_k$ let $\mathcal{R}_\kappa$ denote the residue field of the maximal ideal of $\mathcal{R}$ corresponding to κ. Note that for $\kappa \in \mathrm{I}_k$ one has

$$\mathrm{HT}_{\tau,v}(\rho_{\pi_\kappa}) = \{0, n_\kappa + 1\}$$

Moreover, there is an injection $\mathcal{R} \hookrightarrow \prod_{\kappa\in\mathrm{I}_k} \mathcal{R}_\kappa$ since $\mathcal{R}$ is reduced and $\{\mathfrak{m}_\kappa\}_{\kappa\in\mathrm{I}_k}$ is a Zariski dense subset of $\mathrm{Sp}(\mathcal{R})$. Now let M be the dual of $V_{\mathcal{R}}$, let $Y := \phi(U_v)^{-1}$ and let $\{\mathcal{R}_\kappa\}_\kappa$ denote the collection of residue fields of $\mathcal{R}$ corresponding to $\kappa \in \mathrm{Hom}_K(\mathcal{R}, \overline{\mathbb{Q}}_p)$ such that $\kappa \in \mathrm{I}_k$ for some $k \geq 1$.

For $\kappa \in \mathrm{I}_k$ the representation $\pi_{\kappa,v}$ is an unramified principal series representation and hence it follows from Hypothesis 4.1.1 (i) that

$$\mathrm{D}_{\mathrm{cris}}(\rho_{\mathcal{R}_\kappa}|_{G_{F_v}})^{\varphi_{F_v} = \kappa\circ\phi(U_v)} \neq (0)$$

It follows now from Lemma 3.6 that there is an isomorphism

$$\begin{array}{c}(\mathrm{B}^+_{\mathrm{cris}}\widehat{\otimes}_{\mathbb{Q}_p}(M\otimes_{\mathcal{R}}\mathcal{R}_\kappa))^{G_{F_v},\varphi_{F_v}=\kappa\circ\phi(U_v)}\otimes_{F_{v,0}}F_v\\ \big\downarrow\wr\\ (\mathrm{B}^+_{\mathrm{dR}}/t^k\mathrm{B}^+_{\mathrm{dR}}\widehat{\otimes}_{\mathbb{Q}_p}(M\otimes_{\mathcal{R}}\mathcal{R}_\kappa))^{G_{F_v}}\end{array}$$

Hence all the assumptions of Theorem 3 are satisfied and it follows that

$$\mathrm{D}_{\mathrm{cris}}(\rho_{\pi,\ell,\iota}|_{G_{F_v}})^{\varphi_{F_v}=\chi(\varpi_v)q^{1/2}}\neq(0)$$

Let $\alpha:=\chi_v(\mathrm{diag}(\varpi_v,1))$ and $\beta:=\chi_v(\mathrm{diag}(1,\varpi_v))$. Let $\sigma\in W_{F_v}$ be such that $\nu(\sigma)=[F_{v,0}:\mathbb{Q}_p]$ and let S_σ denote the multiset of eigenvalues of σ acting on the Weil-Deligne representation $\mathrm{D}_{\mathrm{pst}}(\rho_{\pi,\ell,\iota}|_{G_{F_v}})$. By using Lemma 3.5 it follows from the above that $S_\sigma=\{\alpha q^{1/2},y\}$ for some y. To conclude the proof of the proposition we will now use the local-global compatibility for the determinant of $\rho_{\pi,\ell,\iota}$.

Let χ_π denote the central character of π and let

$$r_{\ell,\iota}(\chi_\pi|\cdot|^{-1}_{\mathbb{A}_F^\times}):\mathrm{Gal}(\overline{F}/F)\longrightarrow\overline{\mathbb{Q}}_\ell^\times$$

denote the ℓ-adic character associated to $\chi_\pi|\cdot|^{-1}_{\mathbb{A}_F^\times}$ as defined for example in [CHT, Lem. 4.1.3]. It follows from the Chebotarev density theorem and Hypothesis 4.1.1 that

$$\det\rho_{\pi,\ell,\iota}\cong r_{\ell,\iota}(\chi_\pi|\cdot|^{-1}_{\mathbb{A}_F^\times})$$

Since $\rho_{\pi,\ell,\iota}$ is potentially semi-stable at v one has

$$\wedge^2\mathrm{D}_{\mathrm{pst}}(\rho_{\pi,\ell,\iota}|_{G_{F_v}})\cong\mathrm{D}_{\mathrm{pst}}(\wedge^2\rho_{\pi,\ell,\iota}|_{G_{F_v}})$$

Moreover, $\chi_{\pi,v}$ is unramified and $\mathrm{D}_{\mathrm{cris}}(\wedge^2\rho_{\pi,\ell,\iota}|_{G_{F_v}})^{\varphi_{F_v}=\alpha\beta q}\neq(0)$ and hence by Lemma 3.5 one has $\alpha\beta q=\alpha q^{1/2}y$. It follows that $y=\beta q^{1/2}$ and

$$\mathrm{Trace}\ \mathrm{WD}(\rho_{\pi,\ell,\iota})(\sigma)=\mathrm{Trace}\ \mathrm{rec}_{\mathrm{GL}_2}(\pi_v\otimes|\det|^{-\frac{1}{2}})(\sigma)$$

for all $\sigma\in W_{F_v}$ with $\nu(\sigma)\geq 0$. The result follows now from [SAI, Lem. 1]. ☐

Remark 4. The methods of this proof can also be used to obtain local-global compatibility results for Galois representations associated to Hilbert modular forms of partial weight one. Moreover, one can use base change methods to use the previous proposition to obtain local-global compatibility results even if the local component of the automorphic representation is not Iwahori-spherical.

4.2.1 Monodromy operators

We now illustrate how modularity lifting theorems can be used to calculate local monodromy operators. To avoid a detailed analysis of what type of local-global compatibility results are used in the proofs of particular modularity lifting theorems, and hence the possibility of circular arguments, we will treat such modularity lifting results as hypotheses in this work. See Section 9.1 for some discussion of this. Note also that more general modularity lifting results than the ones used in this work are known. We simply give an example of how modularity lifting theorems can be used to deduce results for local monodromy operators of automorphic Galois representations.

Hypothesis 4.2.2. Let F be a totally real field and $\rho:\mathrm{Gal}(\overline{F}/F)\longrightarrow\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ a continuous Galois representation unramified outside of a finite set of places. Suppose that

- $\ell\geq 7$ and ℓ is unramified in F

- ρ is crystalline at all $v|\ell$
- $\overline{\rho}$ is absolutely irreducible and $\overline{\rho} \cong \overline{\rho}_{\pi',\ell,\iota}$ for some cuspidal cohomological automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $\kappa = ((k_\tau)_\tau; w)$ and some $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$
- the Hodge-Tate weights of ρ at $v|\ell$ with respect to any $\tau \in \mathrm{Hom}(F, \mathbb{R})$ giving rise to v via ι are

$$(-\frac{w-k_\tau}{2}, -\frac{w+k_\tau-2}{2})$$

- $\overline{\rho}|_{G_{F(\zeta_\ell)}}$ is absolutely irreducible where ζ_ℓ denotes a primitive ℓ'th root of unity

Then there is a cuspidal cohomological automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight κ such that

- $\rho \cong \rho_{\tilde{\pi},\ell,\iota}$
- $\tilde{\pi}_v$ is unramified for any finite place $v \nmid \ell$ of F such that $\rho|_{G_{F_v}}$ and π'_v are unramified

This hypothesis will be assumed for the rest of this chapter. It will be used to obtain information about local properties of ℓ-adic automorphic Galois representations. This works if ℓ is such that the ℓ-adic Galois representation is amenable to Galois deformation theory. Hence we make the following definition:

Definition 6. Let F be a totally real field and let π be a cuspidal cohomological automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\mathcal{B}_\pi$ denote the set of pairs (ℓ, ι) consisting of a rational prime ℓ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ such that

- π_v is spherical for all places $v|\ell$ of F
- $\overline{\rho}_{\pi,\ell,\iota}|_{G_{F(\zeta_\ell)}}$ is absolutely irreducible where ζ_ℓ denotes a primitive ℓ'th root of unity
- $\ell \geq 7$ and ℓ is unramified in F

Remark 5. Note that for the calculation of local semi-simplifications of automorphic Galois representations we have assumed a certain independence of ℓ hypothesis. However, we will not assume an independence of ℓ hypothesis for local monodromy operators in this work. If such a hypothesis were assumed then the non-triviality results for local monodromy operators could be extended beyond those ℓ-adic Galois representations to which one can apply the modularity lifting theorem methods. Note however that in some cases one can rather easily obtain independence of ℓ results for monodromy operators. For example in the next proposition the assumption on the weight allows to realize the Galois representation in the Galois representation of an abelian variety and one can use the good reduction criterion of Serre and Tate.

As in Proposition 4.2, the restriction on the weight in the next proposition is not necessary and is included only to be able to appeal directly to the potential level-lowering results of [SW]. Results for Hilbert modular forms of more general weight will be obtained in Chapter 6.

Proposition 4.3. *Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $(2, \cdots, 2; 2)$. Assume $(\ell, \iota) \in \mathcal{B}_\pi$ and let $v \nmid \ell$ of F be a finite place such that π_v is Iwahori-spherical and write $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} = (r, N)$. Then N is non-trivial if and only if it is predicted to be non-trivial by the local Langlands correspondence.*

Proof. If the local Langlands correspondence predicts, in the above notation, that N is trivial then π_v is a principal series representation and by 4.1.1 (i) it follows that N is indeed trivial. Hence suppose now that the local Langlands correspondence predicts that N is non-trivial. Then $\pi_v \cong \mathrm{St}_2(\chi)$ for some unramified character χ. Suppose for contradiction that N is trivial. Then there exists a totally real solvable extension F'/F such that $\rho_{\pi,\ell,\iota}|_{G_{F'}}$ is unramified at all places of F' above v and $\mathrm{BC}_{F'}(\pi)$ is cuspidal and $\ell \in \mathcal{B}_{\mathrm{BC}_{F'}(\pi)}$, where $\mathrm{BC}_{F'}(-)$ denotes the base change to an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F'})$ as constructed in [AC]. By [SW] there exists a solvable totally real extension L/F' such that $\mathrm{BC}_L(\pi)$ is cuspidal and there is a cuspidal cohomological automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_L)$ of the same weight as $\mathrm{BC}_L(\pi)$ such that

- $\overline{\rho}_{\mathrm{BC}_L(\pi),\ell,\iota} \cong \overline{\rho}_{\tilde{\pi},\ell,\iota}$
- there is a place w of L above v such that $\tilde{\pi}_w$ is an unramified principal series representation
- $\tilde{\pi}$ is an unramified principal series at all places above ℓ
- $(\ell,\iota) \in \mathcal{B}_{\tilde{\pi}}$

It then follows from Hypothesis 4.2.2 with $\rho := \rho_{\pi,\ell,\iota}|_{G_L}$ and $\pi' := \tilde{\pi}$ that there is a cuspidal cohomological automorphic representation π_2 of $\mathrm{GL}_2(\mathbb{A}_L)$ of the same weight as $\mathrm{BC}_L(\pi)$ such that $\rho_{\mathrm{BC}_L(\pi),\ell,\iota} \cong \rho_{\pi_2,\ell,\iota}$ and such that π_2 is an unramified principal series at all places of L above v. From the local-global compatibility assumption at unramified principal series places it follows that for all but finitely many finite places u of L one has $\mathrm{BC}_L(\pi)_u \cong \pi_{2,u}$. By strong multiplicity one for cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_L)$, see for example [PS], it follows that in particular for all places u of L above v one has $\mathrm{BC}_L(\pi)_u \cong \pi_{2,u}$. Since for such places $\pi_{2,u}$ is an unramified principal series this is a contradiction since $\mathrm{rec}_{\mathrm{GL}_2}(\mathrm{BC}_L(\pi)_u)$ has a non-trivial monodromy operator but $\mathrm{rec}_{\mathrm{GL}_2}(\pi_{2,u})$ has trivial monodromy operator. □

4.4 Non-cohomological weight

Suppose F is a totally real field and f is a Hilbert modular form of partial weight one. Jarvis has shown in [JAR] the existence of associated Galois representations but he was not able to show the non-triviality of monodromy operators at Steinberg places. Generalizing our earlier methods from cohomological to non-cohomoligcal weights we are able to prove the desired non-triviality under some assumptions. Note that, since we are proving new result, in this chapter we allow ourselves to use the known fact of compatibility of local Langlands correspondence with base change.

4.4.1 Classical and p-adic Hilbert modular forms

We will recall some background on Hilbert modular forms, following the discussion in [HID3, Section 4.3.1].

Throughout this chapter fix a totally real number field F with ring of integers $\mathfrak{O}_F$ and let $F_+^\times$ denote the set of totally positive elements. Let $\mathfrak{d}$ denote the different of F over $\mathbb{Q}$ and let I denote the set of embeddings $F \hookrightarrow \mathbb{R}$. We denote by $\mathbb{Z}[I]$ the set of collections of integers indexed by I and let $t = (1, \cdots, 1) \in \mathbb{Z}[I]$. For $k \in \mathbb{Z}[I]$ and $x \in F$ we let $x^k := \prod_{\sigma\in I} \sigma(x)^{k_\sigma}$ and for $k_1, k_2 \in \mathbb{Z}[I]$ we say $k_1 \geq k_2$ if $k_{1,\sigma} \geq k_{2,\sigma}$ for all $\sigma \in I$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, let Z denote the center of G and let $G(\mathbb{R})^+$ denote the connected component of the identity of $G(\mathbb{R})$.

Let T_G be the diagonal torus of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ and let $T = \mathrm{Res}_{\mathfrak{O}_F/\mathbb{Z}}\mathrm{GL}_1$. Identify $\mathbb{Z}[I]^2$ with $\mathrm{Hom}(T_G, \mathrm{GL}_1)$ by taking $(\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2$ to the morphism

$$\begin{bmatrix} a & \\ & d \end{bmatrix} \mapsto a^{\kappa_1} d^{\kappa_2}$$

for $a, d \in F$. Fix a square root $\sqrt{-1}$ of -1 in $\mathbb{C}$ and let $\mathcal{H}$ denote the corresponding complex upper half-plane and and let $\mathbf{i} := (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathcal{H}^I$. Let $C_{\mathbf{i}}$ denote the stabilizer of $\mathbf{i}$ in $G(\mathbb{R})^+$. Let $\kappa = (\kappa_1, \kappa_2) \in \mathrm{Hom}(T_G, \mathrm{GL}_1)$ for κ_1 and κ_2 in $\mathbb{Z}[I]$. For $g \in G(\mathbb{R})$ and $z \in \mathcal{H}^I$ let

$$j(g,z) = (c_\sigma z_\sigma + d_\sigma)_{\sigma\in I} \in \mathbb{C}^I \text{ and } J_\kappa(g,z) := \det(g)^{\kappa_1 - I} j(g,z)^{\kappa_2 - \kappa_1 + I}$$

Throughout this chapter fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and for each rational prime p fix an embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathcal{W}$ be a subring of $\overline{\mathbb{Q}}$ which is a discrete valuation ring and contains the pre-image under i_p of the valuation ring of the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. Let $W = \varprojlim_m \mathcal{W}/p^m\mathcal{W}$ and let $W_m := W/p^mW$. Throughout this section fix a prime p and assume that it is unramified in F.

For an integral ideal $\mathfrak{N}$ of $\mathfrak{O}_F$ let

$$\widehat{\Gamma}_0(\mathfrak{N}) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{N}\}$$

Since it simplifies the form of Fourier expansions, in [HID3] one works with a slightly modified level structure: Let $d \in \mathbb{A}_F^\times$ be such that $d\mathfrak{O}_F = \mathfrak{d}$ and $d^{(\mathfrak{d})} = 1$. For an integral ideal $\mathfrak{N}$ of $\mathfrak{O}_F$ let

$$S_0(\mathfrak{N}) = \begin{bmatrix} d & \\ & 1 \end{bmatrix}^{-1} \widehat{\Gamma}_0(\mathfrak{N}) \begin{bmatrix} d & \\ & 1 \end{bmatrix}$$

Let $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_+)$ be a nebencharacter in the sense that there is a continuous character $\epsilon : T_G(\hat{\mathbb{Z}}) \longrightarrow \mathcal{W}^\times$ given by

$$\epsilon(\begin{bmatrix} a & \\ & d \end{bmatrix}) = \epsilon_1(a)\epsilon_2(d)$$

and $\epsilon_+ : Z(\hat{\mathbb{Z}}) \longrightarrow \mathcal{W}^\times$ given by $\epsilon_+(z) = \epsilon_1(z)\epsilon_2(z)$ and $\epsilon^- : T(\hat{\mathbb{Z}}) \longrightarrow \mathcal{W}^\times$ by $\epsilon^-(z) = \epsilon_2^{-1}(z)\epsilon_1(z)$ and assume that ϵ_+ can be extended to a Hecke character $\epsilon_+ : Z(\mathbb{A})/Z(\mathbb{Q}) \longrightarrow \mathbb{C}^\times$ such that

$$\epsilon_+(x_\infty) = x_\infty^{-(\kappa_1+\kappa_2)+I}$$

Define the character $\epsilon_\Delta^\infty : S_0(\mathfrak{N}) \longrightarrow \mathbb{C}^\times$ by

$$\epsilon_\Delta^\infty(s) = \epsilon_2(\det(s))\epsilon^-(a_\mathfrak{N})$$

where

$$s = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Definition 7. For an integral ideal $\mathfrak{N}$ of $\mathfrak{O}_F$ let $S_\kappa(\mathfrak{N}, \epsilon, \mathbb{C})$ denote the space of functions

$$f : \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

such that the following holds:

- For all $\alpha \in G(\mathbb{Q})$, $z \in Z(\mathbb{A})$, $u \in S_0(\mathfrak{N})C_{\mathbf{i}}$ and $x \in \mathrm{GL}_2(\mathbb{A}_F)$ one has

$$f(\alpha x u z) = \epsilon_+(z)\epsilon_\Delta^\infty(u) f(x) J_\kappa(u_\infty, \mathbf{i})^{-1}$$

- For each $z \in \mathcal{H}^I$ choose $u \in G(\mathbb{R})$ such that $u(\mathbf{i}) = z$ and for each $g \in G(\mathbb{A}^{(\infty)})$ the function

$$f_g : \mathcal{H}^I \longrightarrow \mathbb{C}$$

  given by $f_g(z) = f(gu_\infty)J_\kappa(u_\infty, \mathbf{i})$ is holomorphic on $\mathcal{H}^I$ and exponentially decreasing as $\mathrm{Im}(z) \longrightarrow \infty$.

Remark 6. Note that $S_\kappa(\mathfrak{N}, \epsilon, \mathbb{C}) = \{0\}$ unless there is an integer $[\kappa_1 + \kappa_2]$ such that $\kappa_1 + \kappa_2 = [\kappa_1 + \kappa_2]t$ and hence we will assume from now on that this holds.

Remark 7. The space $M_\kappa(\mathfrak{N}, \epsilon, \mathbb{C})$ of not necessarily cuspidal Hilbert modular forms is defined analogously but with the exponential decrease assumption replaced by an assumption on so-called moderate growth.

Remark 8. For simplicity of notation we will also denote by $S_\kappa(\mathfrak{N}, 1, \mathbb{C})$ the space of modular forms as defined above with nebencharacter ϵ such that ϵ_1 and ϵ_2 are trivial. Similar conventions will be used later on in definitions of spaces of p-adic modular forms.

We now recall from [HID3] how the above defined spaces of Hilbert modular forms can be described geometrically via sheaves on Shimura varieties: Let

$$\overline{S}_0(\mathfrak{N}) = S_0(\mathfrak{N})Z(\mathbb{A}^{(\infty)})$$

and let $\mathrm{Sh}_{\overline{S}_0(\mathfrak{N})}$ be the Hilbert modular Shimura variety of level $\overline{S}_0(\mathfrak{N})$. Fix κ and a nebencharacter ϵ. In [HID3, Section 4.2.6] a sheaf $\underline{\omega}_{\kappa,\epsilon}$ on the above Shimura variety is defined which by loc. cit. (p.194) satisfies

$$\mathrm{H}^0(\mathrm{Sh}_{\overline{S}_0}(\mathfrak{N}), \underline{\omega}_{\kappa,\epsilon/\mathbb{C}}) \cong S_\kappa(\mathfrak{N}, \epsilon, \mathbb{C})$$

To define spaces of Hilbert modular forms defined over, for example, number fields one can use Fourier expansions: As described in loc. cit. (p.196), every $f \in S_\kappa(\mathfrak{N}, \epsilon, \mathbb{C})$ has a Fourier expansion

$$f(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) = |y|_{\mathbb{A}} \sum_{\xi \in F_+^\times} a_\infty(\xi y, f)(\xi y_\infty)^{-\kappa_1} \mathbf{e}_F(\sqrt{-1}\xi y_\infty)\mathbf{e}_F(\xi x)$$

where for $\overline{x} = (x_i) \in \mathbb{C}^d$ $(d \geq 1)$ one defines

$$\mathbf{e}_F(x) = \exp(2\pi\sqrt{-1}\sum_i x_i)$$

Definition 8. For any $\mathbb{Q}$-algebra R contained in $\mathbb{C}$ and containing the image of ϵ_1, ϵ_2 and κ, define $S_\kappa(\mathfrak{N}, \epsilon, R)$ to be the space of $f \in S_\kappa(\mathfrak{N}, \epsilon, \mathbb{C})$ such that $a_\infty(y, f) \in R$ for all $y \in \mathbb{A}_F^\times$. It is then known that

$$S_\kappa(\mathfrak{N}, \epsilon, R) = \mathrm{H}^0(\mathrm{Sh}_{\overline{S}_0(\mathfrak{N})}, \underline{\omega}_{\kappa,\epsilon/R})$$

Definition 9. For a $\overline{\mathbb{Q}}_p$-algebra R the p-adic q-expansion coefficients are defined as

$$a_p(y, f) = y_p^{-\kappa_1} a_\infty(y, f)$$

and the formal q-expansion of f is the element in $R[[q^\xi]]_{\xi \in F_+^\times}$ given by

$$f(y) = \sum_{\xi \in F_+^\times} a_p(\xi y, f) q^\xi$$

We now define various Hecke operators acting on spaces of Hilbert modular forms. To do so, fix an integral ideal $\mathfrak{N}$ of $\mathfrak{O}_F$ and let

$$D = \{\begin{bmatrix} a & \\ & d \end{bmatrix} \in M_2(\mathfrak{O}_{F,\mathfrak{N}}) \big| a \in \mathfrak{O}_{F,\mathfrak{N}}^\times, d \in \mathfrak{O}_{F,\mathfrak{N}}\}$$

Let $\Delta_0(\mathfrak{N}) = \Delta_0(\mathfrak{N})_\mathfrak{N} \times \Delta_0(\mathfrak{N})^{(\mathfrak{N})}$ where

$$\Delta_0(\mathfrak{N})^{(\mathfrak{N})} = M_2(\widehat{\mathfrak{O}}_F) \cap \mathrm{GL}_2(\mathbb{A}^{(\mathfrak{N}\infty)}) \text{ and } \Delta_0(\mathfrak{N})_{(\mathfrak{N})} = (S_0(\mathfrak{N}) D S_0(\mathfrak{N})) \cap \mathrm{GL}_2(F_\mathfrak{N})$$

For an invertible matrix y write $y^\iota = \det(y) y^{-1}$. Write

$$S_0(\mathfrak{N}) y^\iota S_0(\mathfrak{N}) = \bigsqcup_{u,t} u t S_0(\mathfrak{N})$$

with $u \in U(\hat{\mathbb{Z}})$, where U denotes the upper triangular unipotent subgroup of G, and $t \in T_G(\mathbb{A}^{(\infty)})$ such that $\det t = \det y$. Then define the operator $[S_0(\mathfrak{N}) y^\iota S_0(\mathfrak{N})]$ by

$$(f|[S_0(\mathfrak{N}) y^\iota S_0(\mathfrak{N})])(x) := \sum_{u,t} \epsilon_\Delta^\infty((ut)^\iota) f(xut)$$

and define the modified Hecke operator

$$[S_0(\mathfrak{N}) y^\iota S_0(\mathfrak{N})]_p := \det(y_p)^{-\kappa_1} [S_0(\mathfrak{N}) y^\iota S_0(\mathfrak{N})]$$

As described in [HID3, p.196] , the action of $y \in \Delta_0(\mathfrak{N})$ on $S_\kappa(\mathfrak{N}, \epsilon, R)$ via $[S_0(\mathfrak{N})y^\iota S_0(\mathfrak{N})]$ coincides withe the action of the geometrically defined Hecke operator associated to y as in loc. cit. (Section 4.2.6).

Definition 10. For every finite place $\mathfrak{q}$ of F fix a uniformizer $\varpi_\mathfrak{q}$ and let $T(\varpi_\mathfrak{q}) = [S_0(\mathfrak{N})y^\iota S_0(\mathfrak{N})]$ for

$$y = \begin{bmatrix} 1 & \\ & \varpi_\mathfrak{q} \end{bmatrix}$$

If $\mathfrak{q}|\mathfrak{N}$ then denote $T(\varpi_\mathfrak{q})$ also by $U(\varpi_\mathfrak{q})$. Similarly, let $S(\varpi_\mathfrak{q})$ be defined by choosing instead

$$y = \begin{bmatrix} \varpi_\mathfrak{q} & \\ & \varpi_\mathfrak{q} \end{bmatrix}$$

The operators $T_p(\varpi_\mathfrak{q})$, $U_p(\varpi_\mathfrak{q})$ and $S_p(\varpi_\mathfrak{q})$ are obtained by the above described modification, namely by multiplying by $\det(y_p)^{-\kappa_1}$.

Note that it is shown in [HID2] that these operators are in fact elements in $\mathrm{End}_W(S_\kappa(\mathfrak{N}, \epsilon, W))$.

Definition 11. For a prime $\mathfrak{p}|p$ the ordinary projector $e_\mathfrak{p}$ in $\mathrm{End}_W(S_\kappa(\mathfrak{N}, \epsilon; W))$ is defined to be $\lim_{n\to\infty} U(\varpi_\mathfrak{p})^{n!}$ if $\mathfrak{p}|\mathfrak{N}$ and $\lim_{n\to\infty} T(\varpi_\mathfrak{p})^{n!}$ if $\mathfrak{p} \nmid \mathfrak{N}$. Also, let $e_p := \prod_{\mathfrak{p}|p} e_\mathfrak{p}$.

Definition 12. For $z \in Z(\mathbb{A}^{(p\infty)})$ define the associated normalized diamond operator $\langle z \rangle = |z|_\mathbb{A}^{-2}[S_0(\mathfrak{N})zS_0(\mathfrak{N})]$. Note that with this normalization, $f \in S_\kappa(\mathfrak{N}, \epsilon, W)$ satisfies

$$f|\langle z \rangle = \epsilon_+(z)f$$

We now describe p-adic Hilbert modular forms as well as Λ-adic modular forms. For each $m \geq 1$ let

$$S_m = \mathrm{Sh}^{(p)}(\mathrm{PGL}_2, X)/S_0(\mathfrak{N})[\frac{1}{E}]_{/W_m}$$

where E is a lift of the Hasse invariant, see [HID3, p.186] for precise definitions of the Shimura variety and level structure. Following [HID3, Section 4.1.6, 4.1.7] let $T_{m,n}$ be the piece of the Igusa tower over S_m as defined in loc. cit.. Let $V_{m,n}^{\mathrm{cusp}}$ denote the space of the global section of the structure sheaf of $T_{m,n}$ that vanish at the cusps and let $V_{m,\infty}^{\mathrm{cusp}} = \cup_n V_{m,n}^{\mathrm{cusp}}$ and define

$$V_{\mathrm{cusp}}(\mathfrak{N}; W) = \varprojlim_m V_{m,\infty}^{\mathrm{cusp}}$$

$$\mathcal{V}_{\mathrm{cusp}}(\mathfrak{N}; W) = \varinjlim_m V_{m,\infty}^{\mathrm{cusp}}$$

Define the spaces $V_{\mathrm{cusp}}(\mathfrak{N}, \epsilon; W)$ and $\mathcal{V}_{\mathrm{cusp}}(\mathfrak{N}, \epsilon, W)$ by imposing the central character and nebencharacter given by ϵ. For later use in describing modularity lifting results we define the Hecke algebras $\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N}, \epsilon, W[[T(\mathbb{Z}_p)]])$ to be the $W[[T(\mathbb{Z}_p)]]$-subalgebra of $\mathrm{End}(\mathcal{V}_{\mathrm{cusp}}^{\mathrm{ord}}(\mathfrak{N}, \epsilon, W))$ generated over $W[[T(\mathbb{Z}_p)]]$ by $T_p(y)$ and $\langle y \rangle_p$ as y ranges over integral ideles, where $\mathcal{V}_{\mathrm{cusp}}^{\mathrm{ord}}(\mathfrak{N}, \epsilon, W))$ denotes the image of the ordinary projector acting on $\mathcal{V}_{\mathrm{cusp}}(\mathfrak{N}, \epsilon, W))$. Here the operator $\langle y \rangle_p$ is obtained by associating to y the corresponding element in $Z(\mathbb{A}_\mathbb{Q})$ and then the associated diamond operator as defined in [HID3, p.173] and the Hecke operator $T_p(y)$ is defined analogously to Definition 10 by associating to y the matrix

$$\begin{bmatrix} 1 & \\ & y \end{bmatrix}$$

Let $\Lambda = W[[\Gamma]]$ where $T(\mathbb{Z}_p) = \Gamma \times \Delta^{(p)}$ where Γ is the maximal p-profinite subgroup. We now describe Λ-adic modular forms as p-adic modular forms defined over Λ. By taking the base change of the Igusa tower $T_{m,n}$ one defines the space $V_{\mathrm{cusp}}(\mathfrak{N}, \epsilon; \Lambda)$ of p-adic modular forms over Λ in an analogous manner to $V_{\mathrm{cusp}}(\mathfrak{N}, \epsilon; W)$. The space of Λ-adic modular forms can then be defined as the subspace of $V(\mathfrak{N}, \epsilon; \Lambda)$ where the Λ-module structure agrees with the Λ-module structure coming from the diamond operators. See for example [HID2, Section 3.4.1] for a comparison in the case of $F = \mathbb{Q}$ with other definitions of Λ-adic modular forms. For $f \in V(\mathfrak{N}, \epsilon; \Lambda)$ and $\kappa \in \mathrm{Hom}(T_G, \mathrm{GL}_1)$ the weight κ

specialization $\mathcal{F}_\kappa$ of $\mathcal{F}$ is the p-adic modular form over W obtained from f by specializing the $T(\mathbb{Z}_p)$-action via κ_1. Note that ϵ is fixed and through this the second weight κ_2 intervenes.

4.4.2 Monodromy Operators

We first discuss a type of rigidity result for p-adic modular forms that will replace our appeal to strong multiplicity one results that we used in Section 4.2.1 to deal with Hilbert modular forms of cohomological weight. In order to conform with conventions used in other parts of this memoir, for the rest of this chapter we will denote the prime p of the previous sections by ℓ. We will assume throughout the remainder of this chapter that $\ell \geq 3$. Fix a nebencharacter ϵ. Let B be a discrete valuation ring which is a subring of $\overline{\mathbb{Q}}$ and an extension of $\mathbb{Z}_{(\ell)}$ such that $B^\times$ contains the values of ϵ. Let W be the ℓ-adic completion of B.

Definition 13. A form $f \in V_{\text{cusp}}(\mathfrak{N}, 1; W)$ will be called an eigenform with respect to $\mathfrak{N}$ if it is an eigenform for $T(\varpi_v)$ for all $v \nmid \mathfrak{N}$ and an eigenform for $U(\varpi_v)$ for all $v|\mathfrak{N}$.

Let $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_2 - \kappa_1 + t \geq t$ in the sense that $(\kappa_2 - \kappa_1 + t)_\sigma \geq 1$ for all $\sigma \in I$. Let $\mathfrak{N}$ be a square-free ideal of $\mathfrak{O}_F$ and let $f \in V^{\text{ord}}_{\text{cusp}}(\mathfrak{N}, \epsilon; W)$ be an eigenform with respect to $\mathfrak{N}$ with weight $\kappa \in \text{Hom}(T_G, \text{GL}_1)$. For a Hecke operator T let $\theta_f(T)$ denote the corresponding eigenvalue. Then, see [JAR] for references, there exists a continuous Galois representation

$$\rho_{f,\ell} : \text{Gal}(\overline{F}/F) \longrightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that in particular:

- $\rho_{f,\ell}$ is unramified outside $\mathfrak{N}\ell$
- If $v \nmid \mathfrak{N}\ell$ is a prime of F, then
$$\text{Trace } \rho_{f,\ell}(\text{Fr}_v) = \theta_f(T(\varpi_v))$$
- $$\det \rho_{f,\ell} = \tilde{\epsilon}_+ \chi_\ell^{[\kappa_1+\kappa_2]}$$
  where $\tilde{\epsilon}_+$ denotes the Galois character associated to to ϵ_+
- If π denotes the automorphic representations associated to f and if $v \nmid \ell$ is a finite place of F such that
$$\pi_v \hookrightarrow \text{Ind}(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$$
  for some unramified character χ then
$$(\rho_{f,\ell}|_{G_{F_v}})^{\text{ss}} \sim \chi|\cdot|^{-1/2} \oplus \chi|\cdot|^{1/2}$$
  where the right hand side is interpreted as a Galois character via local class field theory

Let $\mathfrak{N}$ be an ideal of F and let $v \nmid \mathfrak{N}$ be a finite prime. Recall that in Definition 10 we fixed a uniformizer ϖ_v of F_v. Let $\eta_v \in \text{GL}_2(\mathbb{A}_F)$ be the the element whose local component away from v is the identity and whose component at v is given by

$$(\eta_v)_v := \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix}$$

Consider the map $S_\kappa(\mathfrak{N}, \epsilon, W)^2 \longrightarrow S_\kappa(v\mathfrak{N}, \epsilon, W)$ given by

$$(f, g) = f + g|\eta_v \text{ where } (g|\eta_v)(x) := g(x\eta_v)$$

Definition 14. Let $S^{\text{v-old}}_\kappa(\mathfrak{N}v, \epsilon, W)$ to be the image of the above defined map and $S^{\text{v-new}}_\kappa(\mathfrak{N}v, \epsilon, W)$ be the orthogonal complement of $S^{\text{v-old}}_\kappa(\mathfrak{N}v, \epsilon, W)$ in $S_\kappa(\mathfrak{N}v, \epsilon, W)$ under the Petersson inner product.

Definition 15. A Λ-adic form $\mathcal{F} \in V(\mathfrak{N}, \epsilon, \Lambda)$ will be called v-new for a prime ideal v if $\mathcal{F}_\kappa$ is v-new for all but finitely many classical weights κ with $\kappa_{1,\sigma} - \kappa_{2,\sigma} \geq 1$ for all $\sigma \in I$. If it is not v-new it will be called v-old.

Let $f \in S_\kappa(\mathfrak{N}, \epsilon, W)$ and assume $v \nmid \mathfrak{N}$ is a prime ideal of F. Assume that f is an eigenform for $T(\varpi_w)$ for $w \nmid \mathfrak{N}$ and $U(\varpi_w)$ for $w|\mathfrak{N}$. Suppose α and β are the roots of the Hecke polynomial

$$x^2 - \theta_f(T(\varpi_v))x + (\tilde{\epsilon}_+ \chi_\ell^{[\kappa_1+\kappa_2]})|_{G_{F_v}}(\varpi_v)$$

of f at v. Then

$$f_\alpha := f - \alpha f|\eta_v \in S_\kappa(\mathfrak{N}(v), \epsilon, W)$$

is an eigenform for $T(\varpi_w)$ for $w \nmid \mathfrak{N}v$ and $U(\varpi_w)$ for $w|\mathfrak{N}$ with same eigenvalue as f and an eigenform for $U(\varpi_v)$ with eigenvalue β. The form f_α will be called a v-stabilization of f. Similarly, one can create an eigenform for $T(\varpi_w)$ for $w \nmid \mathfrak{N}v$ and $U(\varpi_w)$ for $w|\mathfrak{N}$ with same eigenvalue as f and an eigenform for $U(\varpi_v)$ with eigenvalue 0. This form will also be called a v-stabilization of f. More generally, suppose now that $\mathfrak{J} = v_1 \cdots v_r$ is a product of distinct prime ideals in F and that it is coprime to $\mathfrak{N}$. A modular form in $S_\kappa(\mathfrak{N}\mathfrak{J}, \epsilon, W)$ will be called an $\mathfrak{J}$-stabilization of f if it is obtained by taking a v_1-stabilization of f, then taking a v_2-stabilization of the resulting form and so on. Such a $\mathfrak{J}$-stabilization of f will be denoted by $f^{\mathfrak{J}\text{-stab}}$. Similar constructions can be made for Λ-adic forms: For example, let $\mathcal{F} \in V(\mathfrak{N}, \epsilon, \Lambda)$ and let $v \nmid \mathfrak{N}$ be a finite place of F. Then, after possibly replacing Λ by a finite extension, there exists $\mathcal{F}|\eta_v \in V(\mathfrak{N}v, \epsilon, \Lambda)$ such that

$$(F|\eta_v)_\kappa = \mathcal{F}_\kappa|\eta_v$$

for all $\kappa \in \mathrm{Hom}(T_G, \mathrm{GL}_1)$. We can now prove:

Lemma 4.5. *Let F be a totally real field and ℓ a rational prime and $\mathfrak{M}, \mathfrak{N}$ square-free integral ideals in F. Let $f \in V_{\mathrm{cusp}}(\mathfrak{N}, 1; W)$ be an eigenform for $\mathfrak{N}$ of some weight $\kappa \in \mathrm{Hom}(T_G, \mathrm{GL}_1)$ such that $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_2 - \kappa_1 + t \geq t$. Let $g \in S_\kappa(\mathfrak{M}, 1; W)$ be a newform. Let $v|\mathfrak{M}$ be a prime and suppose that $\rho_{f,\ell} \cong \rho_{g,\ell}$. Then there exists ideals $\mathfrak{c}_1, \mathfrak{c}_2$ with $\mathfrak{c}_1$ prime to v such that*

$$g^{\mathfrak{c}_1\text{-stab}} = f^{\mathfrak{c}_2\text{-stab}}$$

If f and g are assumed ordinary, then there exists a Λ-adic form deforming $g^{\mathfrak{c}_1\text{-stab}}$ which is v-old.

Proof. Let v be as in the statement of the lemma. By local-global compatibility for f, the $U(\varpi_v)$-eigenvalue γ of f is one of the eigenvalues of the unramified representation $\rho_{f,\ell}|_{G_{F_v}}$ and hence of $\rho_{g,\ell}|_{G_{F_v}}$. It hence is, by local-global compatibility for g, a root of the the Hecke polynomial of g at v. Let δ denote the second root of the Hecke polynomial Hence and consider g_δ and note that

$$U(\varpi_v)g_\delta = \gamma \cdot g_\delta$$

For all other primes $w|\mathfrak{M}\mathfrak{N}$ we take stabilizations of f and g_δ such that the eigenvalues are zero and hence agree. By the assumptions of the proposition and the earlier arguments the resulting stabilizations $g^{\mathfrak{c}_1\text{-stab}}$ and $f^{\mathfrak{c}_2\text{-stab}}$ have the same Hecke eigenvalues at all finite places. It follows that they have the same q-expansions and hence it follows from the q-expansion principle, see for example [HID2, Section 2.3.3] that $g^{\mathfrak{c}_1\text{-stab}} = f^{\mathfrak{c}_2\text{-stab}}$ as desired. Now assume f and g are ℓ-ordinary and let $\mathcal{F}$ be a Λ-adic form deforming f and let $\mathcal{F}^{\mathfrak{c}_2\text{-stab}}$ be the form obtained as a stabilization from $\mathcal{F}$ in the same way as $f^{\mathfrak{c}_2\text{-stab}}$ is obtained from f. Then $\mathcal{F}^{\mathfrak{c}_2\text{-stab}}$ deforms $f^{\mathfrak{c}_2\text{-stab}} = g^{\mathfrak{c}_1\text{-stab}}$ and for sufficiently large weights in $\mathrm{Hom}(T_G, \mathrm{GL}_1)$ its specializations will be v-old since the specializations of $\mathcal{F}$ at such weights are classical and v-old. □

Consider:

Hypothesis 4.5.1. Let F be a totally real field and let ℓ be a rational prime. Let $f \in S_\kappa^{\mathrm{ord}}(\mathfrak{N}, 1, W)$ where $\mathfrak{N}$ is a square-free ideal of $\mathfrak{O}_F$ and assume that f is $\mathfrak{N}$-new and assume $v|\mathfrak{N}$ and $v \nmid (\ell)$ is a prime ideal. Assume that there exists a finite extension K/F such that for all totally real solvable extensions L/F which are linearly disjoint over F with K the following holds: The base change f_L of f to L is cuspidal and for each ordinary (ℓ)-stabilization of every $\mathfrak{c}$-stabilization of f_L for $\mathfrak{c}$ coprime to v, every Hida family passing through it is w-new for all places $w|v$ of L.

Remark. See [HID2, Corollary 3.57] for some relevant rigidity results.

Even though we will not pursue this matter further in this memoir one can modify the previous hypothesis to obtain version of Theorem 4 also for non-ordinary forms. We hope to treat such questions in more detail elsewhere but we state one such possible modified hypothesis in the following:

Hypothesis 4.5.2. Let F be a totally real field and let $f \in S_\kappa(\mathfrak{N}, 1, W)$ where $\mathfrak{N}$ is an integral ideal of $\mathfrak{O}_F$. Assume that there exists a finite extension K/F such that for all totally real solvable extensions L/F which are linearly disjoint over F with K the following holds: The base change f_L of f to L is cuspidal and for every $\mathfrak{c}$-stabilization of f_L for $\mathfrak{c}$ coprime to v whenever it equals the stabilization of an ℓ-adic modular form then the ℓ-adic modular form is classical.

We now follow closely the discussion in [HID2, Section 3.2] adapted to our current situation.

Hypothesis 4.5.3. Let F be a totally real field, ℓ a rational prime and let $f_0 \in S_\kappa^{\mathrm{ord}}(\mathfrak{N}, 1, B)$ be a newform and let π denote the automorphic representation corresponding to f_0. Assume:

- for $v|\ell$ the local component π_v is a principal series representation
- for all $v|\ell$ the Galois representation satisfies

$$\rho_{f_0,\ell}|_{G_{F_v}} \sim \begin{bmatrix} \eta_{1,v}^{-1}(\tilde{\epsilon}_+\chi_\ell^{[\kappa_1+\kappa_2]})|_{G_{F_v}} & * \\ & \eta_{1,v} \end{bmatrix}$$

  where $\eta_{1,v}$ is a character $F_v^\times \longrightarrow B^\times$ such that if $\eta'_{1,v}$ is the corresponding complex character, via our fixed embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$, then

$$\pi_v \hookrightarrow \mathrm{Ind}(\eta'_{1,v}, \eta'_{2,v})$$

  for some character $\eta'_{2,v}$. Let $\overline{\delta}_v := \eta_{1,v} \mod \mathfrak{m}_B$ and assume

$$\overline{\delta}_v^{-1} \det(\overline{\rho}_{f_0,\ell}) \neq \overline{\delta}_v$$

- $\overline{\rho}_{f_0,\ell}$ is absolutely irreducible
- if $v \nmid \ell$ is a finite place such that $\rho_{f_0,\ell}|_{I_{F_v}}$ after restricting to $I_{F'_v}$ for some finite extension F'_v/F_v is non-trivial unipotent then the same holds for $\overline{\rho}_{f_0,\ell}|_{I_{F_v}}$

We now describe modularity lifting results for the residual representation $\overline{\rho}_{f_0}$. Let k be a finite field such that $\overline{\rho}_{f_0,\ell}$ is defined over k and assume from now on that the residue field of B is the finite field k and all Hecke eigenvalues of f_0 for Hecke operators $T(\varpi_\mathfrak{q})$ and $U(\varpi_\mathfrak{q})$ are contained in B. Assume furthermore that if a prime $v \nmid \ell$ divides $\mathfrak{N}$ then

$$\overline{\rho}_{f_0,\ell}|_{I_{F_v}} \sim \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$$

with $*$ non-trivial. We call this last property the condition **(min)**. For every $v|\mathfrak{N}$ write

$$\overline{\rho}_{f_0,\ell}|_{G_{F_v}} \sim \begin{bmatrix} \overline{\epsilon}_v & * \\ & \overline{\delta}_v \end{bmatrix}$$

Let $\mathcal{C}$ denote the category of complete noetherian W-algebras with residue field k. For $A \in \mathcal{C}$ a representation $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ is called a deformation if the reduction modulo the maximal ideal of A is isomorphic to $\overline{\rho}_{f_0,\ell}$. Two such deformations are called strictly equivalent if they are conjugate by an element in $\mathrm{GL}_2(A)$ whose reduction to $\mathrm{GL}_2(k)$ is the identity. Consider the functor $\mathcal{C} \longrightarrow \mathrm{Set}$ which takes an object A of $\mathcal{C}$ to the set of strict equivalence classes of deformations ρ such that

- ρ is unramified away from $\ell\mathfrak{N}$

- For $v|\ell$ one has

$$\rho|_{G_{F_v}} \sim \begin{bmatrix} \epsilon_v & * \\ & \delta_v \end{bmatrix}$$

where ϵ_v and δ_v are characters $G_{F_v} \longrightarrow A^\times$ such that

$$\delta_v \mod \mathfrak{m}_A \cong \overline{\delta}_v$$

and $\delta_v|_{I_{F_v}}$ factors through $\mathrm{Gal}(F_v^{\mathrm{unr}}(\mu_{\ell^\infty})/F_v^{\mathrm{unr}})$

- For all $v|\mathfrak{N}$ that do not divide ℓ one has

$$\rho|_{G_{F_v}} \sim \begin{bmatrix} \epsilon_v & * \\ & \delta_v \end{bmatrix}$$

where ϵ_v and δ_v are characters $G_{F_v} \longrightarrow A^\times$ such that

$$\delta_v \mod \mathfrak{m}_A \cong \overline{\delta}_v \text{ and } \delta_v|_{I_{F_v}} = 1$$

This functor is representable by an object $R = R(N_{\overline{\rho}_{f_0,\ell}})^{\mathrm{ord}}_{\overline{\rho}_{f_0,\ell}}$ in $\mathcal{C}$ and there exists a Galois representation

$$\rho_R : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(R(N_{\overline{\rho}_{f_0,\ell}})^{\mathrm{ord}}_{\overline{\rho}_{f_0,\ell}})$$

such that in particular for $v|\ell$ one has

$$\rho_R|_{G_{F_v}} \sim \begin{bmatrix} \boldsymbol{\epsilon}_v & * \\ & \boldsymbol{\delta}_v \end{bmatrix}$$

where $\boldsymbol{\delta}_v|_{I_{F_v}}$ factors through $\mathrm{Gal}(F_v^{\mathrm{unr}}(\mu_{\ell^\infty})/F_v^{\mathrm{unr}})$ and

$$\boldsymbol{\delta}_v \mod \mathfrak{m}_R = \overline{\delta}_v$$

where $\mathfrak{m}_R$ denotes the maximal ideal of R. Hence, by taking the product over $v|\ell$, one obtains a map $W[[\Gamma]] \longrightarrow R(N_{\overline{\rho}_{f_0,\ell}})^{\mathrm{ord}}_{\overline{\rho}_{f_0,\ell}}$.

Definition 16. Let F be a totally real field and let $f \in S_\kappa(\mathfrak{N}, \epsilon, B)$ be a Hecke eigenform. Define $\mathcal{B}_f$ to be the set of rational primes such that

- f is as in Hypothesis 4.5.3 with respect to ℓ
- $\ell \geq 7$
- ℓ is unramified in F
- $\overline{\rho}_{f,\ell}|_{G_{F(\zeta_\ell)}}$ is absolutely irreducible where ζ_ℓ denotes a primitive ℓ'th root of unity in $\overline{F}$

Let $\mathfrak{N}$ be an ideal of F. Then there is a maximal ideal $\mathfrak{m}$ of the Hecke algebra $\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N}, \epsilon, W[[T(\mathbb{Z}_\ell)]])$ such that there exists a Galois representation

$$\rho_{\mathfrak{h}} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N}, \epsilon, W[[T(\mathbb{Z}_\ell)]])_{\mathfrak{m}})$$

which satisfies the conditions of the above described deformation functor and hence there is a map

$$\pi : R(N_{\overline{\rho}_{f_0,\ell}})^{\mathrm{ord}}_{\overline{\rho}_{f_0,\ell}} \longrightarrow \mathbf{h}^{\mathrm{n.ord}}(\mathfrak{N}, \epsilon, W[[T(\mathbb{Z}_\ell)]])_{\mathfrak{m}}$$

such that $\rho_{\mathfrak{h}}$ is obtained from ρ_R via composition with this map. By work of Fujiwara, see also [HID2, Theorem 3.50] for a similar result in parallel weight, one knows the following: Suppose $f \in S_\kappa^{\mathrm{ord}}(\mathfrak{N}, \epsilon, B)$ is a newform, $\ell \in \mathcal{B}_f$, and if a prime $v \nmid \ell$ divides $\mathfrak{N}$ then condition **(min)** holds. Then the map π is a $W[[\Gamma]]$-algebra isomorphism. We can now prove the main result.

Theorem 4. *Let F be a totally real field and let ℓ be a rational prime and let $f_0 \in S_\kappa^{\mathrm{ord}}(\mathfrak{N}, \epsilon, B)$ be a newform for $\kappa = (\kappa_1, \kappa_2)$ with $\kappa_2 - \kappa_1 + t \geq t$. Assume $\ell \in \mathcal{B}_{f_0}$ and $v|\mathfrak{N}$ is a prime not above ℓ. Assuming Hypothesis 4.5.1, the monodromy operator N of the Weil-Deligne representation associated to $\rho_{f_0,\ell}|_{W_{F_v}}$ is non-trivial if and only if it is predicted to be non-trivial by the local Langlands correspondence.*

Proof. After a solvable base change one can assume that $\mathfrak{N}$ is a square-free ideal and ϵ is such that ϵ_1 and ϵ_2 are trivial. Hence we will now assume this. Let π be the automorphic representation associated to f_0. If the local Langlands correspondence predicts, in the above notation, that N is trivial then π_v is a principal series representation and by Hypothesis 4.1.1 (i) it follows that N is indeed trivial. Hence suppose now that the local Langlands correspondence predicts that N is non-trivial. Then $\pi_v \cong \mathrm{St}_2(\chi)$ for some unramified character χ. Suppose for contradiction that N is trivial. For example by [JAR, Theorem 4.4] there exists $E \in M_{\kappa'}(\mathfrak{c}, 1, \mathbb{Z}_\ell)$ of sufficiently large classical weight κ' and with $\mathfrak{c}$ coprime to $\mathfrak{N}\mathfrak{d}(\ell)$ and such that

$$E \equiv 1 \mod \ell$$

in the sense that there is an integer ring $\mathfrak{O}$ in a finite extension of $\mathbb{Q}_\ell$ such that the Fourier coefficients of E and 1 differ by elements in valuation ring of $\mathfrak{O}$. The product fE^r for r sufficiently large is then an ordinary form of cohomological weight $\tilde{\kappa} = (\tilde{\kappa}_1, \tilde{\kappa}_2)$ with $\tilde{\kappa}_2 - \tilde{\kappa}_1 \geq t$ and $0 \neq e_p(fE^r) \in S_{\tilde{\kappa}}(\mathfrak{N}\mathfrak{c}, 1, W)$. One can deduce from classical results of Deligne-Serre that there exists an ordinary eigenform $d \in S_{\tilde{\kappa}}^{\mathrm{ord}}(\mathfrak{N}\mathfrak{c}, 1, W)$ such that

$$\overline{\rho}_{d,\ell} \cong \overline{\rho}_{f,\ell}$$

Let T be the set of finite places of F such that if $v \in T$ then $v|\mathfrak{N}$ and the Weil-Deligne representation associated to $\rho_{f_0,\ell}|_{G_{F_v}}$ has trivial monodromy operator. Let π_2 denote the automorphic representation associated to d. By the potential level-lowering results of Skinner-Wiles in [SW] there exists a solvable totally real extension L/F such that $\mathrm{BC}_L(\pi_2)$ is cuspidal and there is a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_L)$ of the same weight as $\mathrm{BC}_L(\pi_2)$ such that if $\tilde{g}$ denotes the newform associated to $\tilde{\pi}$ and if f denotes the newform associated to $\mathrm{BC}_L(\pi)$, then

- $\overline{\rho}_{f,\ell} \cong \overline{\rho}_{\tilde{g},\ell}$
- $\tilde{g}$ is of a level which is not divisible by finite places w of L above T
- $\tilde{g}$ is ordinary at all places above ℓ

For suitably chosen L one has $\ell \in \mathcal{B}_{\tilde{g}}$ and we assume this from now on. Furthermore, condition **(min)** holds for $\tilde{g}$. Hence there is an isomorphism of W-algebras

$$R(N_{\overline{\rho}_{\tilde{g},\ell}})^{\mathrm{ord}}_{\overline{\rho}_{\tilde{g},\ell}} \cong \mathbf{h}^{\mathrm{n.ord}}(\mathfrak{M}, 1, W[[T(\mathbb{Z}_\ell)]])_{\mathfrak{m}}$$

where $\mathfrak{m}$ is a maximal ideal corresponding to the residual representation $\overline{\rho}_{\tilde{g},\ell}$ and $\mathfrak{M}$ is an ideal of $\mathfrak{O}_L$ which is prime to places above T. Note that f is w-new for all places $w|v$ of L. Since $\mathfrak{M}$ can be chosen such that $\rho_{f,\ell}$ is unramified at all finite places not dividing $\mathfrak{M}(\ell)$ and since $\rho_{f,\ell}$ is ordinary at ℓ one obtains

$$\xi \in \mathrm{Hom}_{W\text{-alg}}(R(N_{\overline{\rho}_{\tilde{g},\ell}})^{\mathrm{ord}}_{\overline{\rho}_{\tilde{g},\ell}}, W) \cong \mathrm{Hom}_{W\text{-alg}}(\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{M}, 1, W[[T(\mathbb{Z}_\ell)]])_{\mathfrak{m}}, W)$$

such that $\rho_{\pi,\ell}|_{\mathrm{Gal}(\overline{F}/L)}$ is the composition of ρ_R with ξ. Since

$$\mathrm{Hom}_W(\mathbf{h}^{\mathrm{n.ord}}(\mathfrak{M}, 1, W[[T(\mathbb{Z}_\ell)]]), W) \cong V_{\mathrm{cusp}}^{\mathrm{ord}}(\mathfrak{M}, 1; W)$$

there exists $g \in V_{\mathrm{cusp}}^{\mathrm{ord}}(\mathfrak{M}, 1; W)$ which is an eigenform and has a weight and such that $\rho_{g,\ell} \cong \rho_{f,\ell}$. By Proposition 4.5 there exists ideals $\mathfrak{c}_1, \mathfrak{c}_2$ with $\mathfrak{c}_1$ prime to places w above v such that

$$g^{\mathfrak{c}_1\text{-stab}} = f^{\mathfrak{c}_2\text{-stab}}$$

and there exists a Λ-adic form deforming $g^{\mathfrak{c}_1\text{-stab}}$ which is w-old for some places $w|v$ and this contradicts Hypothesis 4.5.1. $\square$

5 Local-global compatibility results via crystalline periods

As a first step towards proving local-global compatibility results for automorphic representations of GL_n and GSp_{2n} we generalize in this chapter the results of Section 4.1.2. This means that we are using unitary group eigenvarieties, as constructed by Chenevier, and Kisin's result on variation of crystalline periods to obtain certain local-global compatibility results for local semi-simplifications. The key result is Lemma 5.5 which concerns local properties of Galois representations associated to automorphic representations of totally definite unitary groups. This is a case that lends itself very well to the construction of eigenvarieties. Via functoriality this has then implications for local-global compatibility for automorphic Galois representations of general linear groups and symplectic groups. In the following chapters we make these implications more explicit.

5.1 Galois representations

We will now state precisely which properties of automorphic Galois representations will be assumed throughout this chapter. Let F be a CM-field and let F^+ denote its maximal totally real subfield and let c denote the non-trivial element in $\mathrm{Gal}(F/F^+)$. An automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is called

- conjugate self-dual if $\Pi^\vee \cong \Pi^c$
- regular algebraic if it has the same infinitesimal character as a finite-dimensional irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$

Let $\mathbb{Z}^{n,+}$ be the set of decreasing n-tuples of integers. For any

$$\overline{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^{n,+}$$

let $W_{\overline{k}}$ denote the algebraic representation of GL_n of highest weight $\overline{k}$, corresponding to the character

$$\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix} \mapsto \prod_{i=1}^{n} t_i^{k_i}$$

of the diagonal torus of GL_n. A finite-dimensional irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$ then corresponds to $\overline{a} = (\overline{a}_\tau)_{\tau \in \mathrm{Hom}(F,\mathbb{C})} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$ and we will say Π is of weight $\overline{a}$ if it has the same infinitesimal character as the finite-dimensional irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)$ corresponding to $\overline{a}$.

Let $\mathrm{rec}_{\mathrm{GL}_n}(-)$ denote a local Langlands correspondence for GL_n such that in particular the following holds. Suppose K is a p-adic field and $\chi = \prod_{i=1}^n \chi_i$ is a character of the diagonal torus of $\mathrm{GL}_n(K)$ such that the principal series representation $\mathrm{Ind}(\chi)$ is irreducible. Then

$$\mathrm{rec}_{\mathrm{GL}_n}(\mathrm{Ind}(\chi)) \cong \bigoplus_{i=1}^{n} \tilde{\chi}_i$$

where $\tilde{\chi}_i$ for $1 \leq i \leq n$ is as defined in the beginning of Chapter 3.

The following hypothesis will be assumed for the rest of this chapter. It will be combined with the deformation theory of automorphic forms to deduce more general local-global compatibility results.

Hypothesis 5.1.1. Let Π be a cuspidal regular algebraic conjugate self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$. For a rational prime ℓ and $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ there is a continuous semi-simple ℓ-adic Galois representation

$$\rho_{\Pi,\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

such that

(i) for all finite places $v \nmid \ell$ of F such that Π_v is a principal series representation there is an isomorphism

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})$$

(ii) for $v|\ell$

- if Π_v is unramified then $\rho_{\Pi,\ell,\iota}$ is crystalline at v and

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})$$

- $\rho_{\Pi,\ell,\iota}$ is Hodge-Tate at each $v|\ell$ and if $\tau \in \mathrm{Hom}(F,\mathbb{C})$ lies above v via ι then the corresponding set of Hodge-Tate weights equals

$$\mathrm{HT}_\tau(\rho_{\Pi,\ell,\iota}|_{G_{F_v}}) = \{-(a_{\tau,i} + n - i) | 1 \le i \le n\}$$

- if Π_v is Iwahori-spherical then $\rho_{\Pi,\ell,\iota}|_{G_{F_v}}$ is potentially semi-stable

(iii) for v a finite place such that Π_v is Iwahori-spherical and for any rational primes ℓ_1, ℓ_2 and isomorphisms $\iota_{\ell_1} : \overline{\mathbb{Q}}_{\ell_1} \xrightarrow{\sim} \mathbb{C}$ and $\iota_{\ell_2} : \overline{\mathbb{Q}}_{\ell_2} \xrightarrow{\sim} \mathbb{C}$

$$\mathrm{WD}_{\iota_{\ell_1}}\left(\rho_{\Pi,\ell_1,\iota_{\ell_1}}|_{W_{F_v}}\right)^{\mathrm{ss}} \cong \mathrm{WD}_{\iota_{\ell_2}}\left(\rho_{\Pi,\ell_2,\iota_{\ell_2}}|_{W_{F_v}}\right)^{\mathrm{ss}}$$

and these representations are unramified.

(iv)

$$\rho_{\Pi,\ell,\iota}^\vee \cong \rho_{\Pi,\ell,\iota}^c \otimes \chi_\ell^{n-1}$$

where χ_ℓ denotes the ℓ-adic cyclotomic character

Remark. In part (iii) we have assumed for simplicity that the semi-simple parts of the Weil-Deligne representations are unramified. This assumption can be avoided but then the statements of main local-global compatibility results have to be somewhat changed. If the Galois representations are realized for example in the cohomology of a Shimura variety then this part of the assumption can be obtained by constructing good integral models for the Shimura-variety with suitable Iwahori level-structure. See for example [TY].

In order to deduce from Hypothesis 5.1.1 local-global compatibility results at places where the local Langlands correspondence predicts ramification we work in this section with Galois representations associated to automorphic forms on totally definite unitary groups. By using functoriality results of [LAB] the results of this section will then later be used to to obtain results for automorphic representations of general linear groups. The advantage of working with totally definite unitary groups is that for such groups families of automorphic forms have been constructed in [CHE]. This allows us to apply the crystalline periods methods described in Chapter 3 in a similar way as in the proof of Proposition 4.2.

Note that contrary to the case of Hilbert modular forms, in the more general setting that we treat in this section there can be multiple accessible refinements of the relevant local component of the automorphic representation and each of these refinements can lead to different information about the local properties of the automorphic Galois representation. In this section we simply determine what information a given accessible refinement yields and only in later sections will we then combine the information coming from all the different accessible refinements to obtain more precise local-global compatibility results.

Let us first describe the Galois representations associated to automorphic representations on totally definite unitary groups.

Let F^+ be a totally real field, let E an imaginary quadratic extension of $\mathbb{Q}$ and let $F := F^+E$ be a CM-field. Let c denote the non-trivial element of $\mathrm{Gal}(F/F^+)$. Let $n \ge 2$ be an integer and assume that F/F^+ is unramified at all finite places and that $n[F^+ : \mathbb{Q}]/2$ is even. Let U_n^* be the quasi-split unitary group over F^+ as defined in [LAB, Sect.

2.1]. As discussed in [CH, Lem. 2.1] there is an inner form G of U_n^* that is quasi-split at all places of F^+ which are inert in F and such that $\prod_{v|\infty} G(F_v^+)$ is compact.

Let π be an automorphic representation of $G(\mathbb{A}_{F^+})$. The weight of π is defined to be the representation ξ of $\prod_{v|\infty} G(F_v^+)$ on $\otimes_{v|\infty}\pi_v$. As explained for example in [CHL, Sect. 2], the representation ξ corresponds to

$$\overline{a}(\xi) \in (\mathbb{Z}^{n,+})^{\{\tau : F \hookrightarrow \mathbb{C}\}}$$

such that

$$\overline{a}(\xi)_{\tau,i} = -\overline{a}(\xi)_{\tau\circ c, n+1-i}$$

for each $1 \leq i \leq n$. For π as above, a base change to an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is constructed in [LAB] and this allows to associate to π a Galois representation. To describe this Galois representation note that if $v \in \mathrm{Spl}_{F/F^+}$ factors as $v = ww^c$ in F then one obtains isomorphisms i_w and i_{w^c} between $G \otimes_{F^+} F_v^+$ and GL_{n,F_w} and $\mathrm{GL}_{n,F_{w^c}}$. The following hypothesis is a consequence of Hypothesis 5.1.1 by results of [GUE].

Hypothesis 5.1.2. Let π be an automorphic representation of $G(\mathbb{A}_{F^+})$ of weight ξ. For a rational prime ℓ and $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ there is a continuous semi-simple Galois representation

$$\rho_{\pi,\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

such that

(i) for all finite places $v \nmid \ell$ of F^+ which split as $v = ww^c$ in F and for which π_v is a principal series representation one has

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_w}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}((\pi_v \circ i_w^{-1}) \otimes |\det|^{\frac{1-n}{2}})$$

(ii) if $v|\ell$ splits as ww^c in F then

- if π_v is spherical then $\rho_{\pi,\ell,\iota}$ is crystalline at w and

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{G_{F_w}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}((\pi_v \circ i_w^{-1}) \otimes |\det|^{\frac{1-n}{2}})$$

- $\rho_{\pi,\ell,\iota}$ is Hodge-Tate at w and if $\tau \in \mathrm{Hom}(F,\mathbb{C})$ lies above w via ι then the corresponding set of Hodge-Tate weights equals

$$\mathrm{HT}_\tau(\rho_{\pi,\ell,\iota}|_{G_{F_w}}) = \{-(\overline{a}(\xi)_{\tau,i} + n - i) | 1 \leq i \leq n\}$$

- if π_v is Iwahori-spherical then $\rho_{\pi,\ell,\iota}|_{G_{F_w}}$ is potentially semi-stable at w

(iii) for v a finite place of F^+ which splits as $v = ww^c$ in F and such that $\pi_v \circ i_w^{-1}$ is Iwahori-spherical and for any rational primes ℓ_1, ℓ_2 and isomorphisms $\iota_{\ell_1} : \overline{\mathbb{Q}}_{\ell_1} \xrightarrow{\sim} \mathbb{C}$ and $\iota_{\ell_2} : \overline{\mathbb{Q}}_{\ell_2} \xrightarrow{\sim} \mathbb{C}$

$$\mathrm{WD}_{\iota_{\ell_1}}(\rho_{\pi,\ell_1,\iota_{\ell_1}}|_{W_{F_w}})^{\mathrm{ss}} \cong \mathrm{WD}_{\iota_{\ell_2}}(\rho_{\pi,\ell_2,\iota_{\ell_2}}|_{W_{F_w}})^{\mathrm{ss}}$$

and these representations are unramified

(iv)

$$\rho_{\pi,\ell,\iota}^\vee \cong \rho_{\pi,\ell,\iota}^c \otimes \chi_\ell^{n-1}$$

where χ_ℓ denotes the ℓ-adic cyclotomic character

(v) if the base change Π of π to $\mathrm{GL}_n(\mathbb{A}_F)$ is cuspidal then

$$\rho_{\pi,\ell,\iota} \cong \rho_{\Pi,\ell,\iota}$$

Moreover, if $\rho_{\pi,\ell,\iota}$ is irreducible then the base change of π to $\mathrm{GL}_n(\mathbb{A}_F)$ is cuspidal

5.2 Eigenvariety preliminaries

In this section fix G, F and F^+ as in Section 5.1. As a preparation for the proof of Lemma 5.5 we will in this section briefly recall some facts about unitary group eigenvarieties as constructed in [CHE].

Let p be a rational prime and fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Fix a decomposition

$$\{\text{places } u|p \text{ of F}^+\} = S_p \sqcup S_p'$$

with $\#S_p = 1$ and assume $S_p = \{v\}$ with $v \in \mathrm{Spl}_{F/F^+}$. In the discussion until Lemma 5.5 fix a place w of F above v and let i_w be the corresponding isomorphism $G(F_v^+) \cong \mathrm{GL}_n(F_w)$. In the following we will identify these groups. Via this identification we let $T_v \subseteq G(F_v^+)$ be the diagonal torus and we let T_v^0 be the maximal compact subgroup of T_v consisting of elements with entries in $\mathcal{O}_{F_w}$. Furthermore, we let B_v denote the upper triangular Borel subgroup of $G(F_v^+)$. Let $\mathbb{G}_m^{\mathrm{rig}}$ denote the analytification, as defined in [BGR, Sect. 9.3.4], of the algebraic variety $\mathbb{G}_m$. View T_v via restriction of scalars as the $\mathbb{Q}_p$-valued points of a torus defined over $\mathbb{Q}_p$ and define the rigid spaces over $\mathbb{Q}_p$ given by $\mathcal{T}_{S_p} := \mathrm{Hom}(T_v, \mathbb{G}_m^{\mathrm{rig}})$ and $\mathcal{W}_{S_p} := \mathrm{Hom}(T_v^0, \mathbb{G}_m^{\mathrm{rig}})$ where in both cases the the homomorphisms are required to be p-adically continuous.

For $u \in S_p \sqcup S_p'$ let $\Sigma(u)$ be the subset of $\mathrm{Hom}(F^+, \overline{\mathbb{Q}}_p)$ consisting of homomorphisms lying above u and let $\Sigma_\infty(u)$ be the subset of $\mathrm{Hom}(F^+, \mathbb{C})$ corresponding to $\Sigma(u)$ via ι.

Let Σ_∞' denote the union of the sets $\Sigma_\infty(u)$ for $u \in S_p'$ and fix an irreducible representation W_∞ of $\prod_{\tau \in \Sigma_\infty'} G(F_\tau^+)$. Let S be a finite set of finite places of F^+ disjoint from S_p. Fix a compact open subgroup

$$K = \prod K_u \leq G(\mathbb{A}_{F^+}^{(\infty)})$$

where u ranges over the finite places of F^+ and K_u is a subgroup of $G(F_u^+)$ which

- is a hyperspecial maximal compact subgroup at all places $u \notin (S_p \sqcup S)$
- equals $G(\mathcal{O}_{F_u^+})$ for almost all places u
- is the Iwahori subgroup corresponding to B_v at $v \in S_p$

Let

$$\mathcal{H}^{(S_p)} = \otimes'_{u \notin (S_p \sqcup S)} \mathcal{H}_u$$

where $\mathcal{H}_u$ for $u \notin (S_p \sqcup S)$ denotes the spherical Hecke algebra consisting of functions on $G(F_u^+)$ which are locally constant, have compact support and are K_u bi-invariant. Let π be an automorphic representation of $G(\mathbb{A}_{F^+})$ such that $\pi^K \neq (0)$ and such that the representation of $\prod_{\tau \in \Sigma_\infty'} G(F_\tau^+)$ on $\otimes_{\tau \in \Sigma_\infty'} \pi_\tau$ is isomorphic to W_∞. To such an automorphic representation one can associate pairs

$$(\psi_\pi, \nu_\pi) \in \mathrm{Hom}(\mathcal{H}^{(S_p)}, \overline{\mathbb{Q}}_p) \times \mathcal{T}_{S_p}(\overline{\mathbb{Q}}_p)$$

in the following way:

Firstly, π gives rise to an algebra homomorphism

$$\psi_\pi : \mathcal{H}^{(S_p)} \longrightarrow \overline{\mathbb{Q}}_p$$

coming from the Hecke action. Secondly, since $\pi^K \neq (0)$ it follows that π_v is Iwahori-spherical for $v \in S_p$. Hence $\pi_v \otimes |\det|^{\frac{1-n}{2}}$ has an accessible refinement as defined in [CHE] and recalled in Section 3.3. Define

$$\nu_\pi := \chi_{\pi,\infty,v} \cdot \chi'_{\pi,v} \cdot \delta_{B_v}^{-1/2} \cdot |\det|^{\frac{n-1}{2}} \in \mathcal{T}_{S_p}(\overline{\mathbb{Q}}_p)$$

where

- $\chi'_{\pi,v}$ is an accessible refinement of $\pi_v \otimes |\det|^{\frac{1-n}{2}}$

- $$\chi_{\pi,\infty,v} = \prod_{u\in\Sigma_\infty(v)} \chi_{\pi_u,u}$$

  where $\chi_{\pi_u,u} : T_v \to \overline{\mathbb{Q}}_p^\times$ is defined as follows. The representation of $G(F_u^+)$ on π_u gives rise, via the embedding $F \hookrightarrow \overline{\mathbb{Q}}_p$ which lies above u via ι and which gives rise to the fixed place w of F above v, to an algebraic representation of GL_n. Via i_w this gives rise to an algebraic representation of $G(F_v^+)$ and $\chi_{\pi_u,u}$ is the corresponding highest weight character

Remark 9. Note that a given automorphic representation π as above yields multiple pairs (ψ_π, ν_π) if π_v has multiple accessible refinements.

Let $\mathcal{Z}'$ denote the collection of all elements $(\psi_\pi, \nu_\pi) \in \mathrm{Hom}(\mathcal{H}^{(S_p)}, \overline{\mathbb{Q}}_p) \times \mathcal{T}_{S_p}(\overline{\mathbb{Q}}_p)$ obtained from automorphic representations π of $G(\mathbb{A}_{F^+})$ as above. In [CHE, Thm. 1.6] an associated eigenvariety is constructed. This is a quadruple (X, ψ, ν, Z') which has the following properties:

- X is a reduced rigid space defined over a finite extension of $\mathbb{Q}_p$ which contains the image of all embeddings $F^+ \hookrightarrow \overline{\mathbb{Q}}_p$
- $\psi : \mathcal{H}^{(S_p)} \longrightarrow \mathcal{O}(X)$ is a ring homomorphism
- $\nu : X \longrightarrow \mathcal{T}_{S_p}$ is a finite analytic map
- $Z' \subseteq X(\overline{\mathbb{Q}}_p)$ is a Zariski dense subset such that for all $z \in Z'$ there is an automorphic representation π_z such that $(\psi_z, \nu(z)) = (\psi_{\pi_z}, \nu_{\pi_z})$ for some choice of ν_{π_z} and (ψ, ν) induces a bijection between $\mathcal{Z}'$ and Z'. Here ψ_z denotes the composition of ψ with the map to the residue field of z

It is important for the application to local-global compatibility questions that the automorphic Galois representations associated to the automorphic representations π_z for $z \in Z'$ are interpolated on the eigenvariety in the following way. By [CHE, Cor. 3.9] there is a continuous n-dimensional pseudo-character

$$T : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathcal{O}(X)$$

such that for each $z \in Z'$ with associated automorphic representation π_z one has

$$T_z = \mathrm{Trace}\ (\rho_{\pi_z,p,\iota})$$

where T_z is the composition of T with the map $\mathcal{O}(X) \to \mathcal{O}_{X,z}/\mathfrak{m}_{\mathcal{O}_{X,z}}$ to the residue field of z.

Let us make some definitions for later use:

Definition 17. Fix a uniformizer $\varpi \in F_v^+$ and let q denote the size of the residue field of F_v^+. For each $1 \le i \le n$ let $u_i \in \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be the diagonal matrix

$$u_i = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \varpi & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

with ϖ in the i'th column and all other entries being 1's. Define for each $1 \le i \le n$

$$\mathrm{F}_i := \nu(u_i) \in O(X)^\times$$

For each $z \in Z'$ let $\chi_{\pi_z} := \chi'_{\pi_z,v} \otimes |\det|^{\frac{n-1}{2}}$ and note that

$$\mathrm{F}_i(z) = \nu_{\pi_z}(u_i) = \iota^{-1}\chi_{\pi_z,i}(\varpi) q^{\frac{n+1}{2}-i} \prod_{\tau\in\Sigma(v)} \tau(\varpi)^{k_{v,\tau,i}}$$

where $(k_{v,\tau,i})_{\tau,i}$ is the element in $(\mathbb{Z}^{n,+})^{\tau\in\Sigma(v)}$ corresponding to $\chi_{\pi_z,\infty,v}$.

Definition 18. As before, for $z \in Z'$ let π_z be the corresponding automorphic representation of $G(\mathbb{A}_{F^+})$. Define Z to be the subset of Z' consisting of those $z \in Z'$ such that $\pi_{z,v}$ is an unramified principal series representation for $v \in S_p$.

We want to use Z and X to obtain local-global compatibility results for Galois representations associated to automorphic representations π_z with $z \in Z'$ but $z \notin Z$. In order to do so we recall in Lemma 5.3 a quantitative version of the fact that automorphic representations π_z for $z \in Z'$ can be approximated in a certain sense by automorphic representations π_z for $z \in Z$. Let us first recall from [BC] some notation for rigid spaces:

Definition 19. A subset U of a rigid space Y is said to accumulate at a point $x \in Y$ if there is a basis of affinoid neighborhoods of x such that U is Zariski dense in each. More generally, U is defined to accumulate at a subset U' of Y if U accumulates at every $u' \in U'$.

The following lemma will eventually allow us to deduce from Hypothesis 5.1.1 local-global compatibility results for automorphic Galois representations at places where ramification is expected. The proof of the lemma essentially follows from the arguments in [CHE2, Prop. 6.4.7] but since only the case of $F^+ = \mathbb{Q}$ is written there we will now recall the proof:

Lemma 5.3. *Z accumulates at Z'.*

Proof. Let $v \in S_p$ be as defined earlier. For $C \in \mathbb{Z}^{\geq 0}$ let $X^*(T_v)^{\mathrm{reg,C}}$ be the subset of $X^*(T_v)$ corresponding to $(k_{v,\tau,i})_{\tau,i} \in (\mathbb{Z}^{n,+})^{\tau\in\Sigma(v)}$ such that

$$k_{v,\tau,i} - k_{v,\tau,j} > C \text{ for all } \tau \in \Sigma(v) \text{ and all } 1 \leq i < j \leq n$$

Let $X^*(T_v)^{\mathrm{reg}} := X^*(T_v)^{\mathrm{reg},0}$. For any Galois extension $L/\mathbb{Q}_p$ such that L contains F_v^+ view $X^*(T_v) \subset \mathcal{W}_{S_p}(L)$. By the same proof as in [CHE, Lem. 2.7], but with $X^*(T_v)^{\mathrm{reg,C}}$ replacing $X^*(T_v)^{\mathrm{reg}}$ in the argument, it follows that $X^*(T_v)^{\mathrm{reg,C}}$ accumulates at $X^*(T_v)$. The relation between $\mathcal{W}_{S_p}$ and the eigenvariety X is now used to also show that Z accumulates at Z': Let $z \in Z'$ and let U' be an admissible open neighborhood of z in X. Let $U \subseteq U'$ be an affinoid neighborhood of z. To show that Z accumulates at Z' it suffices to show that $Z \cap U$ is Zariski dense in U for U sufficiently small. Let κ denote the composition of ν with the natural map $\mathcal{T}_{S_p} \to \mathcal{W}_{S_p}$ and let V be a sufficiently small affinoid neighborhood of $\kappa(z)$. By [CHE, Lem. 2.7], and the accumulation of $X^*(T_v)^{\mathrm{reg,C}}$ at $X^*(T_v)$, for any $C \in \mathbb{Z}^{\geq 0}$ one has that $X^*(T_v)^{\mathrm{reg,C}} \cap V$ is Zariski dense in V. Let q denote the size of the residue field of F_v^+ and in the following use additive notation for the product on $X^*(T_v)$. By using [CHE2, Lem. 6.2.8], to prove that Z accumulates at Z' it suffices to show that for C sufficiently large any point $x \in U$ with $\kappa(x) \in X^*(T_v)^{\mathrm{reg,C}}$ and $\kappa(x) - \kappa(z) \in (q-1)p^N X^*(T_v)$ for some sufficiently large $N \in \mathbb{Z}$ satisfies the following: The point x corresponds to an automorphic representation π_x and the local component $\pi_{x,v}$ is an unramified principal series representation for $v \in S_p$. The former follows for U sufficiently small from the classicality criterion of [CHE, Thm. 1.6]. For the latter note that $\pi_{x,v}$ is Iwahori-spherical and hence $\pi_{x,v} \hookrightarrow \mathrm{Ind}(\chi)$ for some unramified character $\chi = \prod_{i=1}^n \chi_i$. Suppose that $\pi_{x,v}$ is not an unramified principal series representation. Then there is $i < j$ such that

$$\chi_i(\varpi)/\chi_j(\varpi) = q^{\pm 1}$$

Then

$$\frac{\mathrm{F}_i(x)}{\mathrm{F}_j(x)} = q^{\pm 1 + i - j} \prod_{\tau\in\Sigma(v)} \tau(\varpi)^{k_{v,\tau,i} - k_{v,\tau,j}}$$

where $(k_{v,\tau,i})_{\tau,i}$ is the element in $(\mathbb{Z}^{n,+})^{\tau\in\Sigma(v)}$ corresponding to $\chi_{\pi_x,\infty,v}$. As explained in [CHE2, Prop. 6.4.7], since F_i and F_j are in $\mathcal{O}(X)^\times$ there is a constant D such that $|\mathrm{F}_i(x)/\mathrm{F}_j(x)|_p > D > 0$ for all $x \in U$. Hence for C sufficiently large one obtains a contradiction if $\pi_{x,v}$ is not an unramified principal series representation. It follows that Z accumulates at Z'. □

5.4 The key lemma

We will now study crystalline periods of exterior powers of automorphic Galois representations in a somewhat similar way to [BC2, Prop. 6.1]. This will be used to obtain information about local properties of automorphic Galois representations at places of Iwahori-spherical ramification of the automorphic representation. In the following statement the morphism i_w is as defined in the discussion preceding Hypothesis 5.1.2.

Lemma 5.5. *Let G, F and F^+ be as above and let π be an automorphic representation of $G(\mathbb{A}_{F^+})$. Let v be a finite place of F^+ and let q denote the size of the residue field of F_v^+. Suppose v splits as $v = ww^c$ in F and suppose $\pi_v \circ i_w^{-1}$ is Iwahori-spherical. Let ϖ be a uniformizer of F_w. For any lift $\sigma \in W_{F_w}$ of the geometric Frobenius let S_σ denote the multiset of eigenvalues of σ acting on $\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_w}})$. Then for any accessible refinement*

$$\chi_v = \prod_{j=1}^{n} \chi_{v,j}$$

of $\pi_v \circ i_w^{-1}$ and each $1 \leq i \leq n$ there is a multiset

$$\{\alpha_{j_1}, \cdots, \alpha_{j_i}\} \subseteq S_\sigma$$

such that

$$\prod_{k=1}^{i} \alpha_{j_k} = q^{\frac{i(n-1)}{2}} \prod_{k=1}^{i} \chi_{v,k}(\varpi)$$

Moreover, one always has $q^{\frac{n-1}{2}}\chi_{v,1}(\varpi) \in S_\sigma$ and $q^{\frac{n-1}{2}}\chi_{v,n}(\varpi) \in S_\sigma$.

Proof. Let v be a finite place of F^+ as in the statement of the lemma and let p denote its residue characteristic. Fix throughout this proof a finite place w of F lying above v and let $\tilde{\Sigma}(v)$ denote the subset of embeddings in $\mathrm{Hom}(F, \overline{\mathbb{Q}}_\ell)$ which lie above $\Sigma(v)$ and which correspond to w. Fix a uniformizer ϖ of F_w. By Hypothesis 5.1.2 (iii) in order to prove the lemma one can assume that $p = \ell$. This allows the use of the eigenvarieties described in the beginning of Section 5.2. In the notation of the beginning of that section let us make the following definitions:

- Let $S_p := \{v\}$
- Let S be a finite set of finite places of F^+ which is disjoint from S_p and contains all places at which π is ramified except the place v
- Let $K \leq G(\mathbb{A}_{F^+}^{(\infty)})$ be a compact open subgroup of the form described in the beginning of Section 5.2 and such that $\pi^K \neq (0)$
- Let W_∞ denote the representation of $\prod_{\tau\in\Sigma'_\infty} G(F_\tau^+)$ on $\otimes_{\tau\in\Sigma'_\infty}\pi_\tau$ where Σ'_∞ is as defined in the beginning of Section 5.2

Let (X, ψ, ν, Z') be the associated eigenvariety and let Z be the subset of Z' as in Definition 18. The automorphic representation π and the accessible refinement χ_v of the statement of the lemma yield a point

$$x_0 = (\psi_\pi, \nu_\pi) \in X(\overline{\mathbb{Q}}_p)$$

We will now apply the crystalline periods results described in Chapter 3. To be able to do so one must first describe twists of the Galois representations corresponding to points on the eigenvariety X by a suitable character and we will do this now.

Let $z \in Z'$ and as before let $(k_{v,\tau,i})_{\tau,i}$ be the element in $(\mathbb{Z}^{n,+})^{\Sigma(v)}$ corresponding to $\chi_{\pi_z,\infty,v}$. The representation ρ_z is Hodge-Tate at w and for each $\tau \in \tilde{\Sigma}(v)$ and $1 \leq i \leq n$ one has

$$m_{\tau,i}(z) := \mathrm{HT}_\tau^{(1)}(\wedge^i \rho_z|_{G_{F_w}}) = \frac{i(i+1-2n)}{2} - \sum_{j=1}^{i} k_{v,\tau,j}$$

where $\mathrm{HT}_\tau^{(1)}(-)$ is as defined in Section 3.4. Since $\nu : X \to \mathcal{T}_{S_p}$ is an analytic map and since by [CHE, Thm. 1.6] Z' accumulates at X, there is a neighborhood X' of x_0 and $m_{\tau,i} \in \mathcal{O}(X')$ such that the specializations of $m_{\tau,i}$ at points $z \in Z'$ which are also in X' are given by the above formula. For ease of notation, in the the rest of this proof we will denote the intersection of Z' and X' again by Z' and we will denote X' by X. For each $\tau \in \tilde{\Sigma}(v)$ fix a continuous character $\chi_\tau : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ such that

- $\chi_{\tau,w}$ is unramified
- $\chi_\tau|_{(F_\infty^\times)^\circ}((x_\sigma)_{\sigma\in\mathrm{Hom}(F,\mathbb{C})}) = \prod_{\sigma\in\mathrm{Hom}(F,\mathbb{C})} x_\sigma^{a_\sigma}$ with a_σ given by $a_{\iota(\tau)} = 1$ and $a_{\iota(\tau)\circ c} = -1$ and zero otherwise

See for example [CHT, Sect. 4] for the existence of a character with the weight as defined above. Define

$$\chi_\tau^{(\ell)}(x) := \prod_\sigma (\iota^{-1}\sigma)(x_\ell)^{a_\sigma} \iota^{-1}(\prod_\sigma x_\sigma^{-a_\sigma} \chi_\tau(x))$$

By global class field theory the character $\chi_\tau^{(\ell)}$ corresponds to a continuous character

$$r_{\ell,\iota}(\chi_\tau) : \mathrm{Gal}(\overline{F}/F) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

Since $m_{\tau,i} \in \mathcal{O}(X)$ and since Z' accumulates at X one can define a continuous map

$$\mu_i : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathcal{O}(X)$$

such that for all $z \in Z'$ one has

$$\mu_i(g)(z) = \prod_{\tau\in\tilde{\Sigma}(v)} r_{\ell,\iota}(\chi_\tau)(g)^{-m_{\tau,i}(z)}$$

for all $g \in \mathrm{Gal}(\overline{F}/F)$. For $x \in X$ let $\mu_{i,x}$ denote the composition of μ with the map to the residue field of x. Note that for example by [CHT, Lem. 4.1.3] the representation $r_{\ell,\iota}(\chi_\tau)|_{G_{F_w}}$ has Hodge-Tate weight 1 with respect to τ. Furthermore, since $w|v \in \mathrm{Spl}_{F/F^+}$ it follows that $\Sigma_\infty(w) \cap c \circ \Sigma_\infty(w) = \varnothing$ where c denotes the non-trivial element of $\mathrm{Gal}(F/F^+)$. It follows that for each $z \in Z'$ and $\tau \in \tilde{\Sigma}(v)$ one has

$$\mathrm{HT}_\tau(\mu_{i,z}|_{G_{F_w}}) = \{-m_{\tau,i}(z)\}$$

Now let $\mathrm{G}_i \in \mathcal{O}(X)^\times$ be such that for all $z \in Z'$ one has

$$\mathrm{G}_i(z) = \prod_{\tau\in\tilde{\Sigma}(v)} \chi_{\tau,v}(\varpi)^{-m_{\tau,i}(z)} \tau(\varpi)^{m_{\tau,i}(z)}$$

Let U be an admissible open affinoid neighborhood of x_0 in X such that for each $1 \le i \le n$ the inclusion $U \to X$ is $\mathrm{G}_i \prod_{j=1}^i \mathrm{F}_j$-small as defined in [KIS, Sect. 5.2]. For each such i let

$$\mathrm{H}_i := (q^{i(i-1)/2} \prod_{\tau\in\tilde{\Sigma}(v)} (\tau(\varpi)^{-\frac{i(i+1-2n)}{2}}) \mathrm{G}_i \prod_{j=1}^i \mathrm{F}_j)|_U \in \mathcal{O}(U)^\times$$

Using [BC, Lem. 7.8.11] as in [CHE, Sect. 3.15] gives the following: There exists a reduced rigid space U' and a finite map $h : U' \to U$ such that the following holds:

- There is a free $\mathcal{O}(U')$-module M of rank n with a continuous Galois representation

  $$\rho_\mathrm{M} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(\mathrm{M})$$

  For a point x in U' we will denote by E_x its residue field and by $\rho_{\mathrm{M},x}$ the Galois representation on $\mathrm{M} \otimes_{\mathcal{O}(U')} E_x$ obtained from ρ_M

- There is a Zariski dense subset $\tilde{Z} \subseteq U'$ such that $h(\tilde{Z}) \subseteq Z$ and for all $z' \in \tilde{Z}$ one has

$$\rho_{\mathrm{M},z'} \otimes_{E_{z'}} \overline{E}_{z'} \cong \rho_{\pi(h(z')),\ell,\iota}$$

- There is $x_0' \in U'$ such that $h(x_0') = x_0$ and $\rho^{\mathrm{ss}}_{\mathrm{M},x_0'} \cong \rho_{\pi(x_0),\ell,\iota}$

Let $\mu_i' : \mathrm{Gal}(\overline{F}/F) \to \mathcal{O}(U')$ be the pullback to U' of the restriction of μ_i to U and let $\mathcal{O}(U')(\mu_i')$ denote the free rank one $\mathcal{O}(U')$-module with the $\mathrm{Gal}(\overline{F}/F)$-action given by μ_i'. Define

$$(\wedge^i \rho_{\mathrm{M}})(\mu_i) := (\wedge^i \rho_{\mathrm{M}}) \otimes_{\mathcal{O}(U')} \mathcal{O}(U')(\mu_i')$$

Then for all $z \in Z'$ and $\tau \in \tilde{\Sigma}(v)$ one has

- $\mathrm{HT}^{(1)}_\tau((\wedge^i \rho_{\mathrm{M}})(\mu_i)_z|_{G_{F_w}}) = 0$
- $\mathrm{HT}^{(2)}_\tau((\wedge^i \rho_{\mathrm{M}})(\mu_i)_z|_{G_{F_w}}) > 0$

Hence the crystalline periods results of Chapter 3 might be applied to the Galois representation $(\wedge^i \rho_{\mathrm{M}})(\mu_i)$. To do so, fix $k \in \mathbb{Z}^{\geq 1}$ and let

$$\mathrm{I}_k :=$$

$$\{z \in Z \cap U \big| \ \min_{\tau \in \tilde{\Sigma}(v)} \ \mathrm{HT}^{(2)}_\tau((\wedge^i \rho_{\mathrm{M}})(\mu_i)_z|_{G_{F_w}}) > \max(k, [F_v^+ : \mathbb{Q}_p]\mathrm{v}_p(\mathrm{H}_i(x_0)))\}$$

By the arguments used to show that Z accumulates at Z' it also follows that I_k is Zariski dense in U. For each $z \in Z'$ and each $1 \leq i \leq n$ one has

$$\iota^{-1}[q^{i(n-1)/2} \prod_{j=1}^{i} \chi_{\pi(z),j}(\varpi)] = q^{i(i-1)/2} \prod_{j=1}^{i} \mathrm{F}_j(z) \prod_{\tau \in \Sigma(v)} \tau(\varpi)^{-\sum_{j=1}^{i} k_{v,\tau,j}}$$

By Hypothesis 5.1.2 for each $z \in Z$ the representation $\rho_z|_{G_{F_w}}$ is crystalline and hence

$$\wedge^i \mathrm{D}_{\mathrm{cris}}(\rho_z|_{G_{F_w}}) \cong \mathrm{D}_{\mathrm{cris}}(\wedge^i \rho_z|_{G_{F_w}})$$

Therefore, for any $\lambda \in \{q^{i(n-1)/2} \prod_{s=1}^{i} \chi_{\pi(z),j_s}(\varpi) \big| 1 \leq j_1 < \cdots < j_i \leq n\}$ one has

$$\mathrm{D}_{\mathrm{cris}}(\wedge^i \rho_z|_{G_{F_w}})^{\varphi_{F_w} = \lambda} \neq (0)$$

For all $z \in Z$ the character $\mu_{i,z}$ is crystalline at w and

$$\mathrm{D}_{\mathrm{cris}}(\mu_{i,z}|_{G_{F_w}})^{\varphi_{F_w} = \mathrm{G}_i(z) \prod_{\tau \in \Sigma(v)} \tau(\varpi)^{-m_{\tau,i}(z)}} \neq (0)$$

It follows that for all $z \in Z$ one has

$$\mathrm{D}_{\mathrm{cris}}((\wedge^i \rho_{\mathrm{M}})(\mu_i)_z|_{G_{F_w}})^{\varphi_{F_w} = \mathrm{H}_i(z)} \neq (0)$$

It now follows from Lemma 3.6 that for $\mathcal{R} := \mathcal{O}(U')$, the $\mathcal{R}$-module dual of $(\wedge^i \mathrm{M})(\mu_i)$, $\mathcal{R}_i$ the collection of residue fields of $\mathcal{R}$ corresponding to points which map under h to points in I_k for some $k \in \mathbb{Z}^{\geq 1}$, and $Y := \mathrm{H}_i^{-1}$ all assumptions of Theorem 3 are satisfied. It follows that

$$\mathrm{D}_{\mathrm{cris}}((\wedge^i \mathrm{M})(\mu_i)_{x_0}|_{G_{F_w}})^{\varphi_{F_w} = \mathrm{H}_i(x_0)} \neq (0)$$

Hence

$$\mathrm{D}_{\mathrm{cris}}((\wedge^i \rho_{x_0})(\mu_{i,x_0})|_{G_{F_w}})^{\varphi_{F_w} = \mathrm{H}_i(x_0)} \neq (0)$$

and therefore

$$\mathrm{D}_{\mathrm{cris}}((\wedge^i \rho_{x_0}|_{G_{F_w}})^{\varphi_{F_w} = q^{i(n-1)/2} \prod_{j=1}^{i} \chi_{v,j}(\varpi)} \neq (0)$$

where χ_v is the refinement of $\pi_v \circ i_w^{-1}$ as in the statement of the lemma. The first part of the lemma now follows from Lemma 3.5.

To prove the second part of the lemma let σ and S_σ be as in the statement of the lemma. By applying the first part of the lemma with $i = 1$ one obtains $q^{\frac{n-1}{2}} \chi_{v,1}(\varpi) \in S_\sigma$. The first part of the lemma with $i = n-1$ implies that there is a subset $\{x_1, \cdots, x_{n-1}\} \subset S_\sigma$ such that

$$x_1 \cdots x_{n-1} = q^{\frac{(n-1)^2}{2}} \prod_{j=1}^{n-1} \chi_{v,j}(\varpi)$$

The first part of the lemma with $i = n$ implies that if x_n is such that $\{x_1, \cdots, x_n\} = S_\sigma$, then

$$x_1 \cdots x_n = q^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \chi_{v,j}(\varpi)$$

and hence $q^{\frac{n-1}{2}} \chi_{v,n}(\varpi) = x_n \in S_\sigma$. □

Remark 10. Note that one might obtain stronger results than in the previous corollary by using variants of the crystalline period results of [KIS, Prop. 5.14] where more than one weight is fixed.

The above results were obtained for automorphic representations of rather specific unitary groups. Nonetheless, by using base change for GL_n as constructed in [AC] as well as base change and descent between unitary groups and GL_n as constructed in [LAB], one can deduce rather general results for automorphic representations of GL_n over CM-fields:

Lemma 5.6. *Let F be a CM-field and let Π be a regular algebraic cuspidal conjugate self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let v be a finite place of F, let ϖ be a uniformizer of F_v and let q denote the size of the residue field of F_v. Assume that Π_v is Iwahori-spherical. For any lift $\sigma \in W_{F_v}$ of the geometric Frobenius let S_σ denote the multiset of eigenvalues of the action on $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})$. Then for any accessible refinement*

$$\chi_v = \prod_{j=1}^{n} \chi_{v,j}$$

of Π_v and each $1 \leq i \leq n$ there is a multiset

$$\{\alpha_{j_1}, \cdots, \alpha_{j_i}\} \subseteq S_\sigma$$

such that

$$\prod_{k=1}^{i} \alpha_{j_k} = q^{\frac{i(n-1)}{2}} \prod_{k=1}^{i} \chi_{v,k}(\varpi)$$

Moreover, one always has $q^{\frac{n-1}{2}} \chi_{v,1}(\varpi) \in S_\sigma$ and $q^{\frac{n-1}{2}} \chi_{v,n}(\varpi) \in S_\sigma$.

Proof. Choose a CM-field L containing F, with maximal totally real subfield denoted by L^+, such that

(i) L/F is a solvable extension and $L = EL^+$ for some imaginary quadratic field E

(ii) $n[L^+ : \mathbb{Q}]/2$ is even

(iii) v splits completely in L and any place $w|v$ of L lies above Spl_{L/L^+}

(iv) any place of L above ℓ lies above Spl_{L/L^+}

(v) L/L^+ is unramified at all finite places

(vi) $\mathrm{BC}_L(\Pi)$ is cuspidal

Here $\mathrm{BC}_L(-)$ denotes the base change as constructed in [AC]. The existence of such a base change is already used in [CHT]. Note that condition (i) can be achieved by taking the composite of F with an imaginary quadratic field. To arrange certain places to lie above Spl_{L/L^+} is achieved by taking if necessary the composite of L with a suitable totally real field, see for example [HT] (p.228). Now let L be a CM-field satisfying the above conditions. Then there exists a totally definite unitary group G over L^+ as in the beginning of Section 5.1 and by [LAB] there exists an automorphic representation π of $G(\mathbb{A}_{L^+})$ whose base change to $\mathrm{GL}_n(\mathbb{A}_L)$ is isomorphic to $\mathrm{BC}_L(\Pi)$. In particular, at all places u of F^+ which split as u_1u_2 in F one has

$$\pi_u \circ i_{u_1}^{-1} \cong \mathrm{BC}_L(\Pi)_{u_1}$$

Since $\mathrm{BC}_L(\Pi)$ is cuspidal one has

$$\rho_{\pi,\ell,\iota} \cong \rho_{\mathrm{BC}_L(\Pi),\ell,\iota}$$

and since v splits in L the lemma follows from Lemma 5.5. □

Due to results announced very recently by Liu in [LIU] and Kedlaya-Pottharst-Xiao in [KPX] on the existence of global triangulations, one can now often prove stronger results than the ones discussed above. The point is that Kisin's original results on variation of crystalline periods have very recently been strengthened to show very general results on the existence of triangulations for analytic families of Galois representations. We will now state such a recent result proved by Liu in [LIU] where for simplicity we assume that we study representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ instead of representations of the absolute Galois group of a finite extension of K. First, we recall some definitions, see for example [BEN] and [BC] for more details.

Consider the cyclotomic character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathbb{Z}_p^\times$ and let

$$\Gamma := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)/\mathrm{Ker}(\chi)$$

Let $\mathcal{R}_{\mathbb{Q}_p}$ be the Robba ring over $\mathbb{Q}_p$ as defined for example in [BC, Section 2.2.2]. Let A be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{R}_A = \mathcal{R}_{\mathbb{Q}_p} \otimes A$. One has an action of φ and Γ on $\mathcal{R}_A$ by

$$\varphi f(z) = f(z^p) \text{ and } (\gamma f)(z) = f(z^{\chi(\gamma)})$$

A (φ,Γ)-module is a finitely generated $\mathcal{R}_A$-module M which is free as an $\mathcal{R}_{\mathbb{Q}_p}$-module and which has $\mathcal{R}_A$-semi linear actions of φ and Γ which commute and are continuous in a sense defined for example in [BC, Section 2.2.2] and such that $\mathcal{R}_{\mathbb{Q}_p}\varphi(M) = M$.

It is known that there is an equivalence of categories between p-adic representation V and so called (φ,Γ)-modules of slope 0. The (φ,Γ)-module associated to V will be denoted by $D^\dagger_{\mathrm{rig}}(V)$. For a (φ,Γ)-module M one can associate certain finite-dimensional K_0-vector spaces $\mathcal{D}_{\mathrm{cris}}(M)$ and $\mathcal{D}_{\mathrm{st}}(M)$. A (φ,Γ)-module is called semi-stable if

$$\dim_{K_0} \mathcal{D}_{\mathrm{st}}(M) = \mathrm{rank}_{\mathcal{R}(K)}(M)$$

and it is called crystalline if

$$\dim_{K_0} \mathcal{D}_{\mathrm{cris}}(M) = \mathrm{rank}_{\mathcal{R}(K)}(M)$$

These definitions are equivalent to define M semi-stable or crystalline if $M \cong D^\dagger_{\mathrm{rig}}(V)$ and V is semi-stable or crystalline.

Definition 20. A (φ,Γ) module M of rank n over $\mathcal{R}_A$ is called trianguline if there is a filtration

$$(0) \subset F_0M \subset F_1M \subset \cdots \subset F_nM = M$$

where each F_iM is a saturated (φ,Γ) sub-module of M over $\mathcal{R}_A$ and

$$\mathrm{gr}_i M := F_iM/F_{i-1}M$$

is free of rank one for all $1 \leq i \leq n$. A p-adic representation V is called trianguline if and only if $D^\dagger_{\mathrm{rig}}(V)$ is trianguline.

Specialize now to the case $K = \mathbb{Q}_p$ for simplicity. Assume that M is a semi-stable (φ, Γ)-module and suppose A is chosen such that all eigenvalues of $\varphi : \mathcal{D}_{\mathrm{st}}(M) \longrightarrow \mathcal{D}_{\mathrm{st}}(M)$ are contained in A. A refinement of $\mathcal{D}_{\mathrm{st}}(M)$ is defined in [BEN] to be a filtration

$$0 = F_0\mathcal{D}_{\mathrm{st}}(M) \subset F_1\mathcal{D}_{\mathrm{st}}(M) \subset \cdots F_n\mathcal{D}_{\mathrm{st}}(M) = \mathcal{D}_{\mathrm{st}}(M)$$

where each $F_i\mathcal{D}_{\mathrm{st}}(M)$ is an A-subspace of $\mathcal{D}_{\mathrm{st}}(M)$ which is φ-stable and N-stable and such that

$$\mathrm{gr}_i\mathcal{D}_{\mathrm{st}}(M) := F_i\mathcal{D}_{\mathrm{st}}(M)/F_{i-1}\mathcal{D}_{\mathrm{st}}(M)$$

is of dimension one for all $1 \leq i \leq n$. Similarly one defines the refinement of a crystalline representation.

Let $\delta : \mathbb{Q}_p^\times \longrightarrow \mathrm{A}^\times$ be a continuous character. In [BC, Sect. 2.3] the (φ, Γ)-module $R_A(\delta)$ is defined to be the free rank one $\mathcal{R}_A$-module with $\mathcal{R}_A$-semilinear action of φ and Γ defined as follows:

$$\varphi(1) = \delta(p) \text{ and } \gamma(1) = \delta(\gamma) \text{ for all } \gamma \in \Gamma$$

By [BEN, Proposition 1.3.2] there is a bijection between triangulations of M and refinements of $\mathcal{D}_{\mathrm{st}}(M)$. Order the φ-eigenvalues as $\alpha_1, \cdots, \alpha_n$ corresponding to the filtration and similarly order the multiset of Hodge-Tate weights as $k_1, \cdots, k_n$. It is shown in [BEN] that

$$\mathrm{gr}_i M \cong R_A(\delta_i)$$

where $\delta_i : \mathbb{Q}_p^\times \longrightarrow A^\times$ is the character given by

$$\delta_i(p) = \alpha_i p^{-k_i} \text{ and } \delta_i(x) = x^{-k_i} \text{ for } x \in \mathbb{Z}_p^\times$$

The parameter of a triangulation is defined to be the map

$$\mathbb{Q}_p^\times \longrightarrow (A^\times)^n$$

whose components are the characters δ_i.

We now describe Liu's result. Let X be an analytic space and suppose we are given a p-adic family of refined representations, as defined in a more general way in [BC, Section 4.2] . For the following slightly less general case see [LIU, Definition 5.2.3]. There is given a continuous Galois representation

$$G \longrightarrow \mathrm{GL}(V)$$

where V is a free $\mathcal{O}(X)$-module of finite rank n. Suppose for each $1 \leq i \leq n$ there is $F_i \in \mathcal{O}(X)^\times$ and $\kappa_i \in \mathcal{O}(X)$ and suppose there is a Zariski dense subset Z of X such that:

- for every $x \in X$ the multiset of Hodge-Tate weights of V_x is given by $\{\kappa_1(x), \cdots, \kappa_n(x)\}$
- for all $z \in Z$ the representation V_z is crystalline
- for all $z \in Z$ one has $\kappa_1(z) < \cdots < \kappa_n(z)$
- for all $z \in Z$ the eigenvalues of crystalline Frobenius φ acting on $\mathrm{D}_{\mathrm{cris}}(V_z)$ are distinct and given by $\{p^{\kappa_1(z)}F_1(z), \cdots, p^{\kappa_n(z)}F_n($
- let C be any non-negative integer and define the subset Z_C of Z to be

  $$\{z \in Z \big| |\kappa_I(z) - \kappa_J(z)| > C \text{ for all } I, J \subset \{1, \cdots, n\}, \#I = \#J > 0, I \neq J\}$$

  where $\kappa_I = \sum_{i \in I} \kappa_i$. Then Z_C accumulates at all $z \in Z$
- for each $1 \leq i \leq n$ there is a continuous character $\chi_i : \mathbb{Z}_p^\times \mathcal{O}(X)^\times$ whose derivative at 1 is κ_i and whose evaluation at any $z \in Z$ is the map which raises to the $\kappa_i(z)$-th power.

Fix now a p-adic family of refined representations as defined above. For each $1 \le i \le n$ let

$$\Delta_i : \mathbb{Q}_p^\times \longrightarrow \mathcal{O}(X)^\times$$

be given by

$$\Delta_i(p) = \prod_{j=1}^{i} F_j \text{ and } \Delta_i|_\Gamma = \prod_{j=1}^{i} \chi_j$$

It is then shown in [LIU] that on every affinoid subdomain of a Zariski dense subset X_s of X there is a triangulation with parameter whose components are given by

$$(\Delta_{i+1}/\Delta_i)_{1,\cdots,n-1}$$

Note that more is shown about the set X_s in [LIU]. We recall this now briefly. Let V be a crystalline representation of G_p. If the Hodge-Tate weights of V are distinct and given by $k_1 < \cdots < k_n$ then a refinement of V is called non-critical if for all $1 \le i \le n$ one has

$$\mathrm{D}_{\mathrm{cris}}(V) \cong F_i \mathrm{D}_{\mathrm{cris}}(V) \oplus \mathrm{Fil}^{k_{i+1}}(\mathrm{D}_{\mathrm{cris}}(V))$$

Let φ_i be the eigenvalue of φ on F_i/F_{i-1}. Then the refinement is called regular if for all $1 \le i \le n$ $\varphi_1 \cdots \varphi_i$ is an eigenvalue of φ acting on $\mathrm{D}_{\mathrm{cris}}(\wedge^i V)$ of multiplicity one. For $z \in Z$ associate the refinement

$$(p^{\kappa_1(z)} F_1(z), \cdots, p^{\kappa_n(z)} F_n(z))$$

The point $z \in Z$ is called regular if the associated refinement is regular and it is called non-critical if the associated refinement is non-critical. One now defines what it means for a point $x \in X$ to be a saturated point. It is shown that regular non-critical points are saturated and since regular non-critical points are Zariski dense it follows that X_s is Zariski dense.

Suppose now that M is a specialization at a point x of X_s which is semi-stable. By the earlier discussion it follows that the eigenvalues $\alpha_1, \cdots, \alpha_n$ of φ acting on $\mathcal{D}_{\mathrm{st}}(M)$ are given by

$$F_i(p)(x) p^{k_i(x)}$$

for $1 \le i \le n$. In our application this matches the prediction from the local Langlands correspondence. For most of this memoir we focus however on the simpler study of the variation of one crystalline period due to Kisin and this is sufficient for many purposes relating to, for example, Hilbert modular forms and modular forms on GSp_4.

6 Local semi-simplifications: The case of general linear groups

In this chapter we give some examples of how the results of the previous chapter can be used to obtain local-global compatibility results for local semi-simplifications of Galois representations associated to automorphic representations of general linear groups over CM-fields and totally real fields. The calculations of this chapter can be thought of as making explicit, in terms of the classification of Iwahori-spherical representations of GL_n over p-adic fields, the consequences of the variation of one crystalline period in families of automorphic Galois representations of general linear groups.

6.1 Results in dimension at most 4

In this section we focus on the case of low-dimensional representations since it is in this situation that we obtain the most complete local-global compatibility results. In Section 6.4 we then describe how knowledge of monodromy operators can be used in conjunction with the results of the previous chapter to obtain local-global comptiblity for the local semi-simplification of automorphic Galois representations of arbitrary dimension.

Theorem 5. *Let F be a* CM*-field and let Π be a regular algebraic cuspidal conjugate self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for $n \leq 4$. Let v be a finite place of F such that Π_v is Iwahori-spherical. Then*

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})^{\mathrm{ss}}$$

unless $n = 4$ and there is an unramified character $\chi : F_v^\times \to \mathbb{C}^\times$ such that one of the following holds:

(i) $\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \chi|\cdot|^{\pm 1/2} \boxplus \chi|\cdot|^{\pm 1/2}$ *(same sign)*

(ii) $\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \mathrm{St}_2(\chi)$

(iii) $\Pi_v \cong \mathrm{St}_3(\chi) \boxplus \chi|\cdot|^{\pm 1}$

(iv) $\Pi_v \cong \mathrm{St}_4(\chi)$ *and* $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ *has trivial monodromy operator*

Proof. If Π_v is unramified there is nothing to show and hence one can assume that one of the following holds:

- $n = 2$ and $\Pi_v \cong \mathrm{St}_2(\chi)$ for some unramified character χ of $F_v^\times$
- $n = 3$ and one of the following holds:
  - $\Pi_v \cong \mathrm{St}_2(\chi_1) \boxplus \chi_2$ for some unramified characters χ_1 and χ_2 of $F_v^\times$
  - $\Pi_v \cong \mathrm{St}_3(\chi)$ for some unramified character χ of $F_v^\times$
- $n = 4$ and one of the following holds:
  - $\Pi_v \cong \mathrm{St}_4(\chi)$ for some unramified character χ of $F_v^\times$
  - $\Pi_v \cong \mathrm{St}_3(\chi_1) \boxplus \chi_2$ for some unramified characters χ_1 and χ_2 of $F_v^\times$
  - $\Pi_v \cong \mathrm{St}_2(\chi_1) \boxplus \mathrm{St}_2(\chi_2)$ for some unramified characters χ_1 and χ_2 of $F_v^\times$
  - $\Pi_v \cong \mathrm{St}_2(\chi_1) \boxplus \chi_2 \boxplus \chi_3$ for some unramified characters χ_1, χ_2 and χ_3 of $F_v^\times$

Let $\mathrm{Fr}_v \in W_{F_v}$ be any lift of the geometric Frobenius, let ϖ be a uniformizer of F_v and let q denote the size of the residue field of F_v. Let

$$S' = \{x_1, \cdots, x_n\}$$

denote the multiset of eigenvalues of Fr_v acting on $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ and let

$$S = \{x_1 q^{\frac{1-n}{2}}, \cdots, x_n q^{\frac{1-n}{2}}\}$$

Let $\mu = \prod_{i=1}^n \mu_i$ be an accessible refinement of Π_v. By [SAI, Lem. 1] in order to prove the corollary it is enough to show that

$$S = \{\mu_1(\varpi), \cdots, \mu_n(\varpi)\}$$

We will now prove the corollary by working case by case.

- Suppose $\Pi_v \cong \mathrm{St}_2(\chi)$. Applying Lemma 5.6 to the accessible refinement

  $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$$

  yields $S = \{\chi(\varpi)q^{-1/2}, \chi(\varpi)q^{1/2}\}$. This is as predicted by the local Langlands correspondence.

- Suppose $\Pi_v \cong \mathrm{St}_2(\chi_1) \boxplus \chi_2$. Applying Lemma 5.6 with $i = 1$ to the accessible refinement

  $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_1|\cdot|^{1/2} \times \chi_1|\cdot|^{-1/2} \times \chi_2)$$

shows that $\chi_1(\varpi)q^{-1/2} \in S$ and applying the lemma to the accessible refinement

$$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_2 \times \chi_1|\cdot|^{1/2} \times \chi_1|\cdot|^{-1/2})$$

shows that $\chi_1(\varpi)q^{1/2} \in S$. It follows that $S = \{\chi_1(\varpi)q^{-1/2}, \chi_1(\varpi)q^{1/2}, x\}$ for some x. Applying Lemma 5.6 with $i = 3$ to the previous refinement shows that $x = \chi_2(\varpi)$ and hence S is as predicted by the local Langlands correspondence.

- Suppose $\Pi_v \cong \mathrm{St}_3(\chi)$. Applying Lemma 5.6 to the accessible refinement

$$\Pi_v \hookrightarrow \mathrm{Ind}(\chi|\cdot|, \chi, \chi|\cdot|^{-1})$$

shows that $S = \{\chi(\varpi)q^{-1}, \chi(\varpi)q, x\}$ for some x. Applying Lemma 5.6 with $i = 3$ to the same refinement shows that $x = \chi(\varpi)$ and hence S is as predicted by the local Langlands correspondence.

- Suppose $\Pi_v \cong \mathrm{St}_4(\chi)$. Let $\alpha := \chi(\varpi)$. Applying Lemma 5.6 to the accessible refinement

$$\Pi_v \hookrightarrow \mathrm{Ind}(\chi|\cdot|^{3/2}, \chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}, \chi|\cdot|^{-3/2})$$

yields $S = \{\alpha q^{-3/2}, \alpha q^{3/2}, x, y\}$ for some x, y and applying the lemma with $i = 2$ to the same refinement shows that there is a subset $\{a, b\} \subset S$ such that $ab = \alpha^2 q^{-2}$.

  - If
  $$\{a, b\} = \{x, y\}$$
  then applying Lemma 5.6 with $i = 4$ to the same refinement yields a contradiction.
  - If
  $$\{a, b\} = \{x, \alpha q^{-3/2}\}$$
  then $x \cdot \alpha q^{-3/2} = \alpha^2 q^{-2}$ and hence $x = \alpha q^{-1/2}$. Applying Lemma 5.6 with $i = 4$ to the previous refinement yields $y = \alpha q^{1/2}$ and hence S is as predicted by the local Langlands correspondence.
  - If
  $$\{a, b\} = \{x, \alpha q^{3/2}\}$$
  then $x \cdot \alpha q^{3/2} = \alpha^2 q^{-2}$ and hence $x = \alpha q^{-7/2}$. Applying Lemma 5.6 with $i = 4$ to the previous refinement yields $y = \alpha q^{7/2}$. Hence
  $$S = \{\alpha q^{-7/2}, \alpha q^{-3/2}, \alpha q^{3/2}, \alpha q^{7/2}\}$$
  and there are no $c, d \in S$ such that $c/d = q^{\pm 1}$. Write
  $$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong (r, N)$$
  Since
  $$r(\mathrm{Fr}_v) \cdot N \cdot r(\mathrm{Fr}_v)^{-1} = q^{-1}N$$
  it follows that the monodromy operator N has to be trivial.

- Suppose $\Pi_v \cong \mathrm{St}_3(\chi_1) \boxplus \chi_2$. Let $\alpha := \chi_1(\varpi)$ and $\beta := \chi_2(\varpi)$. Applying Lemma 5.6 with $i = 1$ to the accessible refinement

$$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_1|\cdot|, \chi_1, \chi_1|\cdot|^{-1}, \chi_2)$$

and applying the lemma to the accessible refinement

$$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_2, \chi_1|\cdot|, \chi_1, \chi_1|\cdot|^{-1})$$

yields $S = \{\alpha q^{-1}, \alpha q, x, y\}$ for some x, y. Suppose first that $\chi_2 \neq \chi_1|\cdot|^{\pm 1}$. Applying Lemma 5.6 with $i = 1$ to the previous refinement yields $S = \{\alpha q^{-1}, \alpha q, \beta, z\}$ for some z. Applying Lemma 5.6 with $i = 4$ to the same refinement shows that $z = \alpha$ and hence S is as predicted by the local Langlands correspondence. If $\chi_2 = \chi_1|\cdot|^{-1}$ then

$$\{\alpha q^{-2}, \alpha q^{-1}, \alpha q, \alpha q^3\}$$

is not as predicted by the local Langlands correspondence but satisfies all the relations coming from Lemma 5.6. If $\chi_2 = \chi_1|\cdot|^{+1}$ then

$$\{\alpha q^{-3}, \alpha q^{-1}, \alpha q, \alpha q^2\}$$

is not as predicted by the local Langlands correspondence but satisfies all the relations coming from the Lemma 5.6.

- Suppose $\Pi_v \cong \mathrm{St}_2(\chi_1) \boxplus \mathrm{St}_2(\chi_2)$. Let $\alpha := \chi_1(\varpi)$ and $\beta := \chi_2(\varpi)$.
  - Assume first $\chi_1 \neq \chi_2$. Applying Lemma 5.6 to the accessible refinement

    $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_1|\cdot|^{1/2}, \chi_1|\cdot|^{-1/2}, \chi_2|\cdot|^{1/2}, \chi_2|\cdot|^{-1/2})$$

    and applying the lemma to the accessible refinement

    $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_2|\cdot|^{1/2}, \chi_2|\cdot|^{-1/2}, \chi_1|\cdot|^{1/2}, \chi_1|\cdot|^{-1/2})$$

    yields $S = \{\alpha q^{-1/2}, \alpha q^{1/2}, x, y\}$ for some x, y. Applying Lemma 5.6 with $i = 1$ to the previous refinement shows that $\beta q^{-1/2} \in S$. If $\beta q^{-1/2} \neq \alpha q^{\pm 1/2}$ then applying Lemma 5.6 with $i = 4$ to the previous refinement shows that S is as predicted by the local Langlands correspondence. Otherwise, since $\alpha \neq \beta$, one has $\beta q^{-1/2} = \alpha q^{1/2}$. However, this is impossible since the representation Π_v is generic.
  - Assume now $\chi_1 = \chi_2$. Then

    $$\{\alpha q^{-3/2}, \alpha q^{-1/2}, \alpha q^{1/2}, \alpha q^{3/2}\}$$

    is not as predicted by the local Langlands correspondence but satisfies all the relations coming from Lemma 5.6.

- Suppose $\Pi_v \cong \mathrm{St}_2(\chi_1) \boxplus \chi_2 \boxplus \chi_3$. Let $\alpha := \chi_1(\varpi)$ and $\beta := \chi_2(\varpi)$ and $\gamma := \chi_3(\varpi)$. Applying Lemma 5.6 to the accessible refinement

  $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_1|\cdot|^{1/2}, \chi_1|\cdot|^{-1/2}, \chi_2, \chi_3)$$

  and applying the lemma to the accessible refinement

  $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_2, \chi_3, \chi_1|\cdot|^{1/2}, \chi_1|\cdot|^{-1/2})$$

  yields $S = \{\alpha q^{-1/2}, \alpha q^{1/2}, x, y\}$ for some x, y.
  - Assume first that $\chi_2 \neq \chi_1|\cdot|^{\pm 1/2}$. Applying Lemma 5.6 with $i = 1$ to the previous refinement shows that $S = \{\alpha q^{-1/2}, \alpha q^{1/2}, \beta, y\}$ for some y and applying the lemma with $i = 4$ to the same refinement shows that $y = \gamma$ and hence S is as predicted by the local Langlands correspondence.
  - Assume now that $\chi_2 = \chi_1|\cdot|^{\pm 1/2}$.
  - Assume first $\chi_2 \neq \chi_3$. Then $\gamma \notin \{\alpha q^{-1/2}, \alpha q^{1/2}\}$ since this would imply, since $\beta \neq \gamma$, that $\gamma/\beta \in \{q, q^{-1}\}$ which is impossible since the representation Π_v is generic. Hence, applying Lemma 5.6 with $i = 1$ to the accessible refinement

    $$\Pi_v \hookrightarrow \mathrm{Ind}(\chi_3, \chi_2, \chi_1|\cdot|^{1/2}, \chi_1|\cdot|^{-1/2})$$

    yields $S = \{\alpha q^{-1/2}, \alpha q^{1/2}, \gamma, z\}$ for some z and applying the lemma with $i = 4$ to the same refinement shows that $z = \beta$ and hence S is as predicted by the local Langlands correspondence.

– Assume now $\chi_2 = \chi_3 = \chi_1|\cdot|^{\pm 1/2}$. In this case

$$\{\alpha q^{-1/2}, \alpha, \alpha q^{1/2}, \alpha q^{\mp 1}\}$$

is not as predicted by the local Langlands correspondence but satisfies all the relations coming from Lemma 5.6.

$\square$

We will now deduce analogues of the previous results for automorphic representations of general linear groups over totally real fields.

Let F be a totally real field and let Π be a regular algebraic cuspidal essentially self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ as defined in [CHT, p. 134]. In particular, there exists a character

$$\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$$

such that

- $\Pi^\vee \cong \Pi \otimes \chi$
- $\chi_v(-1)$ is independent of $v|\infty$

The existence of the relevant automorphic Galois representations follows then from the CM-case as shown in [CHT, Prop. 3.3.1] by letting CM-fields with maximal totally real subfield F^+ vary. It follows from this construction, by using CM-extensions of the totally real field in which suitable finite places split completely, that the previous local-global compatibility results directly imply the corresponding results for totally real fields:

Corollary 6.2. *Let F be a totally real field and let π be a regular algebraic cuspidal essentially self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for $n \le 4$. Let v be a finite place of F such that π_v is Iwahori-spherical. Then*

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\pi_v \otimes |\det|^{\frac{1-n}{2}})^{\mathrm{ss}}$$

unless $n = 4$ and there is an unramified character $\chi : F_v^\times \to \mathbb{C}^\times$ such that one of the following holds:

- $\pi_v \cong \mathrm{St}_2(\chi) \boxplus \chi|\cdot|^{\pm 1/2} \boxplus \chi|\cdot|^{\pm 1/2}$ *(same sign)*
- $\pi_v \cong \mathrm{St}_2(\chi) \boxplus \mathrm{St}_2(\chi)$
- $\pi_v \cong \mathrm{St}_3(\chi) \boxplus \chi|\cdot|^{\pm 1}$
- $\pi_v \cong \mathrm{St}_4(\chi)$ *and* $\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ *has trivial monodromy operator*

In particular, one obtains the following for Galois representations associated to Hilbert modular forms:

Corollary 6.3. *Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of cohomological weight. Let v be a finite place of F such that π_v is Iwahori-spherical. Then*

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GL}_2}(\pi_v \otimes |\det|^{-\frac{1}{2}})^{\mathrm{ss}}$$

Proof. The result follows from Corollary 6.2 since if χ_π denotes the central character of π then $\pi^\vee \cong \pi \otimes \chi_\pi^{-1}$ and $\chi_{\pi,v}^{-1}(-1)$ is independent of $v|\infty$. $\square$

6.4 Results in arbitrary dimension

We now discuss some implications of the crystalline periods approach to local-global compatibility for Galois representations of arbitrary dimension. After dealing with some specific types of ramification in Section 6.4.1, we prove in Section 6.4.2 that if the local component of the automorphic representation in question is "in general position" then knowledge of the monodromy operators can be combined with the crystalline periods result to obtain semi-simple local-global compatibility results. Note that in Chapter 10 we prove a result - in a symplectic context, but similar results should hold by very similar proofs for GL_n - of how certain generalized potential level-lowering results might be used to obtain strong lower bounds on the rank of monodromy. Hence, via the results of the current section, this gives an approach to showing the full semi-simple local-global compatibility.

6.4.1 Various higher dimensional examples

For some representations of arbitrary large dimension one can still obtain local-global compatibility by the methods of the previous sections:

Suppose π is a local component of an automorphic representation of GL_n over a CM-field or totally real field F as discussed in the previous section. Say it is the local component of Π at a finite place v of F. As before, let $\mathrm{Fr}_v \in W_{F_v}$ be any lift of the geometric Frobenius, let ϖ be a uniformizer of F_v and let q denote the size of the residue field of F_v. Let

$$S' = \{x_1, \cdots, x_n\}$$

denote the multiset of eigenvalues of Fr_v acting on $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ and let

$$S = \{x_1 q^{\frac{1-n}{2}}, \cdots, x_n q^{\frac{1-n}{2}}\}$$

Now suppose for example that

$$\pi \cong \mathrm{St}_3(\chi) \boxplus \mathrm{St}_2(\mu_1) \boxplus \cdots \boxplus \mathrm{St}_2(\mu_a) \boxplus \mathrm{St}_1(\lambda_1) \boxplus \cdots \boxplus \mathrm{St}_1(\lambda_b)$$

Then

$$S = \{\alpha q^{-1}, \alpha q, x, \mu_1, \cdots, \mu_r\}$$

and hence $x = \alpha$ and one is done. However, the more general case where

$$\pi \cong \mathrm{St}_3(\chi_1) \boxplus \cdots \boxplus \mathrm{St}_3(\chi_r) \boxplus \mathrm{St}_2(\mu_1) \boxplus \cdots \boxplus \mathrm{St}_2(\mu_a) \boxplus \mathrm{St}_1(\lambda_1) \boxplus \cdots \boxplus \mathrm{St}_1(\lambda_b)$$

is already more complicated: The crystalline period method can not distinguish the correct set from the set

$$S = \{\alpha_1 q^{-1}, \alpha_1^2/\alpha_{\sigma(1)}, \alpha q, \cdots, \mu_1, \cdots, \mu_r\}$$

where σ is any permutation in the symmetric group S_r.

Note that, depending on the type of ramification the representation Π_v has, the amount of possible sets S can increase quickly as n gets larger. For example, suppose that $\Pi_v \cong \mathrm{St}_5(\chi)$ for some character χ and let $\alpha = \chi(\varpi)$. Then

$$S = \{\alpha q^{-2}, \alpha q^2, x, y, z\}$$

for some numbers x, y, z such that $xyz = \alpha^3$. Via the crystalline periods method it follows that there are two distinct elements $a, b \in S$ such that $ab = \alpha^2 q^{-3}$ and two distinct elements $c, d \in S$ such that $cd = \alpha^2 q^3$. By analyzing the different possibilities one can see that exactly the following 9 multisets cannot be ruled out to be S via the crystalline period method:

- $$\{\alpha q^{-8}, \alpha q^{-2}, \alpha q^2, \alpha q^3, \alpha q^5\}$$

- $$\{\alpha q^{-5}, \alpha q^{-3}, \alpha q^{-2}, \alpha q^2, \alpha q^8\}$$
- $$\{\alpha q^{-5}, \alpha q^{-2}, \alpha, \alpha q^2, \alpha q^5\}$$
- $$\{\alpha q^{-5}, \alpha q^{-2}, \alpha q, \alpha q^2, \alpha q^4\}$$
- $$\{\alpha q^{-4}, \alpha q^{-2}, \alpha q^{-1}, \alpha q^2, \alpha q^5\}$$
- $$\{\alpha q^{-4}, \alpha q^{-2}, \alpha q, \alpha q^2, \alpha q^3\}$$
- $$\{\alpha q^{-3}, \alpha q^{-2}, \alpha q^{-1}, \alpha q^2, \alpha q^4\}$$
- $$\{\alpha q^{-3}, \alpha q^{-2}, \alpha, \alpha q^2, \alpha q^3\}$$
- $$\{\alpha q^{-2}, \alpha q^{-1}, \alpha, \alpha q, \alpha q^2\}$$

Here the last multiset is the one predicted by the local Langlands correspondence.

6.4.2 Using the monodromy operator

We now combine the crystalline period method with assumptions on the rank of the local monodromy operator. We restrict to local components of the automorphic representation in question which satisfy a certain assumption which we now define:

Definition 21. Let K be a finite extension of $\mathbb{Q}_p$ for some prime p and let q denote the size of the residue field. Let π be an Iwahori-spherical irreducible admissible representation of $\mathrm{GL}_n(K)$ for some $n \geq 1$. Write it as

$$\pi \cong \boxplus_{i=1}^{r} \mathrm{St}_{n_i}(\chi_i)$$

We say that the representation π is in general position if

$$\chi_i(\varpi)/\chi_j(\varpi) \notin q^{\mathbb{Z}}$$

for $i \neq j$.

Note that the segments of π as above are unlinked as defined in [ZEL, Section 4.1] and hence a representation which is in general position is generic by [ZEL, Section 9.7].

Definition 22. Let K be a finite extension of $\mathbb{Q}_p$ for some prime p. For $n \in \mathbb{Z}^{\geq 1}$ let $\mathrm{Sp}(n)$ denote the Weil-Deligne representation (r, N) with underlying vector space $\mathbb{C}^n$, with a basis say $\{e_1, \cdots, e_n\}$ and r the unramified representation such that for any $\sigma \in W_K$ one has

$$r(\sigma)e_i = |\sigma|^{n-i} e_i \text{ for all } i$$

and

$$N(e_{i+1}) = e_i \text{ for all } 1 \leq i \leq n-1 \text{ and } N(e_1) = 0$$

Let ρ be an irreducible finite dimensional continuous complex representation of the Weil group W_K. Define $\mathrm{Sp}_n(\rho)$ to be the Weil-Deligne representation

$$\mathrm{Sp}_n(\rho) := \rho \otimes \mathrm{Sp}(n)$$

A Weil-Deligne representation (r, N) is called admissible if r is a semi-simple representation. Note (r, N) is admissible if and only if the image under r of some lift of Frobenius is semi-simple. It is known that any admissible Weil-Deligne representation (r, N) is isomorphic to one of the form

$$\bigoplus_{i=1}^{r} \mathrm{Sp}_{n_i}(\rho_i)$$

and this description is unique up to permutations.

At places where the local component of the automorphic representation is in general position, the variation of one crystalline period together with matching of the rank of monodromy with the prediction from the local Langlands correspondence implies the local-global compatibility for the semi-simple part of the relevant Weil-Deligne representation:

Proposition 6.5. *Let F be a* CM*-field and let Π be a regular algebraic cuspidal conjugate self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let v be a finite place of F such that Π_v is Iwahori-spherical and in general position. Write $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong (r, N)$ and $\mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}}) \cong (r_{\mathrm{rec}}, N_{\mathrm{rec}})$. Under Hypothesis 5.1.1 and assuming that* $\mathrm{rank}\ N = \mathrm{rank}\ N_{\mathrm{rec}}$ *it follows that*

$$(r, N) \cong (r_{\mathrm{rec}}, N_{\mathrm{rec}})$$

Proof. Since Π_v is Iwahori-spherical it is of the form

$$\Pi_v \cong \boxplus_{i=1}^{r} \mathrm{St}_{n_i}(\chi_i)$$

for integers n_i and for characters χ_i of $F_v^\times$. Write

$$(r, N) \cong \bigoplus_{j=1}^{s} \mathrm{Sp}_{m_j}(\rho_j)$$

for integers m_i and irreducible representations ρ_i of W_{F_v}. Note that by Hypothesis 5.1.1 the representation r is unramified and hence the representations ρ_i are unramified characters. One has

$$n - s = \sum_j (m_j - 1) = \mathrm{rank}\ N = \mathrm{rank}\ N_{\mathrm{rec}} = \sum_i (n_i - 1) = n - r$$

and hence $r = s$. Let Fr_v be any lift of Frobenius and let ϖ be a uniformizer of F_v. By Hypothesis 5.1.1 and Lemma 5.6 it follows that for each $1 \le i \le r$ the numbers

$$\chi_i(\varpi) q^{(1-n_i)/2} q^{(n-1)/2} \text{ and } \chi_i(\varpi) q^{(1+n_i)/2} q^{(n-1)/2}$$

are eigenvalues of $r(\mathrm{Fr}_v)$. Suppose that for some i these two numbers occur as eigenvalues of $\mathrm{Sp}_{j_k}(\rho_{j_k})$ with $k \in \{1, 2\}$ and $1 \le j_k \le s$ and $j_1 \ne j_2$. Since Π_v is in general position it follows that

$$r \ge s + 1$$

which is a contradiction. Hence, after possibly reordering the labeling of the j's, one can assume that for all $1 \le i \le r$ the numbers $\chi_i(\varpi) q^{(1-n_i)/2} q^{(n-1)/2}$ and $\chi_i(\varpi) q^{(1+n_i)/2} q^{(n-1)/2}$ are eigenvalues of Fr_v corresponding to $\mathrm{Sp}_{m_i}(\rho_i)$ and hence $m_i \ge n_i$. Since $r = s$ it follows that in fact

$$m_i = n_i$$

for all $1 \le i \le r$. It also follows that $\rho_i = \tilde{\chi}_i \cdot |\cdot|^{(1-n)/2}$ for all such i and since

$$\mathrm{rec}_{\mathrm{GL}_n}(\boxplus_{i=1}^r \mathrm{St}_{n_i}(\chi_i)) \cong \bigoplus_{i=1}^r \mathrm{Sp}_{n_i}(\tilde{\chi}_i)$$

it follows that

$$(r, N) \cong (r_{\mathrm{rec}}, N_{\mathrm{rec}})$$

as desired. □

Note that it makes sense in the above result to restrict to the case where local component π_v is in general position since otherwise one can not obtain very complete results: Assume the set-up of Proposition 6.5 and assume for example that

$$\pi_v \cong \mathrm{St}_2(\chi) \boxplus \mathrm{St}_2(\chi)$$

Let ϖ be a uniformizer of F_v and $\alpha = \chi(\varpi_v)$. It can then be seen that the crystalline periods method cannot rule out that, in the notation of the proof of Proposition 6.5, one has

$$S = \{\alpha q^{-3/2}, \alpha q^{-1/2}, \alpha q^{1/2}, \alpha q^{3/2}\}$$

In other words, one can not rule out the the semi-simple Galois action looks as if the local component were $\mathrm{St}_4(\chi)$. In this case, even if one knows that $N \sim N_{\mathrm{rec}}$ one can not conclude the desired semi-simplified local-global compatibility result.

7 Local semi-simplifications: The case of symplectic groups

In this chapter we use the results of Chapter 5 to prove local-global compatibility results for Galois representations associated to automorphic representations of symplectic groups over totally real fields. Note that the existence of Galois representations attached to certain automorphic representations of GSp_4 over totally real fields as well as the corresponding local-global compatibility were deduced in [SOR] from the corresponding results for GL_4 in [HT] and [TY] by using transfer from GSp_4 to GL_4. Instead of appealing to the results of the latter two references we will in this section use the results of the previous chapter to deduce local-global compatibility results. The main result is Theorem 6 which, in the special case of type (IIa) representations, is of relevance for Conjecture 3.1.7 of [SU]. As described in the introduction, this conjecture would in particular allow in the work of Skinner-Urban in [SU] to avoid the appeal to the difficult work of Kato on Euler systems. In fact, one only needs the upper bound on the rank of monodromy operators for type (IIa) representations and the next theorem proves this desired upper bound in the case of globally generic representations. Essentially the same proof, but using symplectic group eigenvarieties instead of unitary group eigenvarieties, should give the desired upper bound as predicted by the conjecture. See for example [JOR2] where our methods have been used for such purposes in combination with γ-factor arguments.

Let

$$J := \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$$

and let GSp_4 denote the algebraic group over $\mathbb{Q}$ corresponding to those $X \in \mathrm{GL}_4(\mathbb{Q})$ such that

$${}^{\mathrm{T}}X \cdot J \cdot X = c(X) \cdot J$$

for some scalar $c(X)$. Now let F be a totally real field and π a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ of weight

$$((b_{\tau,1}, b_{\tau,2})_\tau; w) \in (\mathbb{Z}^2)^{\mathrm{Hom}(F,\mathbb{R})} \times \mathbb{Z}$$

with $b_{\tau,1} \geq b_{\tau,2} \geq 0$ and $b_{\tau,1} + b_{\tau,2} \equiv w \mod 2$ for all $\tau \in \mathrm{Hom}(F, \mathbb{R})$. This means that if $v|\infty$ is the place of F corresponding to $\tau \in \mathrm{Hom}(F, \mathbb{R})$ then π_v is an essentially discrete series representation of $\mathrm{GSp}_4(F_v)$ with the same infinitesimal character and central character as the finite-dimensional irreducible algebraic representation of GSp_4 of highest weight given by

$$t = \mathrm{diag}(t_1, t_2, t_3, t_4) \mapsto t_1^{b_{\tau,1}} t_2^{b_{\tau,2}} c(t)^{(-w-b_{\tau,1}-b_{\tau,2})/2}$$

An automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_F)$ as above will be called cuspidal regular algebraic. Suppose π is in addition also globally generic as defined for example in [SOR, Sect. 2.3]:

Let ψ be a non-trivial character of $F \backslash \mathbb{A}_F$ and define a character on the unipotent radical N of the upper-triangular Borel of GSp_4 by

$$\begin{bmatrix} 1 & u & * & * \\ & 1 & z & * \\ & & 1 & -u \\ & & & 1 \end{bmatrix} \longrightarrow \psi(u+z)$$

Then π is called globally generic if there exists $f \in \pi$ such that

$$\int_{N(F)\backslash N(\mathbb{A}_F)} f(n)\psi^{-1}(n)dn \neq 0$$

For such a globally generic automorphic representation it follows from the discussion in [SOR] that Hypothesis 5.1.1 implies that for a rational prime ℓ and isomorphism ι between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ there is a continuous semi-simple ℓ-adic Galois representation

$$\rho_{\pi,\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_4(\overline{\mathbb{Q}}_\ell)$$

such that for all finite places $v \nmid \ell$ of F such that π_v is unramified one has

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-\frac{3}{2}})$$

where $\mathrm{rec}_{\mathrm{GSp}_4}(-)$ denotes the local Langlands correspondence for GSp_4 as constructed in [GT]. We will use deformation theory to extend this to places where π is not spherical. To state the local-global compatibility results that we will prove, we first recall some notation about Iwahori-spherical representations of $\mathrm{GSp}_4(K)$ where K is a p-adic field.

Let B denote the Borel subgroup of $\mathrm{GSp}_4(K)$ consisting of upper-triangular matrices and let T denote the diagonal maximal torus. For smooth characters χ_1, χ_2, χ of $K^\times$ let $\mathrm{Ind}_B^G(\chi_1, \chi_2; \chi)$ denote the normalized induction of the extension to B of the character

$$t = \mathrm{diag}(t_1, t_2, t_3, t_4) \mapsto \chi_1(t_1)\chi_2(t_2)\chi(c(t))$$

This induced representation will also be denoted by $\chi_1 \times \chi_2 \rtimes \chi$. Let π_v be a generic Iwahori-spherical representation of $\mathrm{GSp}_4(K)$. In particular there is an injection

$$\pi_v \hookrightarrow \chi_1 \times \chi_2 \rtimes \chi$$

for some characters χ_1, χ_2 and χ of $K^\times$. The generic Iwahori-spherical representations of $\mathrm{GSp}_4(K)$ are divided into six classes in [SCH], namely type (Ia) to type (VIa), and these types can be described in the following manner:

- type (Ia)

Here π_v is isomorphic to an irreducible principal series representation $\chi_1 \times \chi_2 \rtimes \chi$ for some unramified characters χ_1, χ_2 and χ

- type (IIa)

Here π_v is the generic constituent of $\chi_1|\cdot|^{1/2} \times \chi_1|\cdot|^{-1/2} \rtimes \chi_2$ with $\chi_1^2 \notin \{|\cdot|^{\pm 1}, |\cdot|^{\pm 3}\}$

- type (IIIa)

Here π_v is the generic constituent of $\chi_1 \times |\cdot| \rtimes \chi_2|\cdot|^{-1/2}$ with $\chi_1 \notin \{1, |\cdot|^{\pm 2}\}$.

- type (IVa)

Here π_v is the generic constituent of $|\cdot|^2 \times |\cdot| \rtimes |\cdot|^{-3/2}\chi$.

- type (Va)

Here π_v is the generic constituent of $\xi_0|\cdot| \times \xi_0 \rtimes \chi|\cdot|^{-1/2}$ where ξ_0 is the unramified character of $K^\times$ such that $\xi_0^2 = 1$ and $\xi_0 \neq 1$

- type (VIa)

Here π_v is the generic constituent of $|\cdot| \times 1 \rtimes |\cdot|^{-1/2}\chi$.

Let π_v be a generic Iwahori-spherical representations of $\mathrm{GSp}_4(K)$. We now describe explicitly some aspects of the local Langlands correspondence for such a representation. Since π_v is generic it follows, in the notation of [GT, Thm. 5.2], that π_v is of type (B). Hence it follows from the construction of $\mathrm{rec}_{\mathrm{GSp}_4}(-)$ in [GT, Sect. 7] as well as from [GT, Prop. 3.4] that

$$\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)^{\mathrm{ss}} \cong \tilde{\chi} \oplus \tilde{\chi}\tilde{\chi}_1 \oplus \tilde{\chi}\tilde{\chi}_2 \oplus \tilde{\chi}\tilde{\chi}_1\tilde{\chi}_2$$

The local global compatibility results that we will prove in Theorem 6 for the representation $\rho_{\pi,\ell,\iota}$ are obtained by using functoriality transfer of π to an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$. We will only deal with the case of Iwahori-spherical ramification and in this case it will follow from the above description of $\mathrm{rec}_{\mathrm{GSp}_4}(-)$ that the only local components of automorphic representations of $\mathrm{GL}_4(\mathbb{A}_F)$ that will intervene in the calculations are subquotients of representations of the form

$$\mathrm{Ind}(\chi, \chi\chi_1, \chi\chi_2, \chi\chi_1\chi_2)$$

for some characters χ_1, χ_2 and χ as above.

Theorem 6. *Let F be a totally real field and let π be a cuspidal globally generic regular algebraic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Let v be a finite place of F such that π_v is Iwahori-spherical but not of type* (VIa)*. Then*

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-\frac{3}{2}})^{\mathrm{ss}}$$

unless π_v is of type (IVa) *and $\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ has trivial monodromy operator.*

Proof. By [GT] the transfer of π to an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$ is either cuspidal or an isobaric sum $\pi_1 \boxplus \pi_2$ with π_1 and π_2 cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. In the latter case the Galois representation $\rho_{\pi,\ell,\iota}$ is constructed from $\rho_{\pi_1,\ell,\iota}$ and $\rho_{\pi_2,\ell,\iota}$ and the theorem follows from Corollary 6.3. Hence assume now that the transfer of π to an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ is cuspidal. Let v be as in the statement of the theorem. Since π_v is Iwahori-spherical there is an injection

$$\pi_v \hookrightarrow \chi_1 \times \chi_2 \rtimes \chi$$

for some unramified characters χ_1, χ_2 and χ of $F_v^\times$. In order to prove that

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-\frac{3}{2}})^{\mathrm{ss}}$$

it is enough, by [SOR] and the discussion preceding the theorem, to show the following: Suppose Π is a cuspidal regular algebraic essentially self-dual automorphic representation $\mathrm{GL}_4(\mathbb{A}_F)$ such that Π_v is the generic constituent of

$$\mathrm{Ind}(\chi\chi_1\chi_2, \chi\chi_1, \chi\chi_2, \chi)$$

where χ_1, χ_2 and χ are as above. Then

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GL}_4}(\Pi_v \otimes |\det|^{-\frac{3}{2}})^{\mathrm{ss}}$$

To show this we will use Theorem 5 and a case by case study of the possible types of π_v.

- type (Ia)

Here π_v is isomorphic to an irreducible principal series representation $\chi_1 \times \chi_2 \rtimes \chi$ for some unramified characters χ_1, χ_2 and χ. By [ST, Lem. 3.2] it follows that $\chi_1 \neq |\cdot|^{\pm 1}$, $\chi_2 \neq |\cdot|^{\pm 1}$, and $\chi_1 \neq |\cdot|^{\pm 1}\chi_2^{\pm 1}$ where in the last expression all sign combinations are allowed. It follows that $\mathrm{Ind}(\chi, \chi\chi_1, \chi\chi_2, \chi\chi_1\chi_2)$ is also irreducible and therefore

$$\Pi_v \cong \mathrm{Ind}(\chi, \chi\chi_1, \chi\chi_2, \chi\chi_1\chi_2)$$

Hence one is done.

- type (IIa)

Here π_v is the generic constituent of $\chi_1|\cdot|^{1/2} \times \chi_1|\cdot|^{-1/2} \rtimes \chi_2$ with $\chi_1^2 \notin \{|\cdot|^{\pm 1}, |\cdot|^{\pm 3}\}$. It follows that

$$\Pi_v \cong \mathrm{St}_2(\chi_1\chi_2) \boxplus \chi_1^2\chi_2 \boxplus \chi_2$$

Hence one is done unless $\chi_1^2\chi_2 = \chi_2 = \chi_1\chi_2|\cdot|^{\pm 1/2}$. But this implies $\chi_1^2 = |\cdot|^{\pm 1}$ which is a contradiction.

- type (IIIa)

Here π_v is the generic constituent of $\chi_1 \times |\cdot| \rtimes \chi_2|\cdot|^{-1/2}$ with $\chi_1 \notin \{1, |\cdot|^{\pm 2}\}$. It follows that

$$\Pi_v \cong \mathrm{St}_2(\chi_1\chi_2) \boxplus \mathrm{St}_2(\chi_2)$$

Hence one is done unless $\chi_1 = 1$ but this is a contradiction.

- type (IVa)

Here π_v is the generic constituent of $|\cdot|^2 \times |\cdot| \rtimes |\cdot|^{-3/2}\chi$. It follows that

$$\Pi_v \cong \mathrm{St}_4(\chi)$$

Hence one is done if $\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ has non-trivial monodromy operator.

- type (Va)

Here π_v is the generic constituent of $\xi_0|\cdot| \times \xi_0 \rtimes \chi|\cdot|^{-1/2}$ where ξ_0 is the unramified character of $K^\times$ such that $\xi_0^2 = 1$ and $\xi_0 \neq 1$. It follows that

$$\Pi_v \cong \mathrm{St}_2(\xi_0\chi) \boxplus \mathrm{St}_2(\chi)$$

Hence one is done since $\xi_0 \neq 1$.

- type (VIa)

Here π_v is the generic constituent of $|\cdot| \times 1 \rtimes |\cdot|^{-1/2}\chi$. It follows that

$$\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \mathrm{St}_2(\chi)$$

Here the desired result does not follow. □

Remark. The local-global compatibility results for local semi-simplifications of automorphic Galois representations obtained in the corollaries 5, 6.2 and 6 can be strengthened if non-triviality results for local monodromy operators are known. In Chapter 9 we will obtain such results.

In the previous chapters we described an approach to obtain local-global compatibility results for semi-simple information via the variation of crystalline periods of Galois representations. Similar ideas have been used before in [SU] with the following difference. In that work one studies extension of automorphic Galois representations by putting them into a family whose generic member is irreducible. The variation results on crystalline periods then show something about the so obtained extension classes. Strictly speaking, this lies outside the framework of local-global compatibility since in the case of reducible Galois representations local-global compatibility does not predict something about the extension classes. However, via the deformation theoretic approach these two questions are seen to be very closely related. The unifying philosophy is the following: In many cases, local-global compatibility is simply part of the information that can be obtained at p by deforming p-adic Galois representations. In the work of Skinner-Urban the same is done, and crystallinity properties of pieces of reducible automorphic Galois representations are obtained.

Let $f \in S_k(\Gamma_0(N))$ be a newform of even weight k which is at least 2, with associated automorphic representation π. Let V_f be the p-adic representation associated to f. The Selmer group is defined as

$$\mathrm{H}^1_f(\mathbb{Q}, V_f(k/2)) := \mathrm{Ker}(\mathrm{H}^1(\mathbb{Q}, V_f(k/2)) \longrightarrow \prod_v \frac{\mathrm{H}^1(\mathbb{Q}_v, V_f(k/2))}{\mathrm{H}^1_f(\mathbb{Q}_v, V_f(k/2))})$$

where for $v \neq p$ one has

$$\mathrm{H}^1_f(\mathbb{Q}_v, V_f(k/2)) = \mathrm{Ker}(\mathrm{H}^1(\mathbb{Q}_v, V_f(k/2)) \longrightarrow \mathrm{H}^1(I_v, V_f(k/2)))$$

where I_v denotes the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$, and for $v = p$ one defines

$$\mathrm{H}^1_f(\mathbb{Q}_v, V_f(k/2)) = \mathrm{Ker}(\mathrm{H}^1(\mathbb{Q}_v, V_f(k/2)) \longrightarrow \mathrm{H}^1(\mathbb{Q}_v, V_f(k/2) \otimes \mathrm{B}_{\mathrm{cris}}))$$

If $\epsilon(1/2, \pi) = -1$, then one knows that there exists a certain special cuspidal automorphic representation $\mathrm{SK}(\pi)$ of $\mathrm{PGSp}_4(\mathbb{A}_{\mathbb{Q}})$. The existence of this special symplectic automorphic representation is then exploited, via deformation theory, to prove in [SU, Theorem 4.1.4] for suitable primes p that

$$\dim \mathrm{H}^1_f(\mathbb{Q}, V_f(k/2)) \geq 1$$

We will now discuss some aspects of local-global compatibility and deformation theory used in the proof. The general principle is to put the Saito-Kurokawa form $\mathrm{SK}(\pi)$ into a p-adic family of automorphic representations with generic member having irreducible Galois representation. A crucial input in the whole strategy is Theorem 4.2.7 of [SU] which says that a certain irreducible component of an eigenvariety is not so called globally endoscopic. To prove this theorem one shows that global endoscopy would imply the existence of a non-trivial extension $*$ of ϵ^{-1} by the trivial representation, where ϵ is the p-adic cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If $*$ can be shown to be everywhere unramified this yields be a contradiction. To control the ramification at p one uses Kisin's result on analytic variation of crystalline periods! With this result on failure of global endoscopy, by a deformation theoretic argument one produces a non-zero class

$$c \in \mathrm{H}^1_f(\mathbb{Q}_p, V_f(2k - 3 - r))$$

where $r = 1 - k$ or $r = 2 - k$ which is a candidate to be in the desired Selmer group. The key is now to show that in fact c yields a non-trivial class in the Selmer group. This uses in particular that the Galois representation is ordinary at p if f is ordinary at p. This has been proven by Kisin via crystalline periods. Now one want to show that $r = 2 - k$ as opposed to $1 - k$. It is a deep result of Kato's that

$$\dim \mathrm{H}^1_f(\mathbb{Q}, V_f(s)) = 0$$

for $s \neq k/2$. A fortiori, it follows that $r = 2 - k$. However, to deduce the last result, really much less is needed. As follows from the discussion in [SU, Section 4.3.4], an upper bound on the monodromy at places of para-spherical ramification suffices and this in turn can be proven - at p - via crystalline periods. More precisely, if $r = 1 - k$ one obtains a non-split extension

$$0 \to L \to \mathcal{K} \to L(\epsilon^{-1}) \to 0$$

where L is some finite extension of $\mathbb{Q}_p$. If one can show that this extension is everywhere unramified one obtains a contradiction. The ramification control at p is obtained via a crystalline periods argument and the desired control of the ramification at a place $q|N$ follows from [SU, Conjecture 3.1.7]. As mentioned in the remark preceding the theorem, the required upper bound on the rank of monodromy can be proven via crystalline period method developed in this memoir. Note that in this case the independence of ℓ hypothesis can be assumed to be known due to the corresponding independence of ℓ theorem for compatible systems of Galois representations associated to suitable automorphic representations of GL_4.

8 Congruences

As described in the introduction, our approach to local monodromy operators of automorphic Galois representations is to combine automorphic congruences with modularity lifting theorems. In the current chapter we discuss an instance of such congruences, namely potential level-lowering results. We prove strengthened versions of the potential level-lowering result of [CHT, Lem. 4.4.1] and [SOR] where we control the residual Hecke action at the place where the level is lowered. One motivation for this is that it allows us to prove in this chapter residual local-global compatibility results in the context of general linear groups, unitary groups, and symplectic groups. This should be useful when proving modularity lifting results since to do so one usually has to use a certain amount of local-global compatibility results. Hence the study of congruences might turn out to be useful for carrying out the previously described principle for the calculation of monodromy operators of automorphic forms on more general reductive algebraic groups than currently approachable cases. The level-lowering proof of the residual local-global compatibility seems significantly simpler than attempting to show this via a study of singularities of Shimura varieties, for example on symplectic groups.

8.1 Potential level-lowering and residual local-global compatibility

We first describe some notation for automorphic forms on unitary groups. Let G, F and F^+ be as in Section 5.1: F^+ is a totally real field, E an imaginary quadratic extension of $\mathbb{Q}$ and $F := F^+E$ is such that F/F^+ is unramified at all finite places and that $n[F^+ : \mathbb{Q}]/2$ is even, $n \geq 2$ is an integer and G is an inner form of the quasi-split unitary group U_n^* discussed earlier, that is quasi-split at all places of F^+ which are inert in F and such that $\prod_{v|\infty} G(F_v^+)$ is compact. If $v \in \mathrm{Spl}_{F/F^+}$ splits as ww^c in F then i_w is the isomorphism discussed in Section 5.1.

Let c be the non-trivial element of $\mathrm{Gal}(F/F^+)$. Let $\ell > n$ be a rational prime and fix throughout this section an isomorphism ι between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$. Fix a subfield K of $\overline{\mathbb{Q}}_\ell$ such that $K/\mathbb{Q}_\ell$ is a finite extension and K contains the image of all embeddings $F \hookrightarrow \overline{\mathbb{Q}}_\ell$. Let $\mathcal{O}$ denote the valuation ring of K and let k denote its residue field. Let S_ℓ denote the set of places of F^+ above ℓ and assume that $S_\ell \subset \mathrm{Spl}_{F/F^+}$. Choose a subset $\tilde{S}_\ell$ of the set of places of F above ℓ such that

$$\{\text{places of } F \text{ above } \ell\} = \tilde{S}_\ell \sqcup \tilde{S}_\ell^c$$

where S_ℓ^c is the collection of elements $x \circ c$ for $x \in \tilde{S}_\ell$. Let

- $G(F_\ell^+) := \prod_{v \in S_\ell} G(F_v^+)$ and identify $G(F_v^+)$ with $\mathrm{GL}_n(F_w)$ where $w \in \tilde{S}_\ell$ divides v
- $G(\mathcal{O}_{F_\ell^+}) := \prod_{v \in S_\ell} G(\mathcal{O}_{F_v^+})$
- $U = \prod_v U_v$ be a compact open subgroup of $G(\mathbb{A}_{F^+}^{(\infty)})$ where v ranges over the finite places of F^+ and U_v is a subgroup of $G(F_v^+)$ and assume that the projection of U to $G(F_\ell^+)$ is contained in $G(\mathcal{O}_{F_\ell^+})$

Fix $\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_\ell)}$ such that

$$\overline{a}_{\tau,i} = -\overline{a}_{\tau\circ c,n+1-i}$$

for all $\tau \in \mathrm{Hom}(F, \overline{\mathbb{Q}}_\ell)$ and all $1 \leq i \leq n$. Let $\tilde{I}_\ell$ denote the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_\ell$ which correspond to places in $\tilde{S}_\ell$. There exists a finite free $\mathcal{O}$-module $\mathrm{M}_{\overline{a}}$ and a representation

$$\kappa : G(\mathcal{O}_{F^+_\ell}) \longrightarrow \mathrm{GL}(\mathrm{M}_{\overline{a}})$$

such that

$$\mathrm{M}_{\overline{a}} \otimes_{\mathcal{O}} K \cong \otimes_{\tau \in \tilde{I}_\ell} (W_{\overline{a}_\tau} \otimes_{\mathbb{Q}} K)$$

where $W_{\overline{a}_\tau}$ is as defined in Section 5.1. For any $\mathcal{O}$-module A define

$$S_\kappa(U, A) =$$

$$\{f : G(F^+)\backslash G(\mathbb{A}^{(\infty)}_{F^+}) \longrightarrow \mathrm{M}_{\overline{a}} \otimes_{\mathcal{O}} A | f(xu) = u_\ell^{-1} \cdot f(x) \text{ for all } x \in G(\mathbb{A}^{(\infty)}_{F^+}), u \in U\}$$

where u_ℓ is the projection of $u \in U$ to $G(F^+_\ell)$. Define an action of $G(\mathbb{A}^{(\infty)}_F)$ on $S_\kappa(U, A)$ by

$$(g \cdot f)(x) = g_\ell \cdot f(xg)$$

for $g \in G(\mathbb{A}^{(\infty)}_F)$ and $f \in S_\kappa(U, A)$. For $f \in S_\kappa(U, \mathcal{O})$ we will denote by $\overline{f}$ the corresponding element of $S_\kappa(U, k)$. For $f_1, f_2 \in S_\kappa(U, \mathcal{O})$ we will write $f_1 \equiv f_2$ if their reductions in $S_\kappa(U, k)$ agree. This will also sometimes be denoted as $\overline{f}_1 \equiv \overline{f}_2$.

Let $\mathcal{A}$ denote the complex vector space of automorphic forms on $G(\mathbb{A}_{F^+})$. Let ξ denote the representation of

$$G(F^+_\infty) := \prod_{v|\infty} G(F^+_v)$$

corresponding to $\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F, \overline{\mathbb{Q}}_\ell)}$ as described in Section 5.1. As described for example in [CHT, p. 101], to each $f \in S_\kappa(U, \mathcal{O})$ corresponds, via ι, an element in

$$\mathrm{Hom}_{G(F^+_\infty)}(\xi^\vee, \mathcal{A})$$

Hence to each such f one can associate in an $G(\mathbb{A}^{(\infty,\ell)}_{F^+})$-equivariant way a vector $\Psi(f)$ of an automorphic representation of $G(\mathbb{A}_{F^+})$.

8.1.1 Lowering the level

In the proof of Proposition 8.2 certain subspaces of $S_\kappa(U, \mathcal{O})$ will be used which we now define:

Fix a finite place $v \in \mathrm{Spl}_{F/F^+}$ such that $v \nmid \ell$. Via the choice of $w|v$ of F identify $G(F^+_v)$ with $\mathrm{GL}_n(F_w)$. Via this identification let T_v be the diagonal torus of $G(F^+_v)$, let I denote the Iwahori subgroup of $G(F^+_v)$ corresponding to the upper triangular Borel subgroup and let I_1 be the subgroup of I as defined in the beginning of Section 3.3. Assume that the compact open subgroup U of $G(\mathbb{A}^{(\infty)}_{F^+})$ as above satisfies $U_v = I_1$. For any character $\theta : I/I_1 \to \mathcal{O}^\times$ let

$$S^\theta_\kappa(U, \mathcal{O}) := \{f \in S_\kappa(U, \mathcal{O}) \big| \ g \cdot f = \theta(g) f \text{ for all } g \in I\}$$

Let $\mathfrak{m}_{\mathcal{O}}$ denote the maximal ideal of $\mathcal{O}$ and let $\overline{\theta} := \theta \mod \mathfrak{m}_{\mathcal{O}}$. Define

$$S^{\overline{\theta}}_\kappa(U, k) := \{f \in S_\kappa(U, k) \big| \ g \cdot f = \overline{\theta}(g) f \text{ for all } g \in I\}$$

For the statement of the next result fix also the following notation. A character $\chi : T_v \to \mathbb{C}^\times$ will be called ℓ-integral if

$$|\iota^{-1}\chi(x)|_\ell \leq 1$$

for all $x \in T_v$. Moreover, for ℓ-integral characters χ_1 and χ_2 as above we write

$$\chi_1 \equiv \chi_2 \mod \ell$$

if $|\iota^{-1}\chi_1(x) - \iota^{-1}\chi_2(x)|_\ell < 1$ for all $x \in T_v$.

Proposition 8.2. *Let G, F, F^+ and ι be as above. Let π be an automorphic representation of $G(\mathbb{A}_{F^+})$ of weight ν, $\ell > n$ a rational prime such all places of F^+ above ℓ split in F and such that that π_v is spherical at all places above ℓ. Assume v_0 is a finite place of F^+ which splits as ww^c in F and is such that*

- $\mathrm{N}(v_0) \equiv 1 \mod \ell$.
- $\pi_{v_0} \circ i_w^{-1}$ *is Iwahori-spherical, say*
$$(\pi_{v_0} \circ i_w^{-1}) \hookrightarrow \mathrm{Ind}(\chi_1)$$
for some unramified character χ_1

For any finite place $v \neq v_0$ take a compact open subgroup $U'_v \leq G(F_v^+)$ such that

- $\pi_v^{U'_v} \neq (0)$
- $U'_v = G(\mathcal{O}_{F_v^+})$ *for almost all places*

Then there is a place w_0 and an automorphic representation $\tilde{\pi}$ of $G(\mathbb{A}_{F^+})$ of weight ν such that

- *There is an isomorphism*
$$\overline{\rho}_{\tilde{\pi},\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}$$
- *There is an at most tamely ramified character χ such that*
$$(\tilde{\pi}_{v_0} \circ i_w^{-1}) \cong \mathrm{Ind}(\chi)$$
and
$$\chi \equiv \chi_1 \mod \ell$$
- *For all finite places $v \notin \{v_0, w_0\}$ of F^+ one has $\tilde{\pi}_v^{U'_v} \neq (0)$*

Proof. Let us first make some definitions. Fix a place $w_0 \in \mathrm{Spl}_{F/F^+}$ such that π_{w_0} is spherical and such that $w_0 \neq v_0$ and $w_0 \nmid \ell$. Let S be the subset of the set of places of F^+ which is given by the union of the following sets:

- the set finite places of F^+ which do not split in F
- the set of finite places of F^+ in $\mathrm{Spl}_{F/F^+} \backslash \{v_0\}$ at which π is not spherical
- $\{w_0\}$
- the set of all places of F^+ above ℓ
- the set of finite places v of F^+ such that U'_v is not $G(\mathcal{O}_{F_v^+})$, where U'_v is as in the statement of the lemma
- the set of infinite places

For each $v \notin S$ fix an isomorphism $G(F_v^+) \cong \mathrm{GL}_n(F_v^+)$ and note that in the rest of this proof we will identify these groups. We will assume that for $v = v_0$ the isomorphism is given by i_w. Let I be the Iwahori-subgroup of $G(F_{v_0}^+)$ corresponding to the upper triangular Borel B and let I_1 and T be as before. Let

$$U = \prod_v U_v \leq G(\mathbb{A}_{F^+}^{(\infty)})$$

be a compact open subgroup such that

- $U_{v_0} = I_1$
- $U_{w_0} \le G(\mathcal{O}_{F^+_{w_0}})$ has no non-trivial element of finite order
- for all finite places $v \notin \{v_0, w_0\}$ one has $U_v = U'_v$

Note that
$$\pi^U \neq (0)$$
Now let
$$\mathcal{H}_1 = \otimes'_{v \notin S} \mathcal{H}_{1,v}$$
be the Hecke algebra which, in the notation of Section 3.3, at v_0 equals $\mathcal{H}_1^+$ and for $v \neq v_0$ equals the spherical Hecke algebra consisting of functions on $G(F_v^+)$ which are locally constant, have compact support and are $G(\mathcal{O}_{F_v^+})$ bi-invariant. Let $\nu^\vee$ denote the dual representation of ν. As discussed in Section 5.1, the representation $\nu^\vee$ corresponds to some
$$\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$$
In the notation of the discussion preceding the lemma, choose any set $\tilde{S}_\ell$ and let $\kappa : G(\mathcal{O}_{F_\ell^+}) \longrightarrow \mathrm{GL}(\mathrm{M}_{\overline{a}})$ be the corresponding representation.

Let $f'_1 \in \pi^{U^{(v_0)}I}$ be an eigenform for the $\mathcal{H}_1$-action, this exists due to the commutativity of the action, such that there exists a finite extension $K/\mathbb{Q}_\ell$ with valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{m}_\mathcal{O}$ and residue field k_ℓ such that
$$f_1 := \Psi^{-1}(f'_1) \in S_\kappa(U, \mathcal{O})$$
and such that $\overline{f}_1 \in S_\kappa(U, k_\ell)$ is non-zero. After possibly taking a finite extension we will now assume that K contains a primitive ℓ-th root of unity ζ_ℓ. Let $\mathcal{O}_{F^+_{v_0}}$ denote the valuation ring of $F^+_{v_0}$ and let k_{v_0} denote the residue field. Let x be a generator of $k_{v_0}^\times$ and define a character
$$\xi : T(\mathcal{O}_{F^+_{v_0}}) \to 1 + \mathfrak{m}_\mathcal{O}$$
by the composition of the reduction map with the character of $T(k_{v_0})$ defined by
$$\mathrm{diag}(1, \cdots, 1, x, 1 \cdots, 1) \mapsto \zeta_\ell^i$$
where x is at the i'th entry. Note that since $\ell > n$ the character ξ is regular. Via the isomorphism of I/I_1 with $T(k_{v_0})$ one also obtains from the above definitions a character ρ of I/I_1. Since U_{w_0} has no non-trivial element of finite order it follows from [CHT, Lem. 2.3.1] that $S_\kappa(U, \mathcal{O})$ is a free $\mathcal{O}[I/I_1]$-module and hence there exists $f_\xi \in S_\kappa^\rho(U, \mathcal{O})$ such that
$$\overline{f}_\xi \equiv \overline{f}_1$$
The representation of $G(\mathbb{A}_{F^+})$ generated by $\Psi(f_\xi)$ decomposes as a direct sum $\pi_1 \oplus \cdots \oplus \pi_r$ for some automorphic representations π_i of $G(\mathbb{A}_{F^+})$. We will show in the rest of this proof that at least one of these automorphic representations can be taken as the $\tilde{\pi}$ in the statement of the lemma.

First note that one can write
$$\Psi(f_\xi) = \sum_{i=1}^r g_i$$
for some $g_i \in \pi_i$. Since $f_\xi \in S_\kappa^\rho(U, \mathcal{O})$ it follows that for all $v \notin S \cup \{v_0\}$ each g_i is fixed by the action of $G(\mathcal{O}_{F_v^+})$ and furthermore it follows that each g_i transforms under I by ξ. In particular one has
$$\pi_{i,v_0}^\rho \neq (0)$$
for each $1 \le i \le r$. Here and also later in the proof we identify ρ with the $\mathbb{C}$-valued character corresponding to it via ι. It follows from [ROC] that for each i the representation π_{i,v_0} is a subquotient of $\mathrm{Ind}(\chi_2)$ for some character χ_2,

depending on i, such that

$$(\iota^{-1}\chi_2)|_{T(\mathcal{O}_{F^+_{v_0}})} = \xi$$

Since ξ is regular it follows from [ROD, p. 419] that $\mathrm{Ind}(\chi_2)$ is in fact irreducible and hence

$$\pi_{i,v_0} \cong \mathrm{Ind}(\chi_2)$$

We already know that $\pi^{\rho}_{i,v_0} \neq (0)$ and we will now show that in fact

$$\dim_{\mathbb{C}} \pi^{\rho}_{i,v_0} = 1$$

Note that we will use in the following the description of principal series representations given in section 3.3. Let $g \in \pi^{\rho}_{i,v_0}$ and let $h \in T(\mathcal{O}_{F^+_{v_0}})$. Let W denote the Weyl group of $G(F^+_{v_0})$ and let $w \in W$. Then one has

$$\xi(h)g(w) = g(wh) = g((whw^{-1})w) = \chi_2(whw^{-1})g(w) = \xi(whw^{-1})g(w)$$

By the regularity of ξ it follows that $g(w) = 0$ unless $w = 1$. By the Iwasawa decomposition

$$G(F^+_{v_0}) = BWI$$

it follows that the space π^{ρ}_{i,v_0} is one-dimensional. A generator is for example the function $G(F^+_{v_0}) \to \mathbb{C}$ which is given by

$$bh \mapsto \chi_2(b)\xi(h)$$

for $b \in B$ and $h \in I$ and which is zero on BwI for any non-trivial $w \in W$.

Now let $\mathcal{H}_2 = \otimes'_{v \notin S} \mathcal{H}_{2,v}$ denote the Hecke algebra such that $\mathcal{H}_{2,v}$ is the spherical Hecke algebra for all $v \neq v_0$ and, in the notation of Section 3.3, $\mathcal{H}_{v_0} \cong \mathcal{H}^+_{\rho}$. It follows from the above discussion that each g_i is an eigenvector for the $\mathcal{H}^+_{\rho}$-action. By using the one-dimensionality of spherical vectors in unramified representations it follows that in fact each g_i is an eigenform for the $\mathcal{H}_2$-action. Let $g'_i := \Psi^{-1}(g_i)$ and note that after possibly taking a finite extension of K one can assume that $g'_i \in S_{\kappa}(U, \mathcal{O})$ for all i.

For each $R \in \mathcal{H}_v$ for $v \notin S$ let $\alpha(R)$ be given by $R \cdot f_1 = \alpha(R) f_1$ and for each i let $\alpha_i(R)$ be given by $R \cdot g'_i = \alpha_i(R) g'_i$. We will now show that there exists g'_i such that

$$\alpha_i(R) \equiv \alpha(R) \mod \mathfrak{m}_{\mathcal{O}}$$

for every R as above. Suppose for contradiction that for each j there exists a finite place v_j and $R \in \mathcal{H}_{v_j}$ such that $\alpha_j(R) \not\equiv \alpha(R)$. First note that since $f_{\xi} \equiv f_1$ one has

$$\sum_{i=1}^{r} \alpha(R) \cdot \overline{g}'_i \equiv \alpha(R) \cdot \overline{f}_{\xi} \equiv R \cdot \overline{f}_{\xi} \equiv \sum_{i=1}^{r} \alpha_i(R) \cdot \overline{g}'_i \mod \mathfrak{m}_{\mathcal{O}}$$

For example for $j = 1$ one deduces that

$$\overline{f}_{\xi} \equiv \sum_{2 \leq i \leq r} \frac{\alpha_i(R) - \alpha_1(R)}{\alpha(R) - \alpha_1(R)} \cdot \overline{g}'_i \mod \mathfrak{m}_{\mathcal{O}}$$

By applying to the above description of $\overline{f}_{\xi}$ the corresponding calculation for $j = 2$ and continuing in this way yields that $\overline{f}_{\xi} \equiv 0$ which is a contradiction. Hence there exists g'_i such that $\alpha_i(R) \equiv \alpha(R) \mod \mathfrak{m}_{\mathcal{O}}$ for every R as above. Choose one such g'_i and let $\tilde{\pi}$ denote the corresponding automorphic representation of $G(\mathbb{A}_{F^+})$. We will now show that $\tilde{\pi}$ satisfies all the properties stated in the lemma.

First note that by construction at every place $v \notin \{v_0, w_0\}$ one has

$$\tilde{\pi}_v^{U'_v} \neq (0)$$

Moreover, note that for all $v \notin S \cup \{v_0\}$ for any $R \in \mathcal{H}_v$ the eigenvalues of R acting on f_ξ and f_1 have the same image in k_ℓ. We have hence shown that there exists a valuation ring $\mathcal{O}'$ in a finite extension of $\mathbb{Q}_\ell$ such that for almost all finite places v in Spl_{F/F^+} one has $\tilde{\pi}_v \cong \mathrm{Ind}(\mu_1)$ and $\pi_v \cong \mathrm{Ind}(\mu_2)$ for some characters μ_1 and μ_2, depending on v, such that $\iota^{-1}\mu_1$ and $\iota^{-1}\mu_2$ take values in $\mathcal{O}'$ and their reductions in the residue field of $\mathcal{O}'$ agree. It hence follows that

$$\overline{\rho}_{\tilde{\pi},\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}$$

Let us now show the remaining property of $\tilde{\pi}$ stated in the lemma. As explained earlier, one knows that

$$\tilde{\pi}_{v_0} \cong \mathrm{Ind}(\chi_2)$$

for some character χ_2 of $T(F^+_{v_0})$ with

$$(\iota^{-1}\chi_2)|_{T(\mathcal{O}_{F^+_{v_0}})} = \xi$$

We will now show that there is w in the Weyl group such that $\chi_2^w \equiv \chi_1 \mod \ell$. Let $t \in T(F^+_{v_0})$ and write

$$\mathrm{ItI} = \bigsqcup_{b \in S} \mathrm{bI}$$

for some finite set S of elements in $G(F^+_{v_0})$. One can choose $b \in S$ to be of the form $b = i \cdot t$ for some $i \in I$. Write $i = i_1 i'$ with $i_1 \in I_1$ and $i' \in T(k_{v_0})$. Since $t \in T(F^+_{v_0})$ it follows that one can choose $b \in I_1 \cdot t$ and since

$$\phi_t^\xi(t) = 1$$

it follows that $\phi_t^\xi(b) = 1$. We will now assume that all the elements $b \in S$ are chosen to be of this form. Fix a Haar measure on $G(F^+_{v_0})$ such that I has volume 1. Then

$$\begin{aligned}(\phi_t^\xi f_\xi)(x) &= \int_{G(F^+_{v_0})} \phi_t^\xi(y) f_\xi(xy)dy \\ &= \sum_{b \in S} f_\xi(xb)\end{aligned}$$

Similarly, for the characteristic function $\mathrm{char}(ItI) = \phi_t^1 \in \mathcal{H}_1^+$ one has

$$(\phi_t^1 f_1)(x) = \sum_{b \in S} f_1(xb)$$

For any $t \in T(F^+_{v_0})$ let $\lambda_\xi(t)$ and $\lambda_1(t)$ denote the eigenvalue of ϕ_t^ξ and ϕ_t^1 acting on f_ξ and f_1, respectively. Then it follows from the above that $\lambda_\xi(t)$ and $\lambda_1(t)$ have the same image in the residue field k_ℓ since $\overline{f}_\xi \equiv \overline{f}_1$.

Fix a uniformizer of F_{v_0} and let $T(F_{v_0})^+$ be the corresponding subset of $T(F_{v_0})$ as defined in Chapter 3. Recall the $T(F_{v_0})^+$-equivariance, as described in Chapter 3, of the isomorphism

$$\pi_{v_0}^I \longrightarrow J(\pi_{v_0})^{T(\mathcal{O}_{F_v})} \otimes \delta_B^{-1}$$

as well as of the isomorphism

$$\tilde{\pi}_{v_0}^\rho \longrightarrow J(\tilde{\pi}_{v_0})^\xi \otimes \delta_B^{-1}$$

By [ZEL, Thm. 1.2] there are maps of $\mathbb{C}[T(\mathcal{O}_{F^+_{v_0}})]$-modules

$$J(\pi_{v_0})^{\mathrm{ss}} \hookrightarrow \bigoplus_{w\in W} \chi_1^w \cdot \delta_B^{1/2}$$

as well as

$$J(\tilde{\pi}_{v_0})^{\mathrm{ss}} \hookrightarrow \bigoplus_{w\in W} \chi_2^w \cdot \delta_B^{1/2}$$

where W denotes the Weyl group of $G(F_{v_0})$. Since $\Psi(f_1)$ and $\Psi(f_\xi)$ are eigenvectors for the $\mathbb{C}[T(\mathcal{O}_{F^+_{v_0}})]$-action their images in

$$J(\pi_{v_0}) \otimes \delta_B^{-1} \text{ and } J(\tilde{\pi}_{v_0}) \otimes \delta_B^{-1}$$

respectively span lines of the form

$$\chi_1^{w_1} \cdot \delta_B^{-1/2} \text{ and } \chi_2^{w_2} \cdot \delta_B^{-1/2}$$

for some elements $w_1, w_2 \in W$. Since we have already shown that $\lambda_\xi(t)$ and $\lambda_1(t)$ have the same image in the residue field k_ℓ it follows that $\chi_1^{w_1} \equiv \chi_2^{w_2} \mod \ell$ and hence $\chi_1 \equiv \chi_2^w \mod \ell$ for some w in the Weyl group. Hence $\tilde{\pi}$ satisfies all the properties of the lemma if one takes $\chi := \chi_2^w$. □

One can deduce from Proposition 8.2 residual local-global compatibility results for automorphic Galois representations. Such results can be useful in proving modularity lifting theorems where the only assumed properties of automorphic Galois representations are as in Hypothesis 5.1.1. See the beginning of Section 9.1 for a brief discussion of this.

Corollary 8.3. *Let G, F, F^+ and ι be as above and let π be an automorphic representation of $G(\mathbb{A}_{F^+})$. Let $\ell > n$ be a rational prime and let v be a finite place of F^+ which splits as ww^c in F such that $\mathrm{N}(v) \equiv 1 \mod \ell$ and $\pi_v \circ i_w^{-1}$ is Iwahori-spherical. Then $(\overline{\rho}_{\pi,\ell,\iota})|^{\mathrm{ss}}_{G_{F_w}}$ is as predicted by the local Langlands correspondence.*

Proof. Let v and w be as in the statement of the corollary and identify throughout this proof $G(F_v^+)$ with $\mathrm{GL}_n(F_w)$ via i_w. By Lemma 8.2 there exists an automorphic representation π' of $G(\mathbb{A}_{F^+})$ and characters $\mu_i : F_w^\times \to \mathbb{C}^\times$ and $\rho_i : F_w^\times \to \mathbb{C}^\times$ for $1 \le i \le n$ such that the following holds: Firstly, one has

$$\overline{\rho}_{\pi,\ell,\iota} \cong \overline{\rho}_{\pi',\ell,\iota}$$

Secondly, one has

$$\pi_v \hookrightarrow \mathrm{Ind}(\mu_1, \cdots, \mu_n)$$

as well as

$$\pi'_v \cong \mathrm{Ind}(\rho_1, \cdots, \rho_n)$$

and there is a finite extension $K/\mathbb{Q}_\ell$ such that the characters $\iota^{-1}\mu_i$ and $\iota^{-1}\rho_i$ take values in $\mathcal{O}_K$ and

$$\iota^{-1}\mu_i \equiv \iota^{-1}\rho_i \mod \mathfrak{m}_{\mathcal{O}_K}$$

for all $1 \le i \le n$. Here $\mathcal{O}_K$ denotes the valuation ring of K and $\mathfrak{m}_\mathcal{O}$ denotes the maximal ideal of $\mathcal{O}$. By Hypothesis 5.1.2 one has

$$\rho^{ss}_{\pi',\ell,\iota}|_{G_{F_w}} \cong (\iota^{-1}(\tilde{\rho}_1|\tilde{\cdot}|^{\frac{1-n}{2}} \oplus \cdots \oplus \tilde{\rho}_n|\tilde{\cdot}|^{\frac{1-n}{2}})) \otimes_{\mathcal{O}_K} \overline{\mathbb{Q}}_\ell$$

and since

$$\overline{\rho}^{ss}_{\pi,\ell,\iota}|_{G_{F_w}} \cong \overline{\rho}^{ss}_{\pi',\ell,\iota}|_{G_{F_w}}$$

the lemma follows. □

Results like Proposition 8.2 also allow to deduce level-lowering results for automorphic representations of GL_n over CM-fields. This method is due to Clozel-Harris-Taylor, see [CHT] (Lemma 4.4.1). We state it in a form convenient for our later use:

Corollary 8.4. *Let F be a* CM*-field and* Π *a cuspidal regular algebraic conjugate self-dual automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$ *and* $\ell > n$ *a rational prime such that* $\overline{\rho}_{\Pi,\ell,\iota}$ *is irreducible. Let* $v \nmid \ell$ *be a finite place of* F *such that* π_v *is Iwahori-spherical. Then there is a solvable* CM*-extension* F'/F *such that the base change of* Π *to* F' *remains cuspidal and there is a cuspidal regular algebraic conjugate self-dual automorphic representation* $\tilde{\Pi}$ *of* $\mathrm{GL}_n(\mathbb{A}_{F'})$ *such that*

$$\overline{\rho}_{\mathrm{BC}_{F'}(\Pi),\ell,\iota} \cong \overline{\rho}_{\tilde{\Pi},\ell,\iota}$$

and $\tilde{\Pi}_w$ *is an unramified principal series representation for some* $w|v$*. The extension* F' *can be chosen to be linearly disjoint over* F *from any given finite extension of* F*.*

Proof. By making a base change to a suitable solvable imaginary CM-extension of F which is linearly disjoint over F with $\overline{F}^{\ker \overline{\rho}_{\Pi,\ell,\iota}}$ and such that the base change of Π remains cuspidal one can assume that

- $\mathrm{N}(v) \equiv 1 \mod \ell$
- v lies above a place v_0 of Spl_{F/F^+}
- all places of F above ℓ lie above places in Spl_{F/F^+}
- F/F^+ is unramified at all finite places
- $n[F^+ : \mathbb{Q}]/2$ is even

In particular, there exists a unitary group G over F^+ as described in Section 5.1. It follows from [LAB] that there is an automorphic representation π of $G(\mathbb{A}_{F^+})$ whose base change to $\mathrm{GL}_n(\mathbb{A}_F)$ is Π. Then π, G and v_0 satisfy the conditions in Lemma 8.2. Hence there is an automorphic representation π' of $G(\mathbb{A}_{F^+})$ such that $\overline{\rho}_{\pi',\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}$ and such that π'_{v_0} is a principal series representation. By [LAB] one can base change π' to an automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$ and Π' is cuspidal since $\overline{\rho}_{\Pi,\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}$ is irreducible. Now take a solvable CM-extension such that the base change of Π' and Π remain cuspidal and such that at all places $w|v$ the base change of Π' is an unramified principal series representation. This representation satisfies all the requirements of the lemma. □

By essentially the same proofs as above, one can prove corresponding level-lowering and residual local-global compatibility result for symplectic groups. This in particular generalizes the symplectic potential level-lowering results of Sorensen in [SOR2]. We simply state the results:

Proposition 8.5. *Let* F *be a totally real field and let* G *be an inner form of* $\mathrm{GSp}_{2n,F}$ *(for some* $n \geq 1$*) such that* G_∞ *is compact mod center (and* ι *be as above). Let* π *be an automorphic representation of* $G(\mathbb{A}_F)$ *with infinity type* ξ*. Suppose* $\ell > 2n$ *is a rational prime. Assume* v_0 *is a finite place of* F *such that*

- $\mathrm{N}(v_0) \equiv 1 \mod \ell$
- π_{v_0} *is Iwahori-spherical, say* $\pi_{v_0} \hookrightarrow \mathrm{Ind}(\chi_1)$ *for some unramified character* χ_1

For any finite place $v \neq v_0$ *take a compact open subgroup* $U'_v \leq G(F_v^+)$ *such that*

- $\pi_v^{U'_v} \neq (0)$
- $U'_v = G(\mathcal{O}_{F_v^+})$ *for almost all places*

Then there is a place w_0 *and an automorphic representation* $\tilde{\pi}$ *of* $G(\mathbb{A}_{F^+})$ *of weight* ξ *such that*

- $\overline{\rho}_{\tilde{\pi},\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}$
- $\tilde{\pi}_{v_0} \cong \mathrm{Ind}(\chi)$ *for an at most tamely ramified character* χ *such that*

$$\chi \equiv \chi_1 \mod \ell$$

- *for all finite places $v \notin \{v_0, w_0\}$ of F^+ one has $\tilde{\pi}_v^{U'_v} \neq (0)$.*

Corollary 8.6. *Let G, F, ι, π, $\ell > 2n$ and v_0 be as above. Then $(\overline{\rho}_{\pi,\ell,\iota})|^{\mathrm{ss}}_{G_{F_{v_0}}}$ is as predicted by the local Langlands correspondence.*

By using functoriality one can deduce analogous results for symplectic groups as in Theorem C of [SOR2]. We refer to loc. cit. for detailed definitions of some of the notions we are now using. Let F be a totally real field. As in loc. cit. we will assume the following:

Hypothesis 8.6.1. Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that there is an integer w such that $\pi \otimes |\det|^{w/2}$ is algebraic. Then for each rational prime ℓ there exists a 4-dimensional Galois representation $\rho_{\pi,\ell}$ associated to π and if π is non-CAP then π_v is tempered for almost all finite places v of F.

If π is as above and non-endoscopic and non-CAP and assuming the previous hypothesis and that $\overline{\rho}_{\pi,\ell}$ is absolutely irreducible one can deduce the analogues of Proposition 8.5 and Corollary 8.6 for automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$ by using Theorem B of [SOR2]. We omit the detailed formulation.

9 Local monodromy operators: The case of general linear groups

In this chapter we prove non-triviality results for local-monodromy operators associated to automorphic representation of GL_n and unitary groups and hence generalize the results of Section 4.2.1 to this more general situation. We also discuss implications of the non-triviality of monodromy to local-global compatibly questions concerning the semi-simple part of local Weil-Deligne representations associated to automorphic Galois representations and prove an analogue of the conjecture of Skinner-Urban that was discussed in Chapter 2.

The main result of this chapter is Theorem 7 which is a generalization of Proposition 4.3 and in which modularity lifting theorems are used to obtain non-triviality results for local monodromy operators of automorphic Galois representations. In the next chapter we then carry out an analogous analysis for monodromy operators of automorphic representations of symplectic groups.

9.1 Monodromy operators

As explained before, the aim of this chapter is to use the deformation theory of automorphic forms to obtain local-global compatibility results for automorphic Galois representations. In particular, we want to use modularity lifting theorems to calculate local monodromy operators of automorphic Galois representations. In proving such modularity lifting theorems it is important to construct certain maps from Galois deformation rings to Hecke algebras or related objects. These maps are constructed by using some local-global compatibility results for automorphic Galois representations. In order to avoid potential instances of circular reasoning, we try to develop the approach to local monodromy operators via modularity lifting theorems in such a way that the local-global compatibility assumptions of Hypothesis 5.1.1 are sufficient for the proofs of the modularity lifting theorems that we use. Hence we will now briefly discuss some of the local properties of automorphic Galois representations that are used in the proofs of relevant modularity lifting results of [CHT], [TAY] and [GUE].

- Firstly:

Consider the crystalline deformation condition as defined in [CHT, Sect. 2.4.1]. The necessary local-global compatibility result is that $\rho_{\pi,\ell,\iota}$ is crystalline at all $v|\ell$ if π_v is spherical for all $v|\ell$. This is assumed in Hypothesis 5.1.1.

- Secondly:

Consider the Taylor-Wiles deformation condition as defined in [CHT, Sect. 1.4.6]. Apart from the local-global compatibility at places for principal series representations one has to show in the proof of [CHT, Prop. 3.4.4 (8)] a certain residual local-global compatibility at places where the local component is a representation of GL_n of the form

$$\chi_1 \boxplus \cdots \boxplus \chi_{n-2} \boxplus \mathrm{St}_2(\chi_{n-1})$$

for some unramified characters χ_i for $1 \le i \le n-1$. This residual local-global compatibility follows from Hypothesis 5.1.1 and Corollary 8.3.

- Thirdly:

Consider the deformation condition as defined in [TAY, Sect. 2]. The necessary local-global compatibility results concern local-global compatibility for principal series representations as well as unipotent inertia action at places of Iwahori-spherical ramification. Both of these results follow from Hypothesis 5.1.1.

In the following, in order to avoid a complete discussion of whether or not Hypothesis 5.1.1 does indeed imply all the necessary local-global compatibility results that are necessary to prove the modularity lifting theorems, we will simply treat modularity lifting theorems as hypotheses. Note also that for ease of exposition we base our hypotheses on the results of [CHT] which are not the most general known modularity lifting theorems that are known. It should be clear, however, how to modify our proofs in order to incorporate stronger modularity lifting results.

Hypothesis 9.1.1. Let F be a CM-field and let

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

be a continuous Galois representation which is unramified outside a finite set of places. Assume there is $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ such that

- $\ell > n$ and ℓ is unramified in F
- $\rho^c \cong \rho^\vee(1-n)$
- for every place $v|\ell$ of F the representation $\rho|_{G_{F_v}}$ is crystalline
- there exists $\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$ such that for all $\tau \in \mathrm{Hom}(F,\mathbb{C})$
  - either
  $$\ell - n - 1 \ge a_{\tau,1} \ge a_{\tau,2} \ge \cdots \ge a_{\tau,n} \ge 0$$
  or the inequalities hold with τ replaced by $\tau \circ c$
  - for all $1 \le i \le n$ one has $a_{\tau\circ c,i} = -a_{\tau,n+1-i}$
  - if $\iota^{-1}\tau \in \mathrm{Hom}(F,\overline{\mathbb{Q}}_\ell)$ gives rise to $v|\ell$ then
  $$\mathrm{HT}_\tau(\rho|_{G_{F_v}}) = \{-(a_{\tau,j} + n - i) \big| 1 \le i \le n\}$$
- $\overline{F}^{\mathrm{Ker}(\mathrm{ad}\overline{\rho})}$ does not contain $F(\zeta_\ell)$ where ζ_ℓ is a primitive ℓ'th root of unity
- the group $\overline{\rho}(\mathrm{Gal}(\overline{F}/F(\zeta_\ell)))$ is big as defined in [CHT, Def. 2.5.1]
- $\overline{\rho}$ is absolutely irreducible and $\overline{\rho} \cong \overline{\rho}_{\Pi_1,\ell,\iota}$ for some regular algebraic cuspidal conjugate self-dual automorphic representation Π_1 of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\overline{a}$

Then there is a regular algebraic cuspidal conjugate self-dual automorphic representation Π' of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\overline{a}$ such that

- $\rho \cong \rho_{\Pi',\ell,\iota}$
- Π'_v is spherical for all finite places $v \nmid \ell$ such that $\rho|_{G_{F_v}}$ is unramified and $\Pi_{1,v}$ is spherical

As in Chapter 4 we will now apply the modularity lifting hypothesis to the residual representations of the automorphic Galois representations for which we try to prove local-global compatibility results. Hence, guided by the previous hypothesis, we make the following definition:

Definition 23. Let F be a CM-field. For a regular algebraic cuspidal conjugate self-dual automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{C})}$ define $\mathcal{B}_\Pi$ to be the set of pairs (ℓ, ι) consisting of a rational prime ℓ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ such that

- $\ell > n$ and ℓ is unramified in F
- for all $\tau \in \mathrm{Hom}(F, \mathbb{C})$ either $\ell - n - 1 \geq a_{\tau,1} \geq a_{\tau,2} \geq \cdots \geq a_{\tau,n} \geq 0$ or the inequalities hold with τ replaced by $\tau \circ c$
- for every place $v|\ell$ of F the representation $\rho_{\Pi,\ell,\iota}|_{G_{F_v}}$ is crystalline
- $\overline{F}^{\mathrm{Ker}(\mathrm{ad}\overline{\rho}_{\Pi,\ell,\iota})}$ does not contain $F(\zeta_\ell)$ where ζ_ℓ is a primitive ℓ'th root of unity
- the group $\overline{\rho}_{\Pi,\ell,\iota}(\mathrm{Gal}(\overline{F}/F(\zeta_\ell)))$ is big as defined in [CHT, Def. 2.5.1]

We will later also prove results for automorphic representations of GL_n over totally real fields and hence we make the following definition:

Definition 24. Let F be a totally real field. For a regular algebraic cuspidal essentially self-dual automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight

$$\overline{a} \in (\mathbb{Z}^{n,+})^{\mathrm{Hom}(F,\mathbb{R})}$$

define $\mathcal{B}_\Pi$ to be the set of pairs (ℓ, ι) consisting of a rational prime ℓ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ such that

- $\ell > n$ and ℓ is unramified in F
- for all $\tau \in \mathrm{Hom}(F, \mathbb{R})$ one has $\ell - n - 1 + a_{\tau,n} \geq a_{\tau,1}$
- for every place $v|\ell$ of F the representation $\rho_{\Pi,\ell,\iota}|_{G_{F_v}}$ is crystalline
- $\overline{F}^{\mathrm{Ker}(\mathrm{ad}\overline{\rho}_{\Pi,\ell,\iota})}$ does not contain $F(\zeta_\ell)$ where ζ_ℓ is a primitive ℓ'th root of unity
- the group $\overline{\rho}_{\Pi,\ell,\iota}(\mathrm{Gal}(\overline{F}/F(\zeta_\ell)))$ is big as defined in [CHT, Def. 2.5.1]

We now prove the main theorem:

Theorem 7. *Let F be a* CM*-field and let Π be a regular algebraic cuspidal conjugate self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $(\ell, \iota) \in \mathcal{B}_\Pi$ and let $v \nmid \ell$ be a finite place of F such that Π_v is Iwahori-spherical. Then the monodromy operator of the Weil-Deligne representation associated to $\rho_{\Pi,\ell,\iota}|_{W_{F_v}}$ is non-trivial if and only if it is predicted to be non-trivial by the local Langlands correspondence.*

Proof. Let N be the monodromy operator of the Weil-Deligne representation associated to $\rho_{\Pi,\ell,\iota}|_{W_{F_v}}$. If the local Langlands correspondence predicts, in the above notation, that N is trivial then Π_v is a principal series representation and by 5.1.1 (i) it follows that N is indeed trivial. Hence suppose now that the local Langlands correspondence predicts that N is non-trivial and assume for contradiction that N is trivial. Let F'/F be a solvable extension such that F' is a CM-field and such that $\mathrm{BC}_{F'}(\Pi)$ is cuspidal and such that $\rho_{\Pi,\ell,\iota}|_{G_{F'}}$ is unramified at all places of F' above v. Note that F' can be chosen linearly disjoint over F to any given finite extension of F. Since $\ell > n$ it follows from Corollary 8.4 that there is a solvable CM-extension L/F' and a regular algebraic cuspidal conjugate self-dual automorphic representation Π_1 of $\mathrm{GL}_n(\mathbb{A}_L)$ of the same weight as $\mathrm{BC}_L(\Pi)$ such that

- $\overline{\rho}_{\mathrm{BC}_L(\Pi),\ell,\iota} \cong \overline{\rho}_{\Pi_1,\ell,\iota}$
- $\Pi_{1,w}$ is an unramified principal series for some place w above v
- Π_1 is spherical at all places above ℓ
- $(\ell, \iota) \in \mathcal{B}_{\Pi_1}$

By Hypothesis 9.1.1 there exists a regular algebraic cuspidal conjugate self-dual automorphic representation Π_2 of $\mathrm{GL}_n(\mathbb{A}_L)$ of the same weight as $\mathrm{BC}_L(\Pi)$ such that $\rho_{\mathrm{BC}_L(\Pi),\ell,\iota} \cong \rho_{\Pi_2,\ell,\iota}$ and such that $\Pi_{2,w}$ is spherical for some place $w|v$ of L. Since Π_2 and $\mathrm{BC}_L(\Pi)$ have the same ℓ-adic Galois representation and since the local-global compatibility is known by Hypothesis 5.1.1 for all but finitely many places it follows that $\Pi_{2,u} \cong \mathrm{BC}_L(\Pi)_u$ for all but finitely many places u of L. Hence by the strong multiplicity one theorem for cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_L)$, see for example [PS], it follows that $\Pi_{2,w} \cong \mathrm{BC}_L(\Pi)_w$ which is a contradiction: The representation $\Pi_{2,w}$ is spherical but $\mathrm{BC}_L(\Pi)_w$ is not since $\mathrm{rec}_{\mathrm{GL}_n}(\mathrm{BC}_L(\Pi)_w)$ has non-trivial monodromy operator. □

We will now deduce an analogous result for automorphic representations of general linear groups over totally real fields. As discussed before Corollary 6.2, it follows from the construction of Galois representations associated to regular algebraic cuspidal essentially self-dual automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ where F is a totally real field, that Proposition 7 implies the following:

Corollary 9.2. *Let F be a totally real field and let Π be a regular algebraic cuspidal essentially self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $(\ell,\iota) \in \mathcal{B}_\Pi$ and let $v \nmid \ell$ be a finite place of F such that Π_v is Iwahori-spherical. Then the monodromy operator of the Weil-Deligne representation associated to $\rho_{\Pi,\ell,\iota}|_{W_{F_v}}$ is non-trivial if and only if it is predicted to be non-trivial by the local Langlands correspondence.*

By the same argument used in the proof of Corollary 6.3 one can deduce the following result for Galois representations associated to Hilbert modular forms:

Corollary 9.3. *Let F be a totally real field and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of cohomological weight. Let $(\ell,\iota) \in \mathcal{B}_\pi$ and let $v \nmid \ell$ be a finite place of F such that π_v is Iwahori-spherical. Then the monodromy operator of the Weil-Deligne representation associated to $\rho_{\pi,\ell,\iota}|_{W_{F_v}}$ is non-trivial if and only if it is predicted to be non-trivial by the local Langlands correspondence.*

We will now deduce an analogous result for automorphic representations of symplectic groups over totally real fields.

Definition 25. Let π be a globally generic regular algebraic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ for some totally real field F. Let Π be the transfer of π to an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$ and if Π is cuspidal define

$$\mathcal{B}_\pi := \mathcal{B}_\Pi$$

Note that whenever we use the set $\mathcal{B}_\pi$ we assume implicitly that the transfer Π of π is cuspidal.

Corollary 9.4. *Let F be a totally real field and let π be cuspidal globally generic regular algebraic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Let $(\ell,\iota) \in \mathcal{B}_\pi$ and let $v \nmid \ell$ be a finite place of F such that π_v is Iwahori-spherical. Then the monodromy operator of the Weil-Deligne representation associated to $\rho_{\pi,\ell,\iota}|_{W_{F_v}}$ is non-trivial if and only if it is predicted to be non-trivial by the local Langlands correspondence.*

9.5 Applications

In this section we will strengthen the local-global compatibility results of Chapter 6 and Section 9.1 by using the interaction between the horizontal and vertical deformation theory of automorphic forms. As a consequence, in Corollary 9.10, we prove a version for globally generic automorphic representations of GSp_4 of Conjecture 3.1.7 in [SU]. Using similar methods as in the proof of this corollary, but using eigenvarieties for symplectic groups instead of unitary groups, leads to results concerning this conjecture even for non-globally generic representations.

Let us first use the results of Section 9.1 to strengthen the local-global compatibility results for local semi-simplifications of automorphic Galois representations that were obtained in Chapter 6.

Corollary 9.6. *Let F be a* CM*-field (totally real field) and let Π be a regular algebraic cuspidal conjugate (essentially) self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for $n \le 4$. Let $(\ell,\iota) \in \mathcal{B}_\Pi$ and let $v \nmid \ell$ be a finite place of F such that Π_v is Iwahori-spherical. Then*

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})^{\mathrm{ss}}$$

unless $n = 4$ and there is an unramified character $\chi : F_v^\times \to \mathbb{C}^\times$ such that one of the following holds:

- $\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \chi|\cdot|^{\pm 1/2} \boxplus \chi|\cdot|^{\pm 1/2}$ *(same sign)*
- $\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \mathrm{St}_2(\chi)$
- $\Pi_v \cong \mathrm{St}_3(\chi) \boxplus \chi|\cdot|^{\pm 1}$

Proof. Assume first F is a CM-field. By Theorem 5 it suffices to show that if $\Pi_v \cong \mathrm{St}_4(\chi)$ for some unramified character χ then $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ has non-trivial monodromy operator. This follows from Proposition 7. Now assume that F is a totally real field. Similarly to the proof in the CM-case the result follows from corollaries 6.2 and 9.2. ☐

For automorphic representations of symplectic groups over totally real fields one obtains:

Corollary 9.7. *Let F be a totally real field and let π be cuspidal globally generic regular algebraic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Let $(\ell,\iota) \in \mathcal{B}_\pi$ and let $v \nmid \ell$ be a finite place of F such that π_v is Iwahori-spherical. If π_v is not of type* (VIa) *then*

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-\frac{3}{2}})^{\mathrm{ss}}$$

Proof. Write $\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong (r, N)$. To prove the corollary it suffices by Corollary 6 to show that if π_v is of type (IVa) then N is non-trivial. This follows from Corollary 9.2. ☐

All of the above corollaries are examples of how the horizontal deformation theory of automorphic forms can strengthen the results for local semi-simplifications of automorphic Galois representations obtained by using the vertical deformation theory. In the reverse direction we will now give examples of how to use the vertical deformation theory to strengthen results obtained by using the horizontal deformation theory.

Corollary 9.8. *Let F be a* CM*-field (totally real field) and let Π be a regular algebraic cuspidal conjugate (essentially) self-dual automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for $n \leq 4$. Let $(\ell,\iota) \in \mathcal{B}_\Pi$ and let $v \nmid \ell$ be a finite place of F such that*

- *Π_v is Iwahori-spherical*
- *$\mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})$ has a monodromy operator of rank at most* 1

Then

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})$$

unless the following two conditions hold:

- *$n = 4$*
- *there is an unramified character $\chi : F_v^\times \to \mathbb{C}^\times$ such that*

$$\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \chi|\cdot|^{\pm 1/2} \boxplus \chi|\cdot|^{\pm 1/2}$$

  (same sign)

Proof. Suppose that $\mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})$ has a monodromy operator of rank at most 1. By corollary 9.6 it follows that

$$\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}_{\mathrm{GL}_n}(\Pi_v \otimes |\det|^{\frac{1-n}{2}})^{\mathrm{ss}}$$

unless $n = 4$ and there is an unramified character $\chi : F_v^\times \to \mathbb{C}^\times$ such that

$$\Pi_v \cong \mathrm{St}_2(\chi) \boxplus \chi|\cdot|^{\pm 1/2} \boxplus \chi|\cdot|^{\pm 1/2}$$

Moreover, by Proposition 7 the representation $\mathrm{WD}_\iota(\rho_{\Pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}}$ has a non-trivial monodromy operator if and only if it is predicted by the local Langlands correspondence and this proves the corollary. ☐

For Hilbert modular forms one obtains the following:

Corollary 9.9. *Let F be a totally real field and let π be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$ *of cohomological weight. Let $(\ell, \iota) \in \mathcal{B}_\pi$ and let $v \nmid \ell$ be a finite place of F such that π_v is Iwahori-spherical. Then*

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GL}_2}(\pi_v \otimes |\det|^{-\frac{1}{2}})$$

Proof. This follows from Corollary 6.3 and Corollary 9.3. □

We will now deduce a result for automorphic representations of symplectic groups over totally real fields.

The next corollary deals with type (Ia) and type (IIa) representations since those types of Iwahori-spherical representations are the only type of the six families of generic representations (Ia) to (VIa) whose local monodromy operator under the local Langlands correspondence has rank at most one: By [ROS, Table A.7] a type (Ia) representation has trivial monodromy operator and a type (IIa) representation has monodromy operator

$$\begin{bmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

and a type (IIIa) representation has monodromy operator

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}$$

and type (Va) and type (VIa) representations have monodromy operator

$$\begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

and a type (Iva) representation has monodromy operator

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}$$

Corollary 9.10. *Let F be a totally real field and let π be cuspidal globally generic regular algebraic automorphic representation of* $\mathrm{GSp}_4(\mathbb{A}_F)$. *Let $(\ell, \iota) \in \mathcal{B}_\pi$ and let $v \nmid \ell$ be a finite place of F such that π_v is Iwahori-spherical and of type* (Ia) *or* (IIa)*. Then*

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-\frac{3}{2}})$$

Proof. In case that π_v is of type (Ia) the result follows from Hypothesis 5.1.1 (i). Hence assume now that π_v is of type (IIa). By arguing as in the proof of Corollary 6, in order to prove the corollary it suffices to check that the corresponding result holds for regular algebraic cuspidal essentially self-dual automorphic representations Π of $\mathrm{GL}_4(\mathbb{A}_F)$ whose local component at v satisfies

$$\Pi_v \cong \mathrm{St}_2(\chi_1\chi_2) \boxplus \chi_1^2\chi_2 \boxplus \chi_2$$

for some unramified characters χ_1 and χ_2 of $F_v^\times$ such that $\chi_1^2 \notin \{|\cdot|^{\pm 1}, |\cdot|^{\pm 3}\}$. Since

$$\chi_1\chi_2|\cdot|^{\pm 1/2} \neq \chi_1^2\chi_2$$

this follows from Corollary 9.8. □

10 Local monodromy operators: The case of symplectic groups

In this chapter we develop a variant of the modularity lifting theorem approach to monodromy operators which is more general than the previously described results in Chapter 9. This variant describes how to obtain general lower bounds on the rank of monodromy and it also applies to automorphic representations of reductive groups where strong multiplicity one results might not be known or might not hold. Note that we develop this approach in the current chapter in the setting of GSp_4, but the method allows to treat higher rank symplectic groups, amongst other cases, as well.

The traditional approach to studying the monodromy operator is based on the study of singularities of algebraic varieties. As mentioned before, the approach we take is very different. The idea is to combine a priori congruences of automorphic representations, such as potential level-lowering results, and combine them with modularity lifting theorems. An obstacle for this strategy for general groups is that strong multiplicity one results might not be available. However, strong multiplicity one is a much stronger result than what is needed and we will in this chapter give an alternative approach to the required rigidity via γ-factors obtained from the doubling method. The latter theory has been developed for general classical groups but for simplicity we focus on automorphic representations of GSp_4 over totally real fields. The main result is Theorem 8 where we show under some strong hypotheses that the rank of monodromy is at least as predicted by the local Langlands correspondence. The hypotheses are strong since, for a result of the above generality, they assume level-lowering principles beyond what has been proved in the potential level-lowering section of the previous chapter and we do not know how to obtain such general congruence results without assuming local-global compatibility in the first place. Similar results could presumably be shown, for example, in the GL_n-case but as mentioned before we restrict to GSp_4 for simplicity. One should note that a much more intricate use of the γ-factors from the doubling methods was used by Jorza in [JOR] to obtain multiplicity one results for automorphic representations of GSp_4 and as a consequence monodromy results are deduce from the known GL_4-cases. Interestingly enough, the eigenvariety arguments that we developed in earlier sections were used there to remove a certain quadratic twist in the results.

10.1 γ-factor arguments

The fine local information of the γ-factors from the doubling method has been proved by Lapid and Rallis in [LR] and we now recall this briefly in the special case of symplectic groups.

Let F be a totally real field. Fix a non-trivial character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}_F$. For a cuspidal automorphic representation π of $\mathrm{Sp}_4(\mathbb{A}_F)$ one obtains via the doubling-method a γ-factor $\gamma(s, \pi, \psi)$. One defines the L-factor $L(s, \pi_v, \psi_v)$ as the numerator of $\gamma(s, \pi, \psi)$ if π is tempered and other wise via the Langlands classification. The relation between the L-factors, ϵ-factors and γ-factors, as defined in [LR], is given as

$$\gamma(s, \pi \times \omega, \psi) = \epsilon(s, \pi \times \omega)\frac{L(1-s, \tilde{\pi} \times \omega^{-1}, \psi)}{L(s, \pi \times \omega, \psi)}$$

where $\tilde{\pi}$ is the contragredient representation of π. By [GT2] (Main theorem, property (i)) the γ-factors obtained by the doubling method agree with the γ-factors of the local Langlands correspondence for Sp_4 as constructed by Gan-Takeda in [GT2]. Since the γ-factors as above are defined for representations for Sp_4, but we are interested in representations of GSp_4, let us recall the relation between the local Langlands correspondences of these two groups. Consider

$$\mathrm{std} : \mathrm{GSp}_4 \longrightarrow \mathrm{PGSp}_4 \cong \mathrm{SO}_5$$

where the last isomorphism is as described for example in [ROS, A.7]. Let π_v be an irreducible smooth admissible representation of $\mathrm{GSp}_4(F_v)$. Let π'_v be some irreducible component of the restriction of π_v to $\mathrm{Sp}_4(F_v)$. Then by [GT2] (Section 1) one has

$$\mathrm{std} \circ \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v) = \mathrm{rec}_{\mathrm{Sp}_4}(\pi'_v)$$

The key properties of the γ-factors for our intended application are:

- Functional equation:

Suppose $\pi \cong \otimes'_v \pi_v$ is an automorphic representation of $\mathrm{Sp}_4(\mathbb{A}_F)$. Let S be a finite set of finite places of F and define

$$L^{(S)}(s, \pi, \psi) := \prod_{v \notin S} L(s, \pi_v, \psi_v)$$

where the product is over finite places of F not in S. Then

$$L^{(S)}(s, \pi, \psi) = \prod_{v \in S} \gamma(s, \pi_v, \psi_v) L^{(S)}(1 - s, \pi_v^\vee, \psi_v)$$

This is [LR, Theorem 4, property 10]. Hence, if π and $\tilde{\pi}$ are two automorphic representations of $\mathrm{Sp}_4(\mathbb{A}_F)$ such that $\pi_v \cong \tilde{\pi}_v$ for all $v \notin S$ where S contains all the archimedean places and all places where π or $\tilde{\pi}$ are ramified then

$$\prod_{v \in S} \gamma(s, \pi_v \times \omega_v, \psi_v) = \prod_{v \in S} \gamma(s, \tilde{\pi}_v \times \omega_v, \psi_v)$$

where $\omega = \prod \omega_v$ is a Hecke character of $F^\times \backslash \mathbb{A}_F^\times$.

- Stability:

The stability of local γ-factors arising from the doubling method has been proved by Rallis-Soudry in [RS]. It implies in our current situation that given π_v there exists a character χ of $F_v^\times$ such that

$$\gamma(s, \pi_v \otimes \chi, \psi_v) = 1$$

10.1.1 Local Langlands correspondence for GSp_4

For later use we describe how the construction of the local Langlands correspondence for GSp_4 for p-adic fields by Gan-Takeda is compatible with the explicit constructions of [ROS] for Iwahori spherical representations. As explained in [GT, Section 7], via the explicit description of certain theta-correspondences in [GT3] it can be seen that the two constructions agree. Since in [GT] this matching is mentioned explicitly only for nondiscrete series representations we recall here the matching also for essentially discrete Iwahori spherical representations. As described in [ROS, Table A.1] these are exactly the type (IVa) and (Va) representations.

The local Langlands correspondence of [GT] is constructed via the theta-correspondence. Let $\mathrm{GSO}_{3,3}$ denote the algebraic group over $\mathbb{Q}$ given by the orthogonal simulate group of a 6-dimensional quadratic form of signature $(3, 3)$ and note that, say for a p-adic field K, there is an isomorphism

$$\mathrm{GSO}_{3,3}(K) \cong (\mathrm{GL}_4(K) \times \mathrm{GL}_1(K)) / \{(z, z^{-2}) \big| z \in K^\times\}$$

where we view $K^\times$ embedded into the diagonal torus of $\mathrm{GL}_4(K)$. Via this description, one writes representations of $\mathrm{GSO}_{3,3}(K)$ as $\Pi \boxtimes \mu$ where Π is a representation of $\mathrm{GL}_4(K)$ and μ a representation of $\mathrm{GL}_1(K)$. As described in [GT], the L-group of $\mathrm{GSO}_{3,3}$ is

$$\{(g, z) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \big| \det g = z^2\}$$

The L-group of GSp_4 should be $\mathrm{GSp}_4(\mathbb{C})$. Consider the map *inc* from the L-group of GSp_4 to the L-group of $\mathrm{GSO}(V)$ given by

$$g \to (g, c(g))$$

where $c(g)$ is the similitude factor. In [GT, Theorem 5.2] an irreducible admissible representation π of $\mathrm{GSp}_4(K)$ for a p-adic field K is defined to be of type (B) if π participates in the theta-correspondence with $\mathrm{GSO}_{3,3}(K)$. By [GT, Theorem 5.6] an irreducible admissible representation of $\mathrm{GSp}_4(K)$ is of type (B) if and only if it is not a non-generic essentially tempered representation of $\mathrm{GSp}_4(K)$. For an irreducible admissible representation of type (B) the Langlands parameter is constructed in the following manner. The theta lift of π to $\mathrm{GSO}_{3,3}(K)$ will be non-zero, say of the form $\Pi \boxtimes \mu$. Let ϕ_Π and ϕ_μ be the Langlands parameters of Π and μ under a local Langlands correspondence for GL_4 and GL_1. Then the parameter of π is defined to be

$$\phi_\Pi \times \phi_\mu : W_K \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

By [GT, Section 7] this map factors through the image of $\mathrm{GSp}_4(\mathbb{C})$ under the inclusion map *inc* and hence one obtains a symplectic Langlands parameter. Note that, as explained for example in [KUD, Section 4], Weil-Deligne representations over $\mathbb{C}$ can also be described via the continuous complex semi-simple representations of $W_K \times \mathrm{SL}_2(\mathbb{C})$.

Let K be a p-adic field. If follows from the discussion in [GT3, Section 5.2] that the Iwahori-spherical discrete series representations of $\mathrm{GSp}_4(K)$ are of the following form. As mentioned in [GT3, Section 5.2.1], all of these representations are in fact generic. Let P denote a Siegel parabolic of $\mathrm{GSp}_4(K)$. The Levi-subgroup is isomorphic to $\mathrm{GL}_2(K)\times\mathrm{GL}_1(K)$ via an isomorphism which takes $(A,u) \in \mathrm{GL}_2(K) \times \mathrm{GL}_1(K)$ to

$$\begin{pmatrix} A & 0 \\ 0 & uA' \end{pmatrix} \in \mathrm{GSp}_4(K)$$

where

$$A' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A^{-1})^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Via this isomorphism, if τ is a representation of $\mathrm{GL}_2(K)$ and μ of $\mathrm{GL}_1(K)$ then we can form the normalized induction $I_P(\tau,\chi)$. We now describe the type (IVa) and type (Va) representations:

- Let μ be a character of $K^\times$. The representation

  $$I_P(st|\cdot|^{3/2}, \mu|\cdot|^{-3/2})$$

  has by [GT3, Lemma 5.2(b)(iii)] a unique irreducible sub-representation which will be denoted by $\mathrm{St}_{\mathrm{PGsp}_4} \otimes \mu$. By [ROS, A.3] this is a type (IVa) representation.

- Let μ be a character of $K^\times$ and ξ_0 is a non-trivial quadratic character of $K^\times$. Then the normalized induction

  $$I_P(\mathrm{St}_2(\xi_0|\cdot|^{1/2}), \mu|\cdot|^{-1/2})$$

  has by [GT3, Lemma 5.2(b)(ii)] a unique irreducible sub-representation which will be denoted by $\mathrm{St}(st_{\xi_0}, \mu)$. By [ROS, A.3] this is a type (Va) representation.

Let θ denote the theta-correspondence from $\mathrm{GSp}_4(K)$ to $\mathrm{GSO}_{3,3}(K)$ and let $P_{2,2}$ denote the standard parabolic subgroup of $\mathrm{GL}_4(K)$ corresponding to the partition $4 = 2+2$. By [GT3, Theorem 8.3(iv)] one has

$$\theta(\mathrm{St}(st_{\xi_0}, \mu)) = I_{P_{2,2}}(st_{\xi_0} \otimes \mu, st \otimes \mu) \boxtimes \mu^2$$

In [ROS, Table A.7] the Langlands parameter associated to a type (IVa) representation is given by (ρ, N) where for $w \in W_K$ one has

$$\rho(w) = \begin{bmatrix} \mu(w)|w|^{1/2} & & & \\ & \mu(w)|w|^{1/2}\xi_0(w) & & \\ & & \mu(w)|w|^{-1/2}\xi_0(w) & \\ & & & \mu(w)|w|^{-1/2} \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Hence the degree 4 Langlands parameters of [GT] and [ROS] agree and hence, in particular, the degree 5 L-factors agree. By [GT3, Theorem 8.3(v)] one has

$$\theta(\mathrm{St}_{\mathrm{PGSp}_4} \otimes \mu) = (\mathrm{St}_{\mathrm{PGL}_4} \otimes \chi) \boxtimes \chi^2$$

In [ROS, Table A.7] the Langlands parameter associated to a type (Va) representations is given by (ρ, N) where for $w \in W_K$ one has

$$\rho(w) = \begin{bmatrix} \mu(w)|w|^{3/2} & & & \\ & \mu(w)|w|^{1/2} & & \\ & & \mu(w)|w|^{-1/2} & \\ & & & \mu(w)|w|^{-3/2} \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{bmatrix}$$

Hence the degree 4 Langlands parameters of [GT] and [ROS] agree and hence the degree 5 L-factors agree.

10.1.2 Rigidity

Let K be a p-adic field and let $N(K)$ be the unipotent radical of the upper-triangular Borel subgroup of $\mathrm{GSp}_4(K)$. For a non-trivial complex character ψ of K define the character, denoted by ψ', of $N(K)$ given by

$$\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \longrightarrow \psi(x+y)$$

Recall that an irreducible admissible representation π of $\mathrm{GSp}_4(K)$ is called generic if

$$\mathrm{Hom}_{N(K)}(\pi, \psi') \neq (0)$$

We can now prove the crucial rigidity lemma:

Lemma 10.2. *Let F be a totally real field. Suppose π_1 and π_2 are cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that $\pi_w \cong \tilde{\pi}_w$ for almost all finite places w of F. Let v be any finite place of F. Then it is not possible that π_v is generic Iwahori-spherical and unramified and $\tilde{\pi}_v$ is generic Iwahori-spherical and ramified.*

Proof. Suppose that π_1 and π_2 are automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that $\pi_w \cong \tilde{\pi}_w$ for almost all finite places w of F. For $i = 1, 2$ the representation π_i can be realized in space of functions on $\mathrm{GSp}_4(\mathbb{A}_F)$. Restricting all of these functions to $\mathrm{Sp}_4(\mathbb{A}_F)$ one obtains a representation of $\mathrm{Sp}_4(\mathbb{A}_F)$. Let π_i' be an irreducible component of this representations. It is a cuspidal automorphic representation of $\mathrm{Sp}_4(\mathbb{A}_F)$. Let w be a finite place of F. Since

$$\mathrm{std} \circ \mathrm{rec}_{\mathrm{GSp}_4}(\pi_w) = \mathrm{rec}_{\mathrm{Sp}_4}(\pi_w')$$

it follows that

$$\mathrm{rec}(\pi_{1,w}') \cong \mathrm{rec}(\pi_{2,w}')$$

for all finite places w of F outside a finite set S of finite places. Hence it follows from the functional equation that

$$\prod_{w \in S} \gamma(s, \pi_w, \psi_w) = \prod_{w \in S} \gamma(s, \pi'_w, \psi_w)$$

Fix a finite place v of F. By stability of γ-factors it follows that

$$\gamma(s, \pi'_{1,v}, \psi_v) = \gamma(s, \pi'_{2,v}, \psi_v)$$

Since, as mentioned before, the γ-factors from the doubling method agree with the γ-factors of the local Langlands correspondence for Sp_4 and since this local Langlands correspondence preserves γ-factors, it follows that

$$\gamma(s, \mathrm{rec}_{\mathrm{Sp}_4}(\pi'_{1,v}), \psi_v) = \gamma(s, \mathrm{rec}_{\mathrm{Sp}_4}(\pi'_{2,v}), \psi_v)$$

By the relate between the local Langlands correspondences for Sp_4 and GSp_4 it should follow that

$$\gamma(s, \mathrm{std} \circ \mathrm{rec}_{\mathrm{GSp}_4}(\pi_{1,v}), \psi_v) = \gamma(s, \mathrm{std} \circ \mathrm{rec}_{\mathrm{GSp}_4}(\pi_{2,v}), \psi_v)$$

The degree five L-factors of generic Iwahori-spherical representations of GSp_4 are listen in table A.10 of [ROS] and by the discussion preceding the lemma, these L-factors agree with the one defined in [GT].

- type (Ia):
$$L(s, \chi_1) L(s, \chi_1^{-1}) L(s, 1_{F^\times}) L(s, \chi_2) L(s, \chi_2^{-1})$$
- type (IIa):
$$L(s, \chi |\cdot|^{1/2}) L(s, \chi^{-1} |\cdot|^{1/2}) L(s, 1_{F^\times})$$
- type (IIIa):
$$L(s, \chi) L(s, \chi^{-1}) L(s, |\cdot|)$$
- type (IVa):
$$L(s, |\cdot|^2)$$
- type (Va):
$$L(s, \xi |\cdot|) L(s, \xi) L(s, 1_{F^\times})$$
- type (VIa):
$$L(s, |\cdot|) L(s, 1_{F^\times}) L(s, 1_{F^\times})$$

One sees that the only generic Iwahori-spherical representation of $\mathrm{GSp}_4(F_v)$ whose L-factor has degree 5 is the unramified irreducible principal series representation. But this contradicts

$$\begin{array}{ccc}
\gamma(s, \mathrm{std} \circ \mathrm{rec}_{\mathrm{GSp}_4}(\pi_{1,v}), \psi_v) & = & \gamma(s, \mathrm{std} \circ \mathrm{rec}_{\mathrm{GSp}_4}(\pi_{2,v}), \psi_v) \\
\| & & \| \\
\epsilon(s, \pi_{1,v}, \psi_v) \frac{L(1-s, \tilde{\pi}_{1,v}, \psi_v)}{L(s, \pi_{1,v}, \psi_v)} & & \epsilon(s, \pi_{2,v}, \psi_v) \frac{L(1-s, \tilde{\pi}_{2,v}, \psi_v)}{L(s, \pi_{2,v}, \psi_v)}
\end{array}$$

Hence the γ-factor distinguishes between ramified and unramified generic Iwahori-spherical representations. □

In fact this argument can be strengthened: The following is a special case of Proposition 6.2 of [JOR]. It will be used when we prove stronger lower bounds on the rank of monodromy than the mere non-triviality.

Lemma 10.3 (Jorza)**.** *Let F be a totally real field. Suppose now that π_1 and π_2 are cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that $\pi_w \cong \tilde{\pi}_w$ for almost all finite places w of F. Let v be any finite place of F. Then* *it is not possible that π_v is generic Iwahori-spherical with $\operatorname{rec}(\pi_v)$ having monodromy of rank r_1 and $\tilde{\pi}_v$ is generic Iwahori-spherical with $\operatorname{rec}(\tilde{\pi}_v)$ having monodromy of rank r_2 with $r_1 \neq r_2$.*

10.4 Monodromy Operators

With the rigidity results of the previous section in hand it is now relatively straightforward to adapt the methods of the proof of Theorem 7 to the symplectic case. It should become clear how to, in principle, generalize the arguments to automorphic representations on other groups as well. We also illustrate how one might prove stronger lower bounds on the rank of monodromy than the mere non-triviality. The following is a generally expected hypothesis, see for example the discussion in [SOR], and we will assume it for the remainder of this chapter:

Hypothesis 10.4.1. Let F be a totally real field and let π be a cuspidal regular algebraic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Then for a rational prime ℓ and isomorphism ι between $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ there is a continuous semi-simple ℓ-adic Galois representation

$$\rho_{\pi,\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$$

such that for all finite places $v \nmid \ell$ of F such that π_v is a principal series representation one has

$$\mathrm{WD}_\iota(\rho_{\pi,\ell,\iota}|_{W_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}_{\mathrm{GSp}_4}(\pi_v \otimes |c|^{-\frac{3}{2}})$$

where c denotes the symplectic similitude character. Moreover, if π is non-endoscopic and non-CAP then π_v is tempered for all finite places v.

Let F_v be a finite extension of $\mathbb{Q}_p$ where p is some prime. Let $\mathcal{O}$ denote the integer ring in F_v and $\mathfrak{p}$ the maximal ideal. We now define, following [ROS, Sect. 2.1], some subgroups of $\mathrm{GSp}_4(F_v)$: The Klingen parahoric J_Q is defined to be the subgroup of $\mathrm{GSp}_4(F_v)$ consisting of matrices A with determinant in $\mathcal{O}^\times$ and

$$A \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{bmatrix}$$

Fix a uniformizer ϖ of F_v. The para-modular group $\tilde{K}$ is defined to be the subgroup of $\mathrm{GSp}_4(F_v)$ given by

$$\tilde{K} = \langle J_Q, \eta J_Q \eta^{-1} \rangle$$

where

$$\eta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \varpi & 0 & 0 & 0 \\ 0 & -\varpi & 0 & 0 \end{bmatrix}$$

Definition 26. If V is a Weil-Deligne representation we denote by N_V the corresponding monodromy operator.

Definition 27. Fix an integer $0 \leq i \leq 3$. Let F_v be a finite extension of $\mathbb{Q}_p$ for some prime p. We say that a pair (K, ϕ) is associated to i over F_v if K is a compact open subgroup $K \leq \mathrm{GSp}_4(F_v)$ and a ϕ is a finite dimensional smooth representation K such that for any irreducible admissible generic Iwahori-spherical representation π_v of $\mathrm{GSp}_4(F_v)$ one has

$$\pi_v^{K,\phi} \neq (0) \Longrightarrow \operatorname{rank} N_{\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)} \leq i$$

Here $\pi_v^{K,\phi}$ denotes the subspace of π_v on which K acts through ϕ.

Remark. Note that there are groups satisfying the properties outlined in definition 27: Fix an integer $0 \leq i \leq 3$ and let F_v be a finite extension of $\mathbb{Q}_p$ for some prime p. Then there exists a pair (K, ϕ) which is associated to i over F_v:

As can be seen for example from the tables in [SOR], one has the following implications:

$$\pi_v^{\mathrm{GSp}_4(\mathcal{O}),1} \neq (0) \Longrightarrow \text{rank } N_{\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)} \leq 0$$

Let $\tilde{K}$ be the para-modular group. Then

$$\pi_v^{K,1} \neq (0) \Longrightarrow \text{rank } N_{\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)} \leq 1$$

Let J_P be a Klingen parahoric subgroup. Then

$$\pi_v^{J_P,1} \neq (0) \Longrightarrow \text{rank } N_{\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)} \leq 2$$

Let I be an Iwahori subgroup. Then

$$\pi_v^{I,1} \neq (0) \Longrightarrow \text{rank } N_{\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)} \leq 3$$

To prove lower bounds on the rank of monodromy operators we will assume strong cyclic base change for GSp_4:

Hypothesis 10.4.2. Let K/F be a cyclic extension of totally real number fields and let π be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Then there exists an automorphic representation $\mathrm{BC}_{K/F}(\pi)$ of $\mathrm{GSp}_4(\mathbb{A}_K)$ such that for all places v of F and places $w|v$ the representation $\mathrm{rec}_{\mathrm{GSp}_4}(\mathrm{BC}_{K/F}(\pi)_w)$ of the Weil-Deligne representation of W_{K_w} is obtained by restriction from the Weil-Deligne representation $\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)$ of W_{F_v}.

The following hypothesis about the existence of congruences between automorphic representations is stronger than the known potential level-lowering results. It will be assumed when proving lower bounds on the rank of monodromy operators which go beyond the non-triviality of monodromy treated in earlier chapter.

Hypothesis 10.4.3. Let F be a totally real field and let π be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Fix a rational prime ℓ and finite place $v \nmid \ell$ of F such that π_v is generic. Assume $\overline{\rho}_{\pi,\ell,\iota}$ is absolutely irreducible. Then there exists a solvable totally real extension F'/F, a place $w|v$ of F' and a cuspidal cohomological automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A}_{F'})$ of the same weight as $\mathrm{BC}_{F'}(\pi)$ such that

- $\overline{\rho}_{\pi',\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}|_{\mathrm{Gal}(\overline{F}/F')}$
- π'_w is generic
- $$(\pi'_w)^{K,\phi} \neq (0)$$
  where (K,ϕ) is associated over F'_w to the rank of $N_{\mathrm{WD}(\rho_{\pi,\ell,\iota}|_{G_{F_v}})}$

We phrase our requirements on a modularity lifting theorem for GSp_4 as a hypothesis. See for example [GET] for some known results. Certainly one can assume weaker modularity lifting results, for example with stronger conditions on the image of the residual Galois representation. The proof of Theorem 8under such a suitably modified hypothesis will essentially be the same.

Hypothesis 10.4.4. Let

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$$

be a continuous representation unramified outside a finite set of finite places and such that there exists a cuspidal regular algebraic automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that

- there is an isomorphism
  $$\overline{\rho}_{\pi,\ell,\iota} \cong \overline{\rho}$$
  and these representations are absolutely irreducible

- π_v is generic and $\pi_v^{K,\phi} \neq (0)$ where (K,ϕ) is associated to

$$i := \text{rank } N_{\mathrm{WD}(\rho|_{G_{F_v}})}$$

over F_v and is as in Hypothesis 10.4.3

Then there exists a cuspidal regular algebraic automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_4(\mathbb{A}_F)$ of the same weight as π and such that

- there is an isomorphism $\rho \cong \rho_{\tilde{\pi},\ell,\iota}$
- $\tilde{\pi}_v$ is generic and $\tilde{\pi}_v^{K,\phi} \neq (0)$

We can now prove the main theorem on lower bounds for the rank of monodromy operators of Galois representations associated to symplectic automorphic representations:

Theorem 8. *Let F be a totally real field and let π be an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ which is cuspidal regular algebraic. Let ℓ be a rational prime and $v \nmid \ell$ a finite place of F such that π_v is generic. Let N_ℓ be the monodromy operator of the Weil-Deligne representation associated to $\rho_{\pi,\ell,\iota}|_{W_{F_v}}$ and let N_{rec} denote the monodromy operator of $\mathrm{rec}(\pi_v)$. Assuming Hypotheses 10.4.2, 10.4.3, and 10.4.4 with $\rho = \rho_{\pi,\ell,\iota}$ it follows that*

$$\text{rank } N_\ell \geq \text{rank } N_{\mathrm{rec}}$$

Proof. Suppose for contradiction that rank $N_\ell <$ rank N_{rec}. After making a suitable solvable base change one can assume that π_v is Iwahori-spherical. By Hypothesis 10.4.3 there exists a solvable totally real extension F'/F and a cuspidal cohomological automorphic representation π' of $\mathrm{GSp}_4(\mathbb{A}_{F'})$ such that $\overline{\rho}_{\pi',\ell,\iota} \cong \overline{\rho}_{\pi,\ell,\iota}|_{\mathrm{Gal}(\overline{F}/F')}$ and

$$(\pi'_v)^{K,\phi} \neq (0)$$

where (K,ϕ) is associated to the rank of $N_{\mathrm{WD}(\rho_{\pi,\ell,\iota}|_{W_{F_v}})}$ over F'_w. By Hypothesis 10.4.4 it follows that

$$\rho_{\pi,\ell,\iota}|_{\mathrm{Gal}(\overline{F}/F')} \cong \rho_{\tilde{\pi},\ell,\iota}$$

for some cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that

$$\tilde{\pi}_v^{K,\phi} \neq (0)$$

It therefore follows that $\mathrm{rec}_{\mathrm{GSp}_4}(\mathrm{BC}_{F'/F}(\pi)_w) \cong \mathrm{rec}_{\mathrm{GSp}_4}(\tilde{\pi}_w)$ for almost all finite places w of F' and

$$\text{rank } N_{\mathrm{rec}(\tilde{\pi}_v)} \leq \text{rank } N_{\mathrm{WD}(\rho_{\pi,\ell,\iota}|_{W_{F_v}})} < \text{rank } N_{\mathrm{rec}}$$

Since for all finite places w of F' which lie above v the rank of the monodromy operator of $\mathrm{rec}_{\mathrm{GSp}_4}(\mathrm{BC}_{F'/F}(\pi_v)_w)$ equals the rank of N_{rec} one obtains from Lemma 10.3 a contradiction. □

Since potential level-lowering results are known, one can deduce a less conditional result for the mere non-triviality of monodromy operators.

Corollary 10.5. *Let F be a totally real field and let π be a cuspidal regular algebraic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$. Let $\ell > 3$ be a rational prime and assume that $\overline{\rho}_{\pi,\ell,\iota}$ is absolutely irreducible. Let v be a finite place of F such that $v \nmid \ell$ and such that π_v is generic. Let N_ℓ be the monodromy operator of the Weil-Deligne representation associated to $\rho_{\pi,\ell,\iota}|_{W_{F_v}}$ and let N_{rec} denote the monodromy operator of $\mathrm{rec}_{\mathrm{GSp}_4}(\pi_v)$. Assuming Hypothesis 10.4.2 and 10.4.4 it follows that $N_\ell \neq 0$ whenever $N_{\mathrm{rec}} \neq 0$.*

Proof. Assume $N_{\mathrm{rec}} \neq 0$ and assume for contradiction that $N_\ell = 0$. Hypothesis 10.4.3 in this case is then implied by the potential level-lowering for symplectic groups which have been proven by Sorensen assuming strong cyclic base change, see [SOR2, Section 1], assuming that $\ell > 3$ and that $\overline{\rho}_{\pi,\ell,\iota}$ is absolutely irreducible. Note that in contrast to [SOR2] we added the temperedness assumption for all finite places for non-endoscopic and non-CAP representations in Hypothesis 10.4.1 since otherwise the arguments of [SOR2] need more justification. The corollary now follows from Theorem 8. □

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