KP Equations, Strings, and the Schottky Problem

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... but it gives the greatest satisfaction
to view the spirit of another age,
to see how wise men thought before our days,
and to rejoice how far we've come at last.

0. Historical Origin

The key point of the relationship between the KP theory and the characterization of jacobians of algebraic curves is the fact that the set A consisting of linear ordinary differential operators which commute with a given ordinary differential operator is itself a commutative algebra of transcendence degree 1 over the ground field. To prove the commutativity of A, we have to introduce fractional powers of differential operators, which are pseudo-differential operators, and a proof based on this idea was given by Gelfand–Dikii [GD] in 1975. Then Krichever [K] studied extensively the algebraic structure of commuting ordinary differential operators, and

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obtained various exact solutions of the KP equation. When Mumford gave a lecture at UCLA about 10 years ago and mentioned the work of Krichever, Coddington pointed out that the essential part of the work had been done by Burchnall and Chaundy as early as in 1922 [BC] (not 1922 B.C. of course!), way before the study of the Russian school. Thus algebraic geometry of commuting differential operators has such a long history, although the early works have been forgotten for more than 50 years.

Krichever's work gives a method of constructing solutions of the KP equations from algebraic curves. But if we want to show, on the contrary, that every solution of the KP equations must come from an algebraic curve, then we need to prove the commutativity of the commutant A of an ordinary differential operator. For a long time, those who worked in these subjects believed that the proof was due to Gelfand-Dikii.

In the fall of 1987, I had a chance to offer a graduate course of about 30 lectures in UCLA on supersymmetry and the KP theory. One day, when I mentioned the theorem of Gelfand-Dikii, to my greatest surprise, R. Steinberg immediately commented, "*It was done by Issai Schur.*" I could not believe it, because I never knew Schur's work before. But actually it was! (I would like to thank Steinberg for drawing my attention to it.) Next day, Varadarajan told me that in his paper [Schu] which appeared in 1905, *Schur proved the commutativity of A by using fractional powers of differential operators!* Schur uses pseudo-differential operators in one variable completely freely. He took the method of pseudo-differential calculus from earlier work of S. Pincherle, Mémoire sur le calcul fonctionnel distributif, *Mathematische Annalen* **49**, 325-382 (1897), especially from its Chapitre IV. At least formal algebraic manipulation of pseudo-differential operators in one variable must have been quite familiar at the end of the 19th century. Pincherle's work could be thought of as one of the origins of algebraic analysis.

Since the fractional power Schur used is essentially a local uniformizing parameter of the curve defined as the projective scheme of the associated graded algebra of A, we could say that *essentially he had the curve too!* Thus the key point of the relationship between the KP theory and the algebraic curves was established in 1905 by Schur. Burchnall-Chaundy was not the starting point. Schur really started. Well, maybe not. Because Schur took the problem of commuting ordinary differential operators from the work of Wallenberg (Über die Vertauschbarkeit homogener linearer Differentialausdrücke, *Archiv der Mathematik und Physik*, Dritte Reihe, Band 4, 252–268) which appeared in 1903. Wallenberg studied the conditions for two linear ordinary differential operators to commute, stated the problem in general and studied some of the special cases such as a pair of order-2 operators, a pair of first-order and n-th order operators, and so on. He also mentioned that the case of two first-order operators had been worked out by Floquet in 1879. Therefore our theory has really a long history—more than 84 years! And Schur's work has been ignored for a long time.

I have experienced a very exciting moment. The sealed book of the past was suddenly opened in front of me, and the great works hidden in the history resurrected from the dead. I looked into them and realized that everything was there.

Of course there is another root in the theory of the KP equations. It is the study of soliton equations which goes back to J. Boussinesq (1872) and D. J. Korteweg-G. deVries (1895). Surprisingly, here again their works were forgotten for more than 50 years!

And the theory, deeply rooted in the history of mathematics, solved a problem which has another long history in mathematics—the *Schottky problem* of finding a characterization of jacobian varieties (Mulase [M1], 1983). It is also used by Shiota [Sh] (1986) in a remarkable way to solve a rather new conjecture of Novikov. And in 1987, it is realized that the theory is very closely related to the brand-new theory of theoretical physics, string theory!

The discovery of Schur's work [Schu] is very impressive to me because the modern theory of the KP equations has its origin in Sato's discovery (1981) of the fact that the *Schur polynomials solve the KP system.* By this theorem Sato was able to analyze the structure of the solution space of the KP system [Sa], and then the Kyoto school, based on Sato's work, discovered (1982) a totally unexpected relation between soliton theory and Kac–Moody algebras [DJKM]. Thus, together with his commutativity theorem mentioned above, Schur's contribution in the theory of the KP equations is really enormous.

In this article, I would like to explain the KP theory, which has many different roots in the long history of mathematics, solved a problem with another long history, and is now giving new dimensions in both mathematics and physics.

I would like to thank all of the audiences of my lectures, especially V. S. Varadarajan, for their great enthusiasm and many stimulating *interruptions!* I also thank K. T. Kim and C. Phillips for their valuable suggestions.

1. Motivations

Let me begin with the following question: *What is the most fundamental difference between classical mechanics and quantum mechanics?* One may imagine non-commutativity of physical quantities, Heisenberg's uncertainty principle, discreteness of energy, or perhaps wave-particle duality. But what was the most fundamental transition from classical physics to quantum physics? In his famous book *The Principles of Quantum Mechanics*, Dirac answered this question and said that it was the *principle of superposition*. In classical mechanics, two different states of motions never mix. But in quantum mechanics, an electron, say, can pass through two different holes on a screen to make a diffraction pattern. Namely, an actual motion in quantum mechanics is a linear combination of many different states of motion. The principle of superposition forces us to introduce the notion of Hilbert spaces of states. Physical quantities are then identified with operators acting on the Hilbert space and hence their non-commutativity follows automatically. The uncertainty principle of Heisenberg and discreteness of some physical quantities are consequences of the non-commutativity.

The principle of superposition, used by Dirac to illuminate beautifully the fundamental transition of 1925, was investigated further by Feynman. According to Feynman, a quantum path can be computed by a linear combination of all classical paths with something like Boltzmann's factors in coefficients.

What does the principle of superposition tell us when we go to string theory? In the classical picture, an orbit of interacting strings is a Riemann surface embedded in space-time together with a metric on it, which is conformally equivalent to the induced metric from space-time. Therefore, to compute quantum effects such as the vacuum–vacuum transition, we have to consider superpositions (i.e., linear combinations) of complete algebraic curves of all different genera. Physicists seem to believe that string theory is the only possible quantum theory of gravity. Therefore, the orbits of strings must not be dealt with as embedded Riemann surfaces in space-time but as abstract algebraic curves, because space-time itself must appear as a classical limit of the solution of the theory. Namely, the principle of superposition in string theory demands a mathematical framework in which we can deal with all algebraic curves of all different genera at one time as well as their infinite linear combinations. Is there any such mathematical theory? If there is one, then it must be necessarily an infinite-dimensional geometry. Therefore the usual algebraic geometry cannot provide such a framework, because by definition algebraic geometry is a science of *finitely* generated *commutative* rings.

During the year of 1987, physicists learned that the KP theory, initiated by Sato and studied mostly by Japanese mathematicians in these years, may give an example of the mathematical framework which is needed in quantum string theory. In connection with the string theory, the key point of the KP theory is that the total hierarchy of the Kadomtsev-Petviashvili equations (KP system) characterizes the Riemann theta functions of all complete algebraic curves (and their degenerated functions) from any other functions (Theorem 3). Usually moduli spaces of algebraic curves are studied in relation to moduli theory of abelian varieties. But in string theory there is no reason to talk about abelian varieties. Therefore it is much more desirable to have a theory of algebraic curves and their moduli spaces without using abelian varieties. The theorem of [M3] tells us that the jacobian varieties, and hence algebraic curves, are completely characterized by the KP system without mentioning even one word on abelian varieties. Moreover, it has been realized by the Japanese physicists Ishibashi-Matsuo-Ooguri [IMO] and Russian mathematicians Beilinson-Manin-Shechtman [BMS] that the Virasoro group acts on the space of regular solutions of the KP system and produces moduli spaces of all algebraic curves regardless of their genera. Therefore, the KP theory could give a desired mathematical framework for string theory.

The reason why string theory is interesting for a mathematician is because it tells us unexpected relations between various disciplines of mathematics itself which did not seem to be related before string theory predicted their relations. For example, the discovery of the mathematical relationship between the Virasoro algebra and the moduli theory of the algebraic curves [ADKP] has one of its motivations in string theory. For another example, Taubes's work on elliptic genera of manifolds with S^1-actions [T] was motivated by Witten's idea stating that the "*index of a Dirac operator on a loop space = a modular form*" [W2] coming from string theory. There are more examples which have been understood mathematically. Also there are interesting predictions which are not yet understood mathematically. As an example I would like to mention the relation between special three-dimensional complex manifolds called the Calabi-Yau spaces appearing in the superstring theory and the super-algebraic curves, which is expected to be understood mathematically by the super-KP theory (cf. Alvarez-Gaumé-Gomes [AG]). It seems that our task is to find, without using string theory, a string of mathematical relations between totally different subjects in mathematics which is predicted and suggested by string theory.

2. Schottky Problem

The Schottky problem, in a general sense, is a problem of finding a *good* characterization of jacobian varieties. Since jacobians form an interesting special class of abelian varieties, historically a characterization always meant a characterization *among* abelian varieties. There has been a substantial amount of work done by many great mathematicians after Schottky's original work [Scho] of 1888. A natural approach to this problem is to perform a case study for low-genus jacobians. If the genus g is less than 4, then moduli of jacobians are open dense in those of abelian varieties and there is no difficulty. The actual problem starts at $g=4$. But already genus 5 is hard enough.

Complete characterizations valid for all genera were discovered only recently. The breakthrough was made by Gunning [G], based on the trisecant relations of Fay and Mumford. He showed that if an abelian variety has one-dimensional trisecants, then it is a jacobian, under an extra condition. When Gunning's paper appeared, Welters in Barcelona immediately recognized that the extra condition was not necessary (summer of 1982). Moreover, Welters succeeded in giving a much stronger *infinitesimal criterion* [We] by March of 1983. In the very same month and year, I discovered, while at MSRI, that the KP system characterizes Riemann theta functions of jacobians *among any other functions*, based on the works of Krichever and Sato. Amazingly enough, again at the very same moment, in Utrecht, van Geemen was completing his work on the geometry of the (small) Schottky locus. His remarkable theorem [vG] says that *the jacobian locus is an irreducible component of the (small) Schottky locus for every genus.* Before him only $g=4$ and $g=5$ were known. Then Arbarello-De Concini [AD] discovered that the differential relations of the theta functions which arise from Welters's infinitesimal criterion are the consequences of the KP system, and thus obtained a characterization of jacobians among abelian varieties by the KP system, independently. Finally, Shiota (1985) realized that if one wants to characterize jacobian theta functions *only among general Riemann theta functions*, then the first equation of the KP system together with a global condition is sufficient. However, we still do not know the explicit form of the Siegel modular forms which vanish exactly at the jacobian locus.

3. Algebraic Curves

Now let me sketch how the KP system is related to algebraic curves and their jacobians. Our goal is to construct an infinite-dimensional space X of all algebraic curves of all genera on which an infinite-dimensional torus $T = \mathbf{C}^{\infty} = \text{ind.lim } \mathbf{C}^n$ and the Virasoro group act, and produce their jacobians and moduli spaces as orbits of their actions.

The first approximation is the following:

1st approximation of X = the set of all linear ordinary differential operators.

Let (R, ∂) be a commutative derivation algebra defined over a field κ of characteristic zero. For simplicity we take $\kappa = \mathbf{C}$, $R = \mathbf{C}[[x]]$ = the formal power series ring and $\partial = d/dx$. We denote by D the set of all linear ordinary differential operators with coefficients in R.

Now how can we associate an algebraic curve to an arbitrary operator $P \in D$? The most naive idea is to take the set of all eigenvalues of P, namely define a curve by Spec P. But since every complex number $\lambda \in \mathbf{C}$ is an eigenvalue of P, Spec $P = \mathbf{C}$ and it is not so interesting. Then how about taking multiplicities into account? Since every eigenvalue of P has multiplicity n, where n = order of P, by resolving the multiplicity we may obtain a more interesting object. Then how can we resolve the multiplicity? The natural idea is to take the maximal set of commuting operators with P and consider the simultaneous eigenspaces. So let

$$A = A_P = \{Q \in D \mid [Q, P] = 0\}.$$

To talk about the simultaneous eigenspaces of A, we have to show the commutativity of A. Since we need a little more preparation, let us postpone the proof of commutativity until a little later. Let us assume for the moment that the eigenspace corresponding to $\lambda \in \text{Spec } P$ decomposes into simultaneous eigenspaces

$$\text{Ker}(P - \lambda) = \mathbf{C}\psi_1 \oplus \mathbf{C}\psi_2 \oplus \cdots \oplus \mathbf{C}\psi_n,$$

where ψ_j is a simultaneous eigenfunction of all operators in A. Define

$$I_j = \{Q \in A \mid Q\psi_j = 0\}.$$

Since $Q_1Q_2 \in I_j$ implies $Q_1Q_2\psi_j = \alpha_1\alpha_2\psi_j = 0$ for some $\alpha_1, \alpha_2 \in \mathbf{C}$, I_j is a prime ideal of A. Therefore, for each point λ of Spec P, there are n "points" $I_1, \ldots, I_n$ sitting above it. Thus we may be able to obtain a desired curve by an n-sheeted covering of $\mathbf{C}$, which is a subset of Spec A = the set of all prime ideals of A in Grothendieck's notation.

But a curve obtained by an n-sheeted covering over $\mathbf{C}$ is a very restricted one. How can we obtain all curves? To this end, we have to get rid of n, the order. From now on let us assume that $P \in D$ is generic. Namely, we assume that P is of the following (normalized) form:

$$P = \partial^n + a_2\partial^{n-2} + a_3\partial^{n-3} + \cdots + a_n.$$

Then we can take its normalized nth root:

$$L = P^{1/n} = \partial + u_2\partial^{-1} + u_3\partial^{-2} \cdots,$$

which is a pseudo-differential operator. We define an associative algebra structure in the set E of pseudo-differential operators with coefficients in R by the commutation relation

$$[\partial^{-1}, f] = -f'\partial^{-2} + f''\partial^{-3} - \cdots,$$

where $f \in R$ and f' denotes the derivative of f. Dealing with a normalized first-order operator $L \in E$ with $L^n \in D$ is equivalent to dealing with all normalized nth-order ordinary differential operators. We can handle all normalized differential operators of all orders at one time by using an arbitrary normalized first-order pseudo-differential operator L.

So define once again

$$A_L = \{Q \in D \mid [Q, L] = 0\}.$$

If $L^n = P$, then $A_L = A_P$. Since L is a first-order operator, we can imagine that its "eigenvalues" have no multiplicity. Therefore, we have

$$\text{Spec } A_L = \text{"Spec } L\text{"}.$$

To be precise, we *define* the analytic notion of Spec L (the set of eigenvalues) by Grothendieck's notion of Spec A_L. As we noted, when $L^n = P \in D$ and eigenspaces decompose into simultaneous eigenlines, the curve Spec A_L is an n-sheeted covering over $\mathbf{C}$. This covering is given by the natural inclusion $\mathbf{C}[P] \subset A_L = A_P$. Taking their affine schemes, we have

$$\text{Spec } A_L \to \text{Spec}_{\text{Grothendieck}}\, \mathbf{C}[P] = \text{Spec}_{\text{Analytic}}\, P = \mathbf{C}.$$

On $(P-\lambda)\in \operatorname{Spec} \mathbf{C}[P]$, there are n points $I_1, I_2, \ldots, I_n$ sitting over it, since they all contain $(P-\lambda)$. Therefore the above map is an n-sheeted covering. To show $\operatorname{Spec} A_L$ is a curve we need the following:

Theorem 1 (Schur). *The commutant A_L of L in the set D of differential operators is itself commutative and has transcendence degree 1 over* $\mathbf{C}$.

This theorem follows immediately from the Lemma:

Lemma. *For any normalized first-order pseudo-differential operator L, there is an invertible monic zeroth-order pseudo-differential operator S such that*

$$S^{-1}LS=\partial.$$

The proof of this lemma is just a computation. Now since $A_0=S^{-1}A_LS$ consists of pseudo-differential operators commuting with ∂, $A_0\subset \mathbf{C}((\partial^{-1}))=\mathbf{C}[\partial]+\mathbf{C}[[\partial^{-1}]]$, and hence $A_L\cong A_0$ is commutative. By an argument in elementary number theory on orders of elements in A_0 (see [M3]), we can show that the transcendence degree of A_0 over $\mathbf{C}$ is 1. The above proof of commutativity is actually Schur's argument. In the above we transformed a differential operator to a formal Laurent series in ∂^{-1} with constant coefficients. Schur expanded it in a formal Laurent series in L^{-1} with constant coefficients. These are exactly the same arguments.

Remark. If we take the commutant $\{Q\in E\,|\,[Q,L]=0\}$ in E, then it is equal to $\mathbf{C}((L^{-1}))$ and hence always isomorphic (conjugate) to the maximal commutative subalgebra $\mathbf{C}((\partial^{-1}))$ of E. Thus it is not interesting at all.

Let $\operatorname{gr}(A_0)$ be the canonical graded algebra associated to the filtration of A_0. We define a complete algebraic curve C by $C=\operatorname{Proj}\operatorname{gr}(A_0)$. Now since $A_0=\{\text{regular functions on } \operatorname{Spec} A_0\}$ and it has a canonical realization as a subring of $\mathbf{C}((\partial^{-1}))$, we can see how C gives a one-point completion of $\operatorname{Spec} A_0$. When A_0 is of rank 1, namely if it has two elements whose orders are coprime, then we have $C=\operatorname{Spec} A_0\cup\{z=0\}$, where we identify $\partial^{-1}=z=$ a local coordinate near the point at infinity. The attached point $p=\{z=0\}$ is a smooth point of C. Since conjugation by S does not change the order of an operator, we have $A_0\cap\mathbf{C}[[z]]=\mathbf{C}$. This means that a regular function on $\operatorname{Spec} A_0$ which is also regular at the point $\{z=0\}$ must be a constant. Therefore $\operatorname{Spec} A_0\cup\{z=0\}$ is complete. In general, we cannot use z as a local parameter at infinity, but a similar argument works. Thus C is always a one-point completion of Spec A_0. Thus we have obtained a curve C, a smooth point $p \in C$ and the linear part of the local coordinate. Namely, we have a unique tangent vector $v = \partial/\partial y$, where y is the local coordinate at p.

In this way we can construct a complete algebraic curve C out of a normalized first-order ordinary pseudo-differential operator L as $C =$ "Spec L." We are already in the second approximation of the space X:

2nd approximation of X = the set of normalized first order pseudo-differential operators L.

4. KP System and Jacobian Varieties

The next natural question is this: how much do we know about L if we know its spectral data $C =$ "Spec L"? This question leads us to the notion of *isospectral deformation* of L. We define a parameter depending $L(t)$ to be an *infinitesimal isospectral deformation* of $L(0)$ if there is another parameter depending differential operator $B(t)$ such that the *Lax equation*

$$\partial L(t)/\partial t = [B(t), L(t)]$$

holds. One reason why we restrict ourselves to a *differential operator* B is because our L is pretended to be an n-th root of a differential operator P. We are looking for an isospectral deformation of P which is a family of differential operators. We do not allow P deforming to a pseudo-differential operator. The other reason is that we want to recover the commutant A_L by stationary (trivial) deformations. As we have observed that the commutant in E is not interesting, we have to restrict B to a differential operator. Since L is normalized, $\operatorname{ord} \partial L(t)/\partial t$ is negative. Hence B must satisfy $\operatorname{ord}[B, L] < 0$. Let $F = \{B \in D \mid \operatorname{ord}[B, L] < 0\}$. Then by a simple argument we can show that F is the linear span of $1, \partial, L_+^2, L_+^3, \ldots$, where L_+^m denotes the differential operator part of L^m. (Schur seems to have known this fact, too.) Therefore, all possible infinitesimal isospectral deformations of L are given by the following system of Lax equations, called the *total hierarchy of the Kadomtsev-Petviashvili equations*, or the KP system:

$$\partial L/\partial t_m = [L_+^m, L], \qquad m = 1, 2, 3, \ldots,$$

where t_m is the deformation parameter corresponding to the conjugation by L_+^m.

Examples.

1. Let $m=1$. Then the Lax equation gives $\partial L/\partial t_1 = \partial L/\partial x$. Namely, it gives the translation of x by t_1.
2. Let $L^2 = \partial^2 + 2u$ be a differential operator. Then the even equations in the KP system are all trivial and the first nontrivial equation is the *KdV equation* $4\partial u/\partial t_3 - u_{xxx} - 12uu_x = 0$. This equation was discovered [KdV] in studies of soliton phenomena of shallow water wave motions.
3. For a general L, the first nontrivial equation among the KP system is the original *KP equation* $3\partial^2 u_2/\partial t_2^2 - (4\partial u_2/\partial t_3 - u_{2,xxx} - 12u_2u_{2,x})_x = 0$, where $L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots$. This equation was introduced by physicists [KP] to study transversal stability of soliton solutions of the KdV equation. Note that the KP equation reduces to the KdV equation if we throw away the t_2 dependence.

$L(t)$ satisfies the KP system if and only if there exists $S(t) = 1 + s_1\partial^{-1} + s_2\partial^{-2} + \cdots$ such that $S(t)^{-1}L(t)S(t) = \partial$, and

$$\partial S/\partial t_m = -L_-^m \cdot S,$$

where $Q_- = Q - Q_+$. We call this equation the *KP system for S.* It is known that about half of the equations of the system are automatically integrated by introducing a new unknown function $\tau(x, t)$ called Hirota-Sato's τ-function, which is something like a potential of S. Let $p_n(t)$ be a polynomial defined by

$$1 + p_1\lambda + p_2\lambda^2 + \cdots = \exp(t_1\lambda + t_2\lambda^2 + \cdots),$$

and let $\partial_t = (\partial_1, \frac{1}{2}\partial_2, \frac{1}{3}\partial_3, \ldots)$, where $\partial_n = \partial/\partial t_n$. Then the coefficients of $S(t)$ are given by

$$s_n(t) = \frac{1}{\tau(x, t)} \cdot p_n(-\partial_t)\tau(x, t).$$

When $\tau(x, t)$ gives a solution $S(t)$ of the KP system, then it is called the *τ-function solution* of the KP system.

Krichever [K] discovered that the Riemann theta functions associated with jacobian varieties give τ-function solutions of the KP system. Namely, for any such theta function $\theta(z)$ defined on $\mathbf{C}^n$, there is a linear transformation ϕ of t-variables to z-variables and a quadratic form q in t such that $e^{q(t)}\theta(\phi(t))$ gives a τ-function solution. Then Sato [Sa] discovered that the Schur polynomials of the tensor irreducible representations of general linear groups give another class of τ-function solutions. Let $t_n = (1/n)$ trace g^n for an element g of a general linear group and $\chi^Y(t)$ be the Schur polynomial corresponding to a Young diagram Y (i.e., the character of the corresponding representation) written in t-variables. Then $\chi^Y(t)$ gives a τ-function solution. On the other hand, I established [M2] that for every initial data $L(0)$ (resp. $S(0)$) there is a unique solution $L(t)$ (resp. $S(t)$) with coefficients in $\mathbf{C}[[x, t_1, t_2, \ldots]]$. Since my theorem [M4] gives an explicit construction of the solution $S(t)$ out of its initial data, we can also establish a convergence condition of the solution: for any $\varepsilon > 0$, there is $\delta > 0$ such that if the modulus of the i-th derivative of the n-th coefficient s_n of $S(0)$ is smaller than $c^i i!\,\delta^n/n!$, then $s_n(t)$ converges absolutely for all $|t_m| < a\varepsilon^m/m!$, where a and c are some positive constants.

Let $X^0 \subset E$ be the set of all normalized first-order pseudo-differential operators. The KP system and its solvability of the initial value problem mentioned above define a vector group $T = \text{ind lim } \mathbf{C}^m$ action on X^0 as the time evolution of L. This action is an analytic action on an open subset of X^0. Every orbit of this action starting at $L \in X^0$ can be thought as a universal moduli space of isospectral deformations of L. Since the space F gives all possible deformations and A_L provides trivial deformations, the tangent space of the orbit at L is canonically isomorphic to the quotient space F/A_L. From now on let us assume that the orbit has finite dimension. (The orbit of L has finite dimension if and only if the rank of A_L is 1.) Then by the covering cohomology technique applied to a neighborhood of the point at infinity of the curve C, we can easily show that

$$F/A_L \cong H^1(C, \mathcal{O}_C),$$

where $\mathcal{O}_C$ is the structure sheaf of the curve C.

To study the global structure of an orbit, we have to define a torsion-free rank-one sheaf on C. If $L^n = P \in D$, then we attach a line $\mathbf{C}\psi_j$ to a point $I_j \in \text{Spec } A_L$. In general $A_0 \cong A_L \subset D$ defines a right A_0-module structure of D. We consider D as a left R- right A_0-bimodule. Its rank is one and $\tilde{D}$ defines a torsion-free rank-one sheaf on $\text{Spec } R \times \text{Spec } A_0$. Restricting $\tilde{D}$ to the unique maximal ideal of $\text{Spec } R$ and extending to C, we obtain a torsion-free rank-one sheaf $\mathcal{L}$ on C. By this construction, we can compute the cohomology of $\mathcal{L}$. It is a generic sheaf of $\deg \mathcal{L} = \dim H^1(C, \mathcal{O}_C) - 1$. (When C is nonsingular, $\mathcal{L}$ corresponds to a point in the complement of the theta divisor of the jacobian of C.) Isospectral deformations of L change the A_0-module structure of D defined by $S(t)A_0S(t)^{-1} \subset D$ and hence give deformations of the sheaf $\mathcal{L}$. Let M_L be the orbit of L under the T-action.

Then we have a covering map $H^1(C, \mathcal{O}_C) \to M_L$ and an injection $M_L \to H^1(C, \mathcal{O}_C^\times)$, and their composition coincides with the cohomology homomorphism $H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C^\times)$ given by the exponential map. Analytically the deformation of $\mathcal{L}$ is described as follows. Let $U_p \subset C$ be a small neighborhood of the point at infinity. Assume C is nonsingular. Then $\mathcal{L}$ is a line bundle and its restrictions on U_p and Spec A_0 are both trivial. Hence $\mathcal{L}$ is defined by a single transition function h on $U_p \backslash \{p\} =$ Spec $A_0 \cap U_p$. Deform h by $\exp(t_1\partial + t_2\partial^2 + t_3\partial^3 + \cdots)h$. This new transition function defines the line bundle corresponding to $L(t) \in M_L$. Note that $t_1\partial + t_2\partial^2 + t_3\partial^3 + \cdots$ gives an element of $H^1(C, \mathcal{O}_C)$ by the identification $\partial = z^{-1}$ and $\exp(t_1\partial + t_2\partial^2 + t_3\partial^3 + \cdots) \in H^1(C, \mathcal{O}_C^\times)$. When we give the precise definition of X as a closure of X^0, we can see that the image of M_L fills up the connected component of $H^1(C, \mathcal{O}_C^\times)$ of degree dim $H^1(C, \mathcal{O}_C) - 1$. Thus we have

Theorem 2 [M3]. *Every finite-dimensional orbit of L under the T-action is isomorphic to the generalized jacobian variety* $H^1(C, \mathcal{O}_C)/H^1(C, \mathbf{Z})$.

If C is nonsingular, then $H^1(C, \mathcal{O}_C)/H^1(C, \mathbf{Z})$ is nothing but the jacobian Jac(C) of C, which we identify with $\mathrm{Pic}^{g-1}(C)$, where $g = \dim H^1(C, \mathcal{O}_C)$ is the genus of C.

If we adopt the theory of τ-functions here, then our theorem states

Theorem 3. *A formal power series* $f(z_1, \ldots, z_n)$ *is a Riemann theta function associated with a jacobian variety of an algebraic curve or its degeneration if and only if there is a linear transformation* $\phi : T \to \mathbf{C}^n$ *and a quadratic form* $q(t_1, t_2, \ldots)$ *in* $t \in T$ *such that* $\exp(q(t))f(\phi(t_1, t_2, \ldots))$ *gives a* τ*-function solution of the KP system at the origin of T.*

This theorem gives a completely local characterization of the jacobian theta functions among all other formal power series. Thus we obtain a characterization of jacobian varieties without mentioning abelian varieties. It is very interesting that a *jacobian variety as a manifold appears as a solution of the KP system.* We do not have to prepare any geometric stage for a jacobian, such as complex tori and abelian varieties. The geometric structure of a jacobian simply *appears* automatically. This gives an unexpected (unwanted?) solution of the Schottky problem.

Of course if we want to characterize jacobian theta functions only among theta functions of abelian varieties, then we do not need the total hierarchy of the KP system but only one differential equation together with a global condition. Actually, Shiota proved the following

Theorem 4 (Shiota [Sh]). *A Riemann theta function θ associated with an abelian variety of dimension n is a jacobian theta function if and only if there are vectors $a_1 \neq 0$, a_2, $a_3 \in \mathbf{C}^n$ and a quadratic form $q(t_1, t_2, t_3)$ such that $\exp(q(t))\theta(t_1a_1 + t_2a_2 + t_3a_3 + b)$ gives a global τ-function solution of the single KP equation in $t \in \mathbf{C}^3$ for all $b \in \mathbf{C}^n$.*

This theorem solved the Novikov conjecture.

Theorem 3 gives a local characterization of jacobian theta functions by a system of nonlinear partial differential equations. There is an interesting open question: *is there any set of differential equations which characterizes arbitrary Riemann theta functions of abelian varieties among any other analytic functions?* Although geometric characterization of abelian varieties among complex tori was established by the beginning of this century, any local characterization is not yet known. Since being abelian means algebraicity of a torus and by definition algebraicity is a global condition, it is not surprising that we still do not have such a characterization. It must be very hard to give a local characterization of global properties. But then why can we obtain a local characterization of jacobians? It is because of the complete integrability of the KP system. Namely, since the system is completely integrable, local information of a solution determines the global structure of the solution. Thus the true question is this: *can we find a completely integrable system which characterizes general theta functions?* The problem of understanding the relation between such a system and the KP system can be called the *adjoint Schottky problem.*

5. Virasoro Action

The solvability of the KP system in terms of $S(t)$ is established in the following way. Let $S(0)$ be an initial value. Then we define

$$U(t) = \exp(t_1\partial + t_2\partial^2 + \cdots) \cdot S(0)^{-1}.$$

Here we consider t as formal parameters and $U(t)$ as a generating function of pseudo-differential operators. We can define a rigorous mathematical framework in which the above expression makes sense as a formal pseudo-differential operator of infinite order.

Theorem 5 [M2, M4]. *There is a unique monic pseudo-differential operator* $S(t)$ *of order* 0 *and an infinite-order invertible differential operator* $Y(t)$ *with* $Y(0)=1$ *such that* $U(t)=S(t)^{-1}\cdot Y(t)$.

This theorem is a generalization of the Birkhoff decomposition of loop groups. Now, by definition, $\partial U/\partial t_n=\partial^n\cdot U$. On the other hand, it is equal to

$$-S^{-1}\cdot\frac{\partial S}{\partial t_n}\cdot S^{-1}\cdot Y+S^{-1}\cdot\frac{\partial Y}{\partial t_n}.$$

Therefore

$$-\frac{\partial S}{\partial t_n}\cdot S^{-1}+\frac{\partial Y}{\partial t_n}\cdot Y^{-1}=S\,\partial^n S^{-1}=L^n.$$

The first term of the equation is a pseudo-differential operator of negative order and the second term is an infinite order differential operator. Therefore,

$$-\frac{\partial S}{\partial t_n}\cdot S^{-1}=L^n_-$$

follows, which is the KP system we wanted to solve.

Thus the essence of the KP system is the generalized Birkhoff decomposition. The decomposition as well as the solvability holds even if we replace $R=\mathbf{C}[[x]]$ by $K=\mathbf{C}((x))[\log x]$. A normalized first order operator L with coefficients in K is said to be *quasi-regular* if there are positive integers m and n such that for every positive integer k, $x^m L^k x^n\in E$, i.e., its coefficients are in R.

Let X be the set of quasi-regular L's. This is the desired space we wanted to construct. Now X is closed under global T-action. We call a point L of X of *finite type* if the orbit of L under the KP flow (i.e., the T-action) has finite dimension. Every point of X gives the following data: $(C, p, v, \mathscr{L})$, where C is an irreducible complete algebraic curve, p is a smooth point on it, v is a non-zero tangent vector at p and $\mathscr{L}$ is a torsion-free finite-rank sheaf on C. Sato proved that the corresponding set of S, monic 0-th order operators with certain singularities at $x=0$, has a structure of an infinite Grassmannian. Let E_- be the set of pseudo-differential operators of order at most -1. Consider the quotient space $W=E/Ex$, where Ex is the left ideal generated by x in E. Following the decomposition $E=D\oplus E_-$, W decomposes $W=W_+\oplus W_-$. For every S corresponding to $L\in X$, define $\{P\in E\,|\,SP\in D\}$ and let $\Sigma\subset W=E/Ex$ be its projection image. Then the composition map $\Sigma \to W \to W_+ = W/W_-$ is Fredholm and has index 0. Therefore Σ defines an element of $\mathrm{Gr}(W_+, W)$, which consists of subspaces of W having the "same" size with W_+ in the above sense of Fredholm. We denote it by Gr and identify S with Σ. Let G_0 be the group of monic 0-th order operators with constant coefficients. Then $\pi: \mathrm{Gr} \to X$, $\pi(S) = S\,\partial S^{-1} = L$ is a principal fiber bundle over X with structure group G_0. Since G_0 is contractible, Gr and X have the same topological structure. Because of the unique solvability of the KP system in terms of S, every finite-dimensional orbit of the T-action on Gr is isomorphic to a generalized jacobian. We define the notion of *finite type* on Gr in the same way. Then every finite-type point S of Gr gives $(C, p, v, \mathscr{L}, S_0)$, where $S_0 \in G_0$ is a local trivialization of $\mathscr{L}$ near p. It is also possible to identify (v, S_0) with a full local parameter y at p. In any case our interpretation of S_0 is not canonical. Krichever's theorem tells us that if we have $(C, p, v, \mathscr{L})$ consisting of a smooth curve of genus g, a point on it, a tangent vector at p and a line bundle of degree $g-1$, then it determines a unique point $L \in X$ of finite type.

Let $G \subset E$ be the set of monic 0-th order pseudo-differential operators with coefficients in R. Then G is the big-cell of Gr. Namely, there is a "divisor" Δ at infinity such that $\mathrm{Gr} = G \cup \Delta$. Let Λ be the line bundle over Gr corresponding to the "divisor" Δ. It is called the determinant line bundle, since it coincides with the dual of the determinant bundle of the universal bundle of Gr. Let τ be a section of Λ which vanishes only on Δ. We have maps

$$T \ni t \mapsto S(t) \in \mathrm{Gr} \ni S \mapsto \tau(S) \in \mathbf{C}.$$

Let $\tau(t) = \tau(S(t))$ be the composition of the above maps. Then this is nothing but the τ-function solution of the KP system: $\tau(x, t) = \tau(x + t_1, t_2, t_3, \ldots)$.

Now let us describe the Virasoro action on Gr. The Virasoro algebra V is a special central extension of the algebra $\mathfrak{g}$ of vector fields on the circle S^1. The $\mathfrak{g}$-action on a point of Gr produces an infinitesimal deformation of the complex structure of the corresponding curve. Let C be the curve corresponding to a point S of Gr. Then $C = \mathrm{Spec}\, A_L \cup U_p$, where $L = S\partial S^{-1}$ and U_p is a small neighborhood of $p \in C$. Since $\mathrm{Spec}\, A_L$ and U_p have no deformations, deformations of the complex structure of C are given by changing the patching of $\mathrm{Spec}\, A_L$ and U_p. Let $\mathscr{X}(\mathrm{Spec}\, A_L \cap U_p)$ be the set of holomorphic vector fields on $\mathrm{Spec}\, A_L \cap U_p$. (Topologically this intersection is a circle.) Then its inductive limit as U_p tends to $\{p\}$ coincides with $\mathfrak{g}$. Therefore there is a natural projection $\mathfrak{g} \to H^1(C, \mathscr{T}_C)$, where $\mathscr{T}_C$ is the tangent sheaf of C. Since $H^1(C, \mathcal{T}_C)$ defines infinitesimal deformations of C by Kodaira-Spencer, we have defined a $\mathfrak{g}$-action on Gr. In other words, a vector field on S^1 determines an infinitesimal change of the patching Spec $A_L \cup U_p$.

We can extend the action to a V-action on the line bundle Λ. The one-dimensional center acts on the fiber. In the group level, the Virasoro action produces moduli spaces of the data (C, p, v).

Let Gr_f be the set of finite-type points of Gr. Then $(\mathrm{Gr}_f, T, \mathcal{V})$ gives a genus-free theory of algebraic curves, where $\mathcal{V}$ is the Virasoro group. Namely, all jacobians are obtained at one time by the T-action and all moduli spaces of curves are given by the $\mathcal{V}$-action. Actually, Gr_f is the moduli space of the data $(C, p, v, \mathcal{L}, S_0)$. To get rid of S_0, we take $X_f = \mathrm{Gr}_f / G_0$. If we also want to eliminate $\mathcal{L}$, then we have to define the quotient space X_f / T. Since an orbit of the T-action has arbitrary dimension, it is very hard to define the orbit space. I do not know how to understand this space. However, ideas of the non-commutative differential geometry might be useful here.

The section τ restricted to each of the compact orbits of the T-action gives all the Riemann theta functions associated with jacobians. On the other hand, restriction of τ to the orbits of the $\mathcal{V}$-action gives modular forms. Schur polynomials also show up as restrictions of τ to special T-orbits. In this way, these important functions can be obtained from a single function τ on the Grassmannian.

The infinite-dimensional orbits are related to infinite-genus situations as well as classification of vector bundles over algebraic curves. For example, the complete family of all different Hirzebruch surfaces appears as an infinite orbit. The geometric study of infinite-dimensional orbits has not been worked out in full generality.

6. Supersymmetrization

Supersymmetry is a new language of global analysis. For example, a supersymmetric quantum mechanics on a compact Riemannian manifold was effectively used to visualize the Morse theory and the Atiyah-Singer index theorem by Witten [W1] and Alvarez-Gaumé [A]. We now know how a harmonic p-form localizes to the critical points of Morse index p in the "strong coupling limit" with a Morse function detecting which critical points contribute to the topology of the manifold and which do not. Or we now know why the A-roof genus of a spin manifold must be of the form of product of $(x/2)/\sinh(x/2)$. These intuitive approaches of global analysis enabled mathematicians to produce new theorems such as a Morse *equality* of Helffer–Sjöstrand [HS] and an analogue of the index theorem on a loop space (cf. Taubes [T]). Since supersymmetry is a language, you do not have to speak it if you do not like. Also, any theorem proved by a supersymmetric technique can be proved by non-supersymmetric regular techniques. Perhaps an advantage of supersymmetry is to visualize global structures and to help one to find a new theorem. It also simplifies drastically the complicated arguments in index theorems and Riemann–Roch-type theorems. There is a formal resemblance between $\mathbf{Z}_2$-graded structures in supersymmetry and the K-theoretic structure in non-commutative differential geometry of Connes. I believe that the geometric structure of the *super-KP system* could be thought as an example of an algebraic version of the non-commutative differential geometry. Algebraic studies of the super-KP equations have been worked out recently by Manin–Radul [MR], Ueno–Yamada [UY] and Mulase [M4]. Geometry, especially in connection with global analysis, of the super-KP system is yet to be done.

It is natural that if we set all "odd" variable evolutions in the super-KP system to be 0, then we recover the original KP system. I discovered in [M4] that if one *eliminates* odd variables in the super-KP system, then one gets the modified KP system. The first nontrivial equation is

$$4f_{xt} - 3f_{yy} + 6f_{xx}f_y - f_{xxxx} + 6f_x^2 f_{xx} = 0.$$

The KP equation is a unification of the KdV equation and the Boussinesq equation. In a similar sense, the super-KP system is a unification of the KP system and the modified KP system. As we have observed, the KP system is a defining equation of the universal family of all isospectral deformations of all normalized differential operators. I still do not have such a simple conceptual definition of the super-KP system. In the KP case, we did not allow a differential operator to deform to a pseudo-differential operator. But in the super-KP case studied in [MR], [M4], even if we start with a super-differential operator, it deforms to a super-pseudo-differential operator. The Birkhoff-type decomposition and the solvability argument in S variable work very well in the supercategory. But the Lax equation does not work well.

If we forget about isospectral deformations and think of the KP system only by Theorem 5, then the super-KP system is the most natural generalization of the theorem to the supersymmetric case. Namely,

$$\text{KP} = \text{Birkhoff decomposition},$$

$$\text{Super-KP} = \text{Super-Birkhoff decomposition}.$$

For more detail about the super-KP system please see [M4]. Since the theory of the KP system and the infinite Grassmannian gives the critical dimension of 26 as a special case of the Riemann-Roch theorem, it is natural to ask if we can deduce 10 from the super-KP theory, which is not known.

> ... the days of history
> make up a book with seven *Siegels*.
> What you call the spirit of an age
> is in reality the spirit of those men
> in which their time's reflected.

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