ALGEBRAIC THEORY OF THE KP EQUATIONS*

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Contents

0. Introduction

The 1980s were the decade of unification in mathematics. The research activities influenced by mathematical physics revealed many unexpected relations between disciplines of mathematics. Such activities include gauge theory, conformal field theory and knot theory. We can also include theories of integrable partial differential equations among them. In this paper, we study a typical example of integrable systems—the Kadomtsev-Petviashvili equation.

The simpler forms of the KP equation such as the Korteweg-de Vries equation and the Boussinesq equation have been known since the last century. These equations were originally proposed to analyze solitary wave propagation observed in a British canal. About the same time in Italy, Pincherle was investigating a ring of pseudo-differential operators in one variable. The formal expansion of the inverse of a linear ordinary differential operator in terms of pseudo-differential operators seems to have been well known before 1900. Schur used this technique in 1905 to study Wallenberg's problem of finding commuting ordinary differential operators. Schur proved that a set of differential operators which commute with a given operator is itself commutative.

These three roots, namely, integrable nonlinear partial differential equations, the theory of pseudo-differential operators in one variable and commuting ordinary differential operators, were put together by the Russian school in the mid 1970s.

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This movement includes Krichever's (re)discovery of a deep relation between the KP theory and algebraic curves. The slow and steady progress is a thing of the past. After 1980, the KP theory has developed at an enormous rate. Let us just enumerate some of the mathematical subjects known to be related with the KP equations: algebraic curves, theta functions, commuting ordinary differential operators, Schur polynomials, infinite-dimensional Grassmannians, affine Kac-Moody algebras, vertex operators, loop groups, Jacobian varieties, the determinant of the Cauchy-Riemann operators, symplectic geometry, string theory, conformal field theory, determinant line bundles on moduli spaces of algebraic curves and their cohomology, vector bundles on curves, Prym varieties, commuting partial differential operators, 2-dimensional quantum gravity, matrix models and intersection theory of cohomology classes of moduli spaces of stable algebraic curves. To make this list complete, we have to add an even larger number of subjects from applied mathematics.

The KP equations are thus as ubiquitous as elliptic functions. Every time people discovered the relation between the KP theory and each of the above mentioned subjects, it was greeted with great surprise by the researchers in the field.

The purpose of this paper is to give a systematic theory of the KP equations and to give some new results including the treatment of spectral curves (Section 2), geometric inverse scattering (Section 3), a construction of commuting partial differential operators (Section 6), and theorems on matrix integrals (Section 8). We also intend to give a *short* review of the algebraic theory of the KP system because, as far as I know, it does not seem to be really covered in any existing articles and books.

We begin this paper by observing that the Weierstrass elliptic function solves the KP equation. We then ask: why do algebraic curves have something to do with the KP equations? The explanation we have is the following: First of all, the KP equations govern all possible iso-spectral deformations of an arbitrary linear ordinary differential operator. On the other hand, every ordinary differential operator defines a unique algebraic curve as a set of eigenvalues with resolved multiplicity. The eigenspace of the operator defines a vector bundle on this curve. Since iso-spectral deformations preserve the eigenvalues, they should correspond to deformations of vector bundles on the curve. In particular, these deformations generate Jacobian varieties, and even arbitrary Prym varieties in a more general setting.

These are the topics of Sections 2–6. In Section 2, we define spectral curves as a set of resolved eigenvalues. In order to make the spectral curves compact, we need pseudo-differential operators, which appear as a local coordinate of the spectral curve around the point at infinity. The Grassmannians are introduced in Section 3 as a tool for determining all the geometric information that a differential operator possesses. To a set of geometric data consisting of an algebraic curve and a vector bundle on it, we can associate a point of the Grassmannian, which is indeed a cohomology group of the vector bundle. By the theory of pseudo-differential operators of Section 2, this cohomology changes holomorphic functions on the curve into ordinary differential operators.

The iso-spectral deformations are defined in Section 4. The KP system is derived as a master equation for all such deformations. In order to establish unique solvability of the KP system, we use infinite-dimensional geometry and a generalization of the Birkhoff decomposition. The geometric counterparts of iso-spectral deformations are studied in Section 5. We prove that a finite-dimensional moduli space of iso-spectral deformations is canonically isomorphic to a Jacobian variety. An algebro-geometric interpretation of this theorem gives a complete characterization of Jacobian varieties (a contribution to the Schottky problem). Section 6 is devoted to generalizing the setting to include matrix coefficients. We see that arbitrary Prym varieties appear naturally in lieu of Jacobian varieties in the original setting. We can finally establish a long-awaited characterization of Prym varieties in terms of the multi-component KP systems. This theory has another byproduct: we give a new construction of commuting partial differential operators that are globally defined on Prym varieties.

We define τ-functions of the KP system in Section 7, and observe its two-fold connections with infinite-determinants. In the final section, we prove that the Hermitian matrix integral is a τ-function of the KP system. The nature of this solution is still unclear to us. Unlike the solutions studied earlier, this solution does not give commuting differential operators. Instead, it defines an embedding of $sl(2,\mathbb{C})$ into the ring of differential operators. The fact that the presentation of Section 8 is computational rather than conceptual shows the current stage of our understanding of this topic.

Some of the material of Section 3, 4, 5 and 6 is taken from [35] [32] [31] and [27] respectively. We do not deal with anything concerning supersymmetry in this paper. We refer to [46] and [36] for this topic.

By the nature of the subject, the bibliography of the KP theory is tremendously large. It is thus impossible to list all the important contributions. I apologize to the authors for any omission of their works in this paper. My intention of the current article is not to give a complete review of the entire KP theory, which is at any rate totally beyond my ability, but to give a systematic presentation of algebraic structure of the KP theory. References are therefore restricted to those which have direct relation to the topics covered in this paper.

1. The KP equation and elliptic functions

The *KP equation* is the following nonlinear partial differential equation

$$\frac{3}{4}\, u_{yy} - \left(u_t - \frac{1}{4}\, u_{xxx} - 3uu_x\right)_x = 0 \tag{1.1}$$

for the unknown function $u = u(x,y,t)$, where the subscripts denote partial derivatives: $u_t = \partial u/\partial t$, etc. One can see immediately that every solution of the *KdV equation*

$$u_t - \frac{1}{4}\, u_{xxx} - 3uu_x = 0 \tag{1.2}$$

gives a solution to the KP equation by giving u trivial y-dependence. In 1895, Korteweg and de Vries [18] derived Eq.(1.2) from the Navier–Stokes equation of fluid dynamics as a special limit to give a model of nonlinear wave motions of shallow water observed in a canal. They showed that the KdV equation admits a *solitary wave* solution, or a one-soliton solution, which we will derive shortly. The motivation in the 1970 paper of Kadomtsev and Petviashvili [22] for introducing the three dimensional nonlinear equation Eq.(1.1) was to study transversal stability of the soliton solutions of the KdV equation.

In a sense, these equations do not describe the real world, because they are derived from a real physical equation by taking a natural but rather unphysical limit. Thus none of the *exact* solutions of these equations can be seen in a real canal. The exact solutions live only in the ideal world, i.e. in the mathematical world. In what follows, we do not study Eqs.(1.1) and (1.2) in their historical settings. We believe that these equations are important not because they represent the shallow water wave motion *approximately*, but because they have rich structures in *exact mathematics* such as in the algebraic geometry of curves, vector bundles, infinite-dimensional Grassmannians, and matrix integrals. For this reason, we will not pay much attention to the water waves any longer, and consider u, x, y and t as *complex* variables.

In the world of complex numbers, the individual coefficients of the KdV equation (1.2) are no longer important. In fact, a suitable re-scaling

$$\begin{cases} u \longmapsto \alpha u \\ x \longmapsto \beta x \\ t \longmapsto \gamma t \end{cases}$$

makes (1.2) equivalent to

$$a u_t + b u_{xxx} + c u u_x = 0$$

as long as $abc \neq 0$. The reason for our choice of the form of (1.2) will soon become clear.

Let us find a simple exact solution of the KdV equation. We put

$$u(x,t) = -\, f(x+ct) + \frac{c}{3} \tag{1.3}$$

to find a wave-like solution that propagates with a constant velocity $-c$. The function f determines the shape of the wave, which is invariant under time evolution. By a simple calculation, one obtains

$$f''' = 12 f f'$$

from the KdV equation, where $f = f(z) = f(x+ct)$ and f' denote the z-derivative of f. This equation has an integral

$$2f'' = 12 f^2 - g_2$$

with a constant of integration g_2. Multiplying both sides by f', we obtain

$$2f''f' = 12 f^2 f' - g_2 f' \ ,$$

which we can integrate again:

$$(f')^2 = 4f^3 - g_2 f - g_3 \ , \tag{1.4}$$

where g_3 is another integration constant. Eq.(1.4) is a rather famous equation called the Weierstrass differential equation. We have chosen the coefficients of Eq.(1.2) so that we obtain the canonical form of the Weierstrass differential equation. It is obvious from the equation that the inverse function of a solution of (1.4) has an integral expression

$$z = \int \frac{df}{\sqrt{4f^3 - g_2 f - g_3}}$$

which is just the *elliptic integral*. Thus, the Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \frac{3g_2g_3}{6160}z^8 + \cdots \tag{1.5}$$

gives a solution of (1.4). The corresponding solution of the KdV equation

$$u(x,t) = -\wp(x+ct) + \frac{c}{3} \tag{1.6}$$

is called a *periodic* solution. For generic values of g_2 and g_3, the periodic solution (1.6) corresponds to an elliptic curve

$$C^\circ = \left\{(X,Y) \mid Y^2 = 4X^3 - g_2X - g_3\right\}. \tag{1.7}$$

Let us see what happens if we set the parameters g_2 and g_3 to special values. The elliptic curve $Y^2 = 4X^3 - g_2X - g_3$ becomes singular when g_2 and g_3 satisfy

$$g_2^3 - 27g_3^2 = 0\,. \tag{1.8}$$

In terms of the speed c of (1.3), the special values of g_2 and g_3 satisfying (1.8) are represented by

$$\begin{cases} g_2 = \frac{4}{3}c^2 \\ g_3 = -\frac{8}{27}c^3. \end{cases}$$

Then the elliptic function $\wp(z)$ degenerates to a trigonometric function

$$\begin{aligned} f(z) &= \frac{1}{z^2} + \frac{c^2}{15}z^2 - \frac{2c^3}{189}z^4 + \frac{c^4}{675}z^6 - \frac{2c^5}{10395}z^8 + \cdots \\ &= c(\operatorname{csch}\sqrt{c}z)^2 + \frac{c}{3}\,, \end{aligned}$$

where we define

$$\operatorname{csch} z = \frac{2}{e^z - e^{-z}}\,.$$

The corresponding solution to the KdV equation becomes

$$u(x,t) = -c\left(\operatorname{csch}\sqrt{c}(x+ct)\right)^2. \tag{1.9}$$

Unfortunately, we have obtained a solution with a singularity at $x + ct = 0$. But if we set $x' = x - \frac{\pi}{2}i$ and consider x' and t as real variables, then the resulting solution

$$u(x',t) = c\left(\operatorname{sech}\sqrt{c}(x'+ct)\right)^2 \tag{1.10}$$

gives the famous *one-soliton* solution that is nonsingular for all real values of x' and t which satisfies

$$|u(x',t)| \to 0 \qquad \text{as} \qquad |x'| \to \infty.$$

Finally, if we let $c \to 0$, then the elliptic function reduces to

$$f(z) = \frac{1}{z^2}$$

and the elliptic curve degenerates to a rational curve with a cubic cusp singularity: $Y^2 = 4X^3$. The corresponding solution

$$u(x,t) = -\frac{1}{x^2}$$

is known as a *rational* solution.

Therefore, we obtain the following dictionary:

$$\begin{aligned}\text{periodic solutions} &= \text{nonsingular curves}\\ \text{soliton solutions} &= \text{curves with rational double points}\\ \text{rational solutions} &= \text{rational curves with more complicated}\\ &\qquad\text{singularities.}\end{aligned}$$

Thus the exact solutions of the KdV equation correspond to algebraic curves, with or without singularities. Now one can ask: why does the KdV (and hence the KP) equation have something to do with algebraic curves? How can we utilize our knowledge of these equations to study algebraic curves? Conversely, can we use algebro-geometric techniques to study these equations?

In the mid 1970s, Krichever [23] produced many exact solutions of the KP equations by using the geometry of algebraic curves. Then in the 1980s, the KP equation was effectively used to study the algebraic geometry of Jacobian varieties. In the following sections, we give an explanation of this (then) mysterious connection between the KP equation and algebraic curves.

2. The spectral curve of a differential operator

In this section, we define a spectral curve associated with a linear ordinary differential operator. This curve emerges as the set of eigenvalues of the differential operator.

Let P be a linear ordinary differential operator. If P is globally defined and acts on a space of functions with global conditions (such as boundary conditions), then the eigenvalues are localized—they may be discrete. Therefore, we do not expect any geometric structures in the set of eigenvalues. For example, take

$$P = -\left(\frac{d}{dx}\right)^2 + \frac{x^2}{4} - \frac{1}{2}$$

and consider the eigenvalue problem

$$P\psi = \lambda\psi \tag{2.1}$$

for L^2 functions ψ defined on $\mathbb{R}^1$. Then the set Spec P of eigenvalues of P coincides with the set of nonnegative integers. But if we think of P as an operator defined on an infinitesimal neighborhood of the origin $0 \in \mathbb{C}$ and acting on the vector space of all formal power series in x with complex coefficients, then every complex number is an eigenvalue of P. Thus, localizing P and ψ makes Spec P global:

$$\text{Spec } P = \mathbb{C}\ .$$

The asymptotic expansion of (2.1) as λ tends to infinity makes it possible to consider ∞ as an eigenvalue of P. Thus we can compactify the space of eigenvalues to obtain $\mathbb{P}^1 = \text{Spec } P \cup \{\infty\}$. No matter how simple it is, we thus have a geometric object.

Let us study another example. This time we take the Schrödinger operator with the Weierstrass elliptic function as its potential:

$$P = \left(\frac{d}{dx}\right)^2 - 2\wp(x+\varepsilon)\ , \tag{2.2}$$

where ε is a small constant to make P regular at $x = 0 \in \mathbb{C}$. If we consider the eigenvalue problem (2.1) for formal power series, then again we have $\mathbb{P}^1$ as the set of eigenvalues. Note that every eigenvalue of P has multiplicity 2. What kind of geometric object do we get if we resolve the multiplicity of P? Since there are two "points" sitting above every eigenvalue, we should get a double covering of the complex projective line with some ramification points. Therefore, we can expect a compact Riemann surface to show up as the set of resolved eigenvalues of an ordinary differential operator.

In order to see the ramified double covering associated with (2.2), we have to actually resolve the multiplicity. The standard way of doing so is to take the set of all differential operators that commute with the given one. So we define

$$B_P = \{P' \text{ a differential operator} \mid [P,\ P'] = 0\}\ ,$$

where the commutator means the difference of the operator products:

$$[P,\ P'] = P \cdot P' - P' \cdot P\ .$$

Note that B_P forms an associative $\mathbb{C}$-algebra. The problem of determining B_P for a given operator P was studied by G. Wallenberg [57] as early as 1903. He showed that for the operator P of (2.2), B_P is equal to the polynomial ring

$$B_P = \mathbb{C}[P,\ Q] \tag{2.3}$$

in two variables P and Q, where

$$Q = \left(\frac{d}{dx}\right)^3 - 3\wp(x+\varepsilon)\frac{d}{dx} - \frac{3}{2}\,\wp'(x+\varepsilon)\ .$$

A simple computation shows that

$$[P,\ Q] = \frac{1}{2}\,\wp''' - 6\wp\wp'.$$

Since $\wp$ satisfies the Weierstrass differential equation (1.4), Q commutes with P. To establish (2.3), let us assume that $\mathbb{C}[P,\ Q]$ contains all operators of order n or less that commute with P. Note that every operator commuting with P has constant leading coefficient. Take $R \in B_P$ of order $n+1 > 1$. Since 2 and 3 are coprime, there are nonnegative integers i and j such that $2i + 3j = n + 1$. Thus there is a constant $c \in \mathbb{C}$ such that the operator

$$R - cP^iQ^j$$

has order strictly less than $n+1$. Since this operator commutes with P, it must be in $\mathbb{C}[P,\ Q]$ by the induction hypothesis. Thus R is an element of $\mathbb{C}[P,\ Q]$. Be careful: this argument is not an induction! Actually, B_P has no element of order 1 (the Weierstrass gap), from which it results that the curve (2.4) below has genus 1. We refer to [35] Section 3 for a more general statement.

By a straightforward computation, we can show that P and Q satisfy a polynomial relation

$$Q^2 = P^3 - \frac{g_2}{4}P - \frac{g_3}{4}\ , \tag{2.4}$$

where g_2 and g_3 are the coefficients of the Weierstrass differential equation (1.4) that our $\wp$ satisfies. Therefore, as an abstract ring, we have

$$B_P \cong \mathbb{C}[X,\ Y]/(Y^2 - X^3 + \frac{g_2}{4}X + \frac{g_3}{4})\ ,$$

where (f) denotes the ideal of $\mathbb{C}[X,\ Y]$ generated by a polynomial f. In particular, B_P is a commutative ring! Certainly the commutativity of B_P is not obvious from its definition. We will come back to this point later.

Now let us resolve the multiplicity of eigenvalues of P. Since B_P is commutative, the simultaneous eigenvalue problem for all the operators in B_P makes sense, and by (2.3), it is equivalent to

$$\begin{cases} P\psi = \lambda\psi \\ Q\psi = \mu\psi \ . \end{cases} \tag{2.5}$$

Because of the polynomial relation (2.4), the eigenvalues of (2.5) satisfy the same relation

$$\mu^2 = \lambda^3 - \frac{g_2}{4}\lambda - \frac{g_3}{4} \ .$$

Conversely, for every (λ, μ) satisfying the above relation, there is a formal power series solution ψ of (2.5). To see this, let V_λ be the two-dimensional eigenspace of P belonging to the eigenvalue λ. Since $QP = PQ$, V_λ is a Q-invariant subspace, hence $Q : V_\lambda \to V_\lambda$ is represented by a two-by-two matrix whose eigenvalues are

$$\mu_\lambda^\pm = \pm\sqrt{\lambda^3 - \frac{g_2}{4}\lambda - \frac{g_3}{4}} \ .$$

From the Jordan canonical form of Q restricted to V_λ, we know that there is a solution $\psi \in V_\lambda$ of (2.5) for each $(\lambda, \mu_\lambda^\pm)$.

This means that the eigenvalue λ of P is labeled by the eigenvalue $\mu_\lambda^\pm$ of Q. So we define a complete elliptic curve

$$C = \left\{ (X : Y : Z) \in \mathbb{P}^2 \ \middle| \ Y^2 Z = X^3 - \frac{g_2}{4} X Z^2 - \frac{g_3}{4} Z^3 \right\} \tag{2.6}$$

as an algebraic subvariety of $\mathbb{P}^2$. This is the natural one-point completion of the affine elliptic curve C° of (1.7). Indeed, we can recover the affine curve by

$$C^\circ = \left\{ \left(\frac{X}{Z}, \frac{Y}{Z} \right) \in \mathbb{C}^2 \ \middle| \ \left(\frac{Y}{Z} \right)^2 = \left(\frac{X}{Z} \right)^3 - \frac{g_2}{4}\frac{X}{Z} - \frac{g_3}{4} \right\} \ .$$

The attached point $C \setminus C^\circ$ is the point $(0 : 1 : 0) \in \mathbb{P}^2$ at infinity. There is a natural holomorphic map

$$\pi : C \longrightarrow \mathbb{P}^1. \tag{2.7}$$

Let us define $\mathbb{P}^1$ as the union of $U_o \cong \mathbb{C}$ and $U_\infty \cong \mathbb{C}$ by the identification

$$U_o \ni \lambda \longleftrightarrow \frac{1}{\lambda} = \xi \in U_\infty .$$

Then for $\left(\frac{X}{Z}, \frac{Y}{Z}\right) \in C^\circ$ we assign $\pi\left(\frac{X}{Z}, \frac{Y}{Z}\right) = \frac{X}{Z} = \lambda$. For the point $(0 : 1 : 0) \in C$ at infinity, we assign $\pi(0 : 1 : 0) = \xi = 0 \in U_\infty$. Certainly, π is holomorphic on the affine part C°. For the points of

$$C \cap \{Y \neq 0\} = \left\{ \left(\frac{X}{Y}, \frac{Z}{Y} \right) \ \middle| \ \frac{Z}{Y} = \left(\frac{X}{Y} \right)^3 - \frac{g_2}{4}\frac{X}{Y}\left(\frac{Z}{Y} \right)^2 - \frac{g_3}{4}\left(\frac{Z}{Y} \right)^3 \right\} ,$$

the map π is defined by

$$\pi\left(\frac{X}{Y}, \frac{Z}{Y} \right) = \frac{Z}{X} = \xi \in U_\infty \ .$$

In terms of a local parameter z of C around $(0:1:0)$, the embedding $C \hookrightarrow \mathbb{P}^2$ is given by $X = \wp(z)$, $Y = \wp'(z)/2$ and $Z = 1$. Hence

$$\pi\left(\frac{X}{Y}, \frac{Z}{Y}\right) = \frac{Z}{X} = \frac{1}{\wp(z)} = z^2 - \frac{g_2}{20}z^6 - \frac{g_3}{28}z^8 + \frac{g_2^2}{600}z^{10} + \cdots ,$$

which is holomorphic in z. Therefore, π is holomorphic everywhere on C.

Except for the point $\lambda = \infty$ and the three zeroes of $\lambda^3 - \frac{g_2}{4}\lambda - \frac{g_3}{4} = 0$, the inverse image of λ by $\pi : C \to \mathbb{P}^1$ consists of two points $\{(\lambda, \mu_\lambda^+),\ (\lambda, \mu_\lambda^-)\}$. Thus we have obtained the desired double sheeted covering. To every simultaneous eigenvalue (λ, μ) of (2.5), π assigns the eigenvalue λ of P. Therefore, the covering map π is indeed the resolution of the multiplicity of eigenvalues of P.

This is an ideal place to explain the relation between the scheme theory of Grothendieck and the set of spectra of ordinary differential operators. Let us consider the polynomial ring $\mathbb{C}[P]$. The set of all prime ideals of $\mathbb{C}[P]$ is denoted by Spec $\mathbb{C}[P]$. (As a convention, by a prime ideal we mean a *proper* prime ideal.) Every prime ideal of this ring other than the zero ideal is of the form

$$\mathbb{C}[P](P - \lambda) = \{f(P) \cdot (P - \lambda) \mid f(P) \in \mathbb{C}[P]\}$$

for some $\lambda \in \mathbb{C}$. Now let ψ be a nontrivial solution of (2.1) and let I_λ be the set of all differential operators in $\mathbb{C}[P]$ that annihilate ψ. Obviously, I_λ is an ideal of $\mathbb{C}[P]$ and we have

$$I_\lambda = \mathbb{C}[P] \cdot (P - \lambda) ,$$

hence it is a *prime* ideal independent of the choice of the eigenfunction ψ. Therefore, we have

$$\text{Spec } P = \text{Spec } \mathbb{C}[P] .$$

Next, let us consider the commutative ring $B_P = \mathbb{C}[P,\ Q]$ of (2.3). We choose a nontrivial solution $\psi_{\lambda,\mu}$ of (2.5) for $(\lambda, \mu) \in C^\circ$ and define

$$I_{\lambda,\mu} = \{R \in B_P \mid R\psi_{\lambda,\mu} = 0\} .$$

We claim that $I_{\lambda,\mu}$ is a prime ideal of B_P. To see this, suppose $Q_1Q_2 \in I_{\lambda,\mu}$. Since $\psi_{\lambda,\mu}$ is a simultaneous eigenfunction for all elements of B_P, there are complex numbers α and β such that

$$\begin{cases} Q_1\psi_{\lambda,\mu} = \alpha\psi_{\lambda,\mu} \\ Q_2\psi_{\lambda,\mu} = \beta\psi_{\lambda,\mu}. \end{cases}$$

Therefore,

$$0 = Q_1Q_2\psi_{\lambda,\mu} = \alpha\beta\psi_{\lambda,\mu},$$

hence $\alpha = 0$ or $\beta = 0$. This means that either Q_1 or Q_2 must be an element of $I_{\lambda,\mu}$. In particular, this prime ideal has the form

$$I_{\lambda,\mu} = B_P \cdot (P - \lambda) + B_P \cdot (Q - \mu) .$$

Conversely, every prime ideal I of B_P other than the zero ideal and the ideals of the form $B_P \cdot (P - \lambda)$ or $B_P \cdot (Q - \mu)$ must be one of the $I_{\lambda,\mu}$s for some $(\lambda, \mu) \in C^\circ$. To establish this fact, first we note that $I \cap \mathbb{C}[P]$ is a prime ideal of $\mathbb{C}[P]$. Thus

$$I \cap \mathbb{C}[P] = \mathbb{C}[P] \cdot (P - \lambda)$$

for some $\lambda \in \mathbb{C}$, or if the intersection is 0, then

$$I \cap \mathbb{C}[Q] = \mathbb{C}[Q] \cdot (Q - \mu)$$

for some $\mu \in \mathbb{C}$. Since the argument is symmetric with respect to P and Q, we consider only the former situation. Therefore, $(P-\lambda) \in I$, and hence $B_P \cdot (P-\lambda) \subset I$. Since I is not equal to this ideal, there is a nonzero element $f(P-\lambda, Q) \in I \setminus B_P \cdot (P-\lambda)$. Expand the element as

$$f(P-\lambda, Q) = f_0(Q) + f_1(Q) \cdot (P-\lambda) + f_2(Q) \cdot (P-\lambda)^2 + \cdots .$$

Then $f_0(Q) \in I$. Since I is prime, one of the factors of $f_0(Q)$, say $(Q-\mu)$, must be in I. Therefore, we have

$$I = B_P \cdot (P-\lambda) + B_P \cdot (Q-\mu) .$$

This ideal is proper (i.e., $I \neq B_P$) if and only if (λ, μ) satisfies the polynomial relation (2.4).

Thus a pair of simultaneous eigenvalues (λ, μ) of the generators P and Q of B_P corresponds injectively to a prime ideal of the ring B_P. If we denote by $\widetilde{\text{Spec}}\ P$ the resolution $\left\{(\lambda, \mu_\lambda^\pm) \mid \lambda \in \text{Spec}\ P\right\}$ of the multiplicity of the spectra of P, then we have

$$\widetilde{\text{Spec}}\ P \subset \text{Spec}\ B_P ,$$

justifying the notation *Spec* of Grothendieck. Even though the prime ideals of the form $B_P \cdot (P-\lambda)$ and $B_P \cdot (Q-\mu)$ do not have corresponding geometric points on the affine curve C°, we *define* the affine algebraic scheme by $C^\circ = \text{Spec}\ B_P$ following Grothendieck's philosophy. The immediate advantage of using the new language is that we do not have to know the generators of B_P to define the curve corresponding to P. In 1905, I. Schur [50] proved that B_P is a commutative ring if P is an arbitrary ordinary differential operator of positive order (Theorem 2.2 below). Thus P gives rise to a canonically defined affine curve Spec B_P. The reason Spec B_P has dimension 1 is that we can always choose an element $Q \in B_P$ such that

$$\begin{cases} \dim_{\mathbb{C}} B_P / \mathbb{C}[P,\ Q] < +\infty \quad \text{and} \\ P \text{ and } Q \text{ satisfy a nontrivial polynomial relation } f(P, Q) = 0 . \end{cases}$$

Thus Spec B_P is a finite cover of the plane curve defined by the polynomial $f(X, Y)$. The existence of a polynomial relation in the general case is due to Burchnall and Chaundy [3]. Let us illustrate their argument here. So let P be a second order monic differential operator and Q be of order three. Note that every operator that commutes with a monic operator of positive order must have constant leading coefficient. Thus the leading coefficient of Q is a constant. Therefore, we can choose a constant $c_1 \in \mathbb{C}$ such that $Q^2 - c_1 P^3$ has order less than 6. Since $Q^2 - c_1 P^3$ commutes with P, its leading coefficient is again a constant. So we can find another constant c_2 so that $Q^2 - c_1 P^3 - c_2 PQ$ becomes an operator of order less than 5. We can continue this procedure until we obtain an operator

$$R = Q^2 - c_1 P^3 - c_2 PQ - c_3 P^2 - c_4 Q - c_5 P$$

of order less than 2. If $\operatorname{ord} R < 1$, then R must be a constant because it commutes with P. If $\operatorname{ord} R = 1$, then we can choose constants c_6 and c_7 such that $P - c_6 R^2 - c_7 R$ becomes a constant. In any case, we have obtained a non-trivial polynomial relation between P and Q.

In the previous example of Wallenberg, the affine curve Spec B_P has a natural one-point completion by a smooth point. Remarkably, This is always true for arbitrary P. To define the canonical completion of Spec B_P, we need the language of pseudo-differential operators. For the elliptic curve of (2.6), we can use

$$\sqrt{\frac{1}{X}} = \sqrt{\frac{1}{\wp(z)}} = z - \frac{g_2}{40}z^5 - \frac{g_3}{56}z^7 + \frac{g_2^2}{1920}z^9 + \cdots$$

as a local coordinate around the point at infinity. Since $X = P$ is a differential operator in our case, the local parameter $P^{-1/2}$ becomes a pseudo-differential operator. So let $E^{(n)}$ denote the set of all formal expressions

$$a_0(x)\partial^n + a_1(x)\partial^{n-1} + \cdots + a_n(x) + a_{n+1}(x)\partial^{-1} + \cdots$$

with coefficients in the ring of formal power series in x, where we use the abbreviation $\partial = d/dx$ throughout this paper. The above expression is called a *pseudo-differential operator* of order n if $a_0(x) \neq 0$, *monic* if $a_0(x) = 1$ and *normalized* if it is monic and $a_1(x) = 0$. Define

$$E = \bigcup_{n\in\mathbb{Z}} E^{(n)}. \tag{2.8}$$

This is the set of all formal ordinary pseudo-differential operators. The Leibniz rule

$$\partial^n \cdot f = \sum_{i=0}^{\infty} \binom{n}{i} f^{(i)}\partial^{n-i} \tag{2.9}$$

holds for all integers $n \in \mathbb{Z}$ if we define

$$\binom{n}{i} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-i+1)}{i!}.$$

(Note that we need $\binom{n}{i}$ for only nonnegative i in (2.9).) In particular, we have

$$\partial^{-1}\cdot f = f\partial^{-1} - f'\partial^{-2} + f''\partial^{-3} - \cdots ,$$

which is just integration by parts! Though we don't teach it in our calculus courses in this form, one has to note that integration by parts is merely a special case of the Leibniz rule. Applying $\partial^{-1}\cdot f$ to a function $g = g(x)$, we have

$$\begin{aligned}
\int f\cdot g dx &= f\int g dx - \int f'\Big(\int g dx\Big)dx \\
&= f\int g dx - f'\int\Big(\int g dx\Big)dx + \int f''\Big(\int\Big(\int g dx\Big)dx\Big)dx \\
&= f\int g dx - f'\int\Big(\int g dx\Big)dx + f''\int\Big(\int\Big(\int g dx\Big)dx\Big)dx \\
&\qquad - \int f'''\Big(\int\Big(\int\Big(\int g dx\Big)dx\Big)dx\Big)dx \\
&= \cdots .
\end{aligned}$$

The filtered set E has the structure of an associative algebra satisfying

$$E^{(m)}\cdot E^{(n)} \subset E^{(m+n)} .$$

Let $P \in E$ be a monic pseudo-differential operator of order n. Then there is a unique inverse of P that is a monic operator of order $-n$, and a unique monic n-th root of P that is an operator of order 1. The reason for introducing E is this larger class of algebraic operations we can do in E. Thus we can imagine E as something like an algebraically closed field and the set D of differential operators as its integer ring. This analogy also suggests that we should not look for the geometric meaning of a general element of E. Only integers (i.e., differential operators) have interesting geometric structures.

Lemma 2.1. *For every normalized first order pseudo-differential operator L, there is a monic zeroth order operator*

$$S = 1 + s_1(x)\partial^{-1} + s_2(x)\partial^{-2} + \cdots$$

such that

$$S^{-1} \cdot L \cdot S = \partial\ .$$

The proof is just a computation applying the Leibniz rule. By this Lemma, we have the following:

Theorem 2.2 (I. Schur)**.** *For every monic differential operator P of positive order, there is an invertible zeroth order operator T such that the set $A_P = T^{-1} \cdot B_P \cdot T$ consists of pseudo-differential operators with constant coefficients:*

$$T^{-1} \cdot B_P \cdot T = A_P \subset \mathbb{C}((\partial^{-1}))\ ,$$

where $\mathbb{C}((\partial^{-1})) = \mathbb{C}[\partial] + \mathbb{C}[[\partial^{-1}]]$ denotes the ring of pseudo-differential operators with constant coefficients. Consequently,

$$B_P \subset T \cdot \mathbb{C}((\partial^{-1})) \cdot T^{-1} = \mathbb{C}\big(\big(P^{-1/n}\big)\big)\ ,$$

and hence B_P is a commutative $\mathbb{C}$-algebra.

Proof. Let $n > 0$ be the order of P. Since P is monic, it has a unique n-th root L that is a monic first order operator. If we write

$$P^{\frac{1}{n}} = L = \partial + a_1(x) + a_2(x)\partial^{-1} + a_3(x)\partial^{-2} + \cdots\ ,$$

then $e^{a(x)} \cdot L \cdot e^{-a(x)}$ is a normalized operator, where

$$a(x) = \int_0^x a_1(x)dx \in \mathbb{C}[[x]]\ .$$

Then by Lemma 2.1, there is a monic zeroth order operator S such that

$$S^{-1} \cdot e^{a(x)} \cdot L \cdot e^{-a(x)} \cdot S = \partial\ .$$

Define $T = e^{-a(x)} \cdot S$. Since every element $Q \in B_P$ commutes with L, $T^{-1} \cdot Q \cdot T \in T^{-1} \cdot B_P \cdot T$ commutes with ∂. Note that

$$\Big[\partial,\ \sum f_n(x)\partial^n\Big] = \sum f_n(x)'\partial^n.$$

Therefore, every element of $A_P = T^{-1} \cdot B_P \cdot T$ is an operator with constant coefficients. This completes the proof. □

If we define the commutant of P in the larger ring E, then we have

$$\widetilde{B}_P = \{P' \in E \mid [P, P'] = 0\} = \mathbb{C}((L^{-1}))\ .$$

Certainly this ring, which is actually a field, does not carry any geometric information on P. This is why we work in the subring $B_P \subset D$ of differential operators. However, the fact that the larger commutant $\widetilde{B}_P$ is always isomorphic to a field of formal Laurent series suggests that the behavior of the algebraic curve Spec P

around the point at infinity is the same for all P. The example of Wallenberg indicates that the operator $L^{-1} = P^{-1/n}$ gives a local coordinate at infinity. Therefore, we can complete the affine curve Spec P by attaching a "point" defined by $L^{-1} = 0$.

In order to make the completion algebraically, let us introduce a new variable

$$z = \partial^{-1} . \tag{2.10}$$

Since the adjoint map

$$ad(T) : E \ni R \longmapsto T \cdot R \cdot T^{-1} \in E$$

preserves the order of R, every element of $A_P = T^{-1} \cdot B_P \cdot T$ has non-negative order. This means that the leading term of any element of A_P is equal to $c \cdot z^{-m}$ for some $m \geq 0$. Since A_P is a $\mathbb{C}$-algebra with identity element, we conclude that

$$A_P \cap \mathbb{C}[[z]] = \mathbb{C} . \tag{2.11}$$

Recall that $C^\circ = \text{Spec } B_P = \text{Spec } A_P$ is an algebraic variety such that the set of all regular functions on it coincides with the algebra A_P. So let us define a complete algebraic curve C_P, which we call the *spectral curve of* P, by

$$C_P = \text{Spec } A_P \cup \text{Spec } \mathbb{C}[[z]] = \text{Spec } A_P \cup \{z = 0\} . \tag{2.12}$$

The relation (2.11) implies that every regular function on Spec A_P that is also regular at $z = 0$ is a constant. Therefore, C_P must be a complete algebraic curve (because it cannot have any missing point), and the attached point $p = \{z = 0\}$ is a nonsingular point of C_P (because it is defined by a local coordinate z). This is the complete algebraic curve we wanted to construct from P. The above construction is lengthy, but it is easily defined as a *projective scheme*

$$C_P = \text{Proj}\big(gr(B_P)\big) ,$$

where the graded algebra $gr(B_P)$ of B_P is defined by the natural filtration of $B_P \subset D$ by the order of operators [38].

The resolution of the multiplicity of eigenvalues of P is a holomorphic map

$$\pi : C_P \longrightarrow \mathbb{P}^1$$

which assigns the point at infinity $\infty \in \mathbb{P}^1$ to $p \in C_P$ and an eigenvalue *without* label to an eigenvalue *with* label. On the affine part, the inclusion map $\mathbb{C}[P] \subset B_P$ induces the map

$$\pi : \text{Spec } B_P \longrightarrow \text{Spec } \mathbb{C}[P] ,$$

which assigns a prime ideal $I \cap \mathbb{C}[P]$ of $\mathbb{C}[P]$ to a prime ideal I of B_P. If B_P contains sufficiently many elements so that we can resolve the multiplicity of P completely, then π is an n-sheeted ramified covering, where n is the order of P. Generically, however, we have always $B_P = \mathbb{C}[P]$. For example, the differential operator we started with in this section (the harmonic oscillator) has this property. Thus by using differential operators alone, we can never resolve the multiplicity of a harmonic oscillator. This is why we have to apply a magnetic field to separate the degeneracy of the spectrum of a hydrogen atom!

The correspondence

$$P \longmapsto C_P \tag{2.13}$$

is therefore not one-to-one. Actually, every generic operator corresponds to the projective line $\mathbb{P}^1$. There are two different directions we can go now. One is to study the inverse image of the map (2.13). Since the curve C_P contains information about the spectrum of P, the inverse image should give *isospectral deformations* of P. Section 4 will be devoted to this topic. There, we will define the KP system as the equation for the "universal" family of isospectral deformations of arbitrary linear ordinary differential operators. The solutions of this system will tell us more about the correspondence (2.13).

The other direction that (2.13) suggests is to find all geometric information we are missing now to make this correspondence one-to-one. Since the spectral curve C_P does not determine the operator P, we need more structure on the curve to construct a unique differential operator out of it. This is the topic we deal with in the next section. Since our goal is to identify an operator from its spectral structure, we may call this inverse correspondence the *geometric inverse scattering method*.

3. Grassmannians and the geometric inverse scattering

In the previous section, we defined the spectral curve C_P for each differential operator P. The fact that the correspondence $P \longmapsto C_P$ is not one-to-one suggests that we are still missing some geometric information that P possesses. Our goal in this section is to identify the missing information of P.

Recall that the curve C_P is the set of "resolved" eigenvalues of P. The second natural geometric object we can assign to P is the *eigenvector bundle*

$$\mathcal{F} = \bigcup_{\lambda \in C_P} \{\psi \mid P\psi = \lambda\psi\}$$

on C_P. We can indeed define a vector bundle $\mathcal{F}$ on the affine part C_P° of the spectral curve by this procedure. However, this bundle is trivial (isomorphic to the product bundle) because every complex holomorphic vector bundle on an affine curve is trivial. Therefore, the global information carried by eigenfunctions of P is concentrated around the point at infinity. In other words, we have to solve the eigenvalue problem (2.1) asymptotically at $\lambda = \infty$ to obtain more information of P.

This is the path Krichever [24] took to define his *algebraic spectral data.* There are infinitely many different ways of extending the trivial bundle on the affine part to a vector bundle on the whole spectral curve. Therefore, one has to be careful to choose a correct interpretation of the eigenvalue problem at infinity. What we see in [24] is a beautiful treatment of the asymptotic behavior of the operator P at infinity, $\lambda = \infty$.

In this section, however, we will take a totally different, purely algebraic approach to this problem. The advantage of our method is that we can generalize the classification theorem of Krichever. The main theorem of [24] is the geometric classification of commutative algebras of ordinary differential operators in *generic position.* By introducing the algebraic method, we can establish a classification theorem for *all* commutative algebras of ordinary differential operators [34] [35]. In our treatment, what corresponds to the asymptotic expansion of Krichever is a *functor* between certain categories.

Although it must sound like a deus ex machina, let us start by defining the *Sato Grassmannian.* This Grassmannian appears as a cohomological key in the connection between the geometry of algebraic curves and the asymptotic behavior of ordinary differential operators. We introduced in Section 2 the ring E of pseudo-differential operators to study the completion of spectral curves. The ring E has two important subalgebras: the algebra D of differential operators and the algebra $E^{(-1)}$ of pseudo-differential operators of order at most -1, i.e., integral operators. We have a natural direct sum decomposition

$$E = D \oplus E^{(-1)} \tag{3.1}$$

as a module. Now we define

Definition 3.1. The Sato Grassmannian SG^+ is the set of right D-submodules $J \subset E$ of E such that

$$E = J \oplus E^{(-1)} .$$

Let $B_P = \{Q \in D \mid [P,\ Q] = 0\}$ be the commutant of $P \in D$ in the ring D. There is a zeroth order operator $T \in E$ such that the ring $T^{-1} \cdot B_P \cdot T = A_P$ consists of operators with constant coefficients, by Theorem 2.2. Certainly, $T^{-1}D = J$ is a point of the Sato Grassmannian SG^+. Since $B_P \subset D$, D is a left B_P-module, and hence $T^{-1}D$ is a left A_P-module.

In order to see the connection between this Grassmannian and algebraic curves, let us introduce a representation of the algebra E. We take a maximal left ideal Ex of E generated by $x \in E$. Define

$$V = E/Ex . \tag{3.2}$$

It is obvious from the definition that V is isomorphic as a vector space to the set $\mathbb{C}((\partial^{-1}))$ of pseudo-differential operators with constant coefficients. As in (2.10), we use the variable z. Then

$$V = E/Ex \cong \mathbb{C}((z)) .$$

The algebra E acts on V from the left by the natural multiplication. We have a direct sum decomposition

$$V = V_+ \oplus V_- \tag{3.3}$$

coming from (3.1), which is just

$$\mathbb{C}((z)) = \mathbb{C}[z^{-1}] \oplus \mathbb{C}[[z]]z .$$

For every vector subspace W of V, we define a map

$$\gamma_W : W \longrightarrow V/V_- \cong V_+ \tag{3.4}$$

by composing the inclusion $W \hookrightarrow V$ and the natural projection $V \to V/V_-$. This map gives us a way of comparing W with a fixed subspace $V_+ = \mathbb{C}[z^{-1}]$ of V. For example, if γ_W has no kernel nor cokernel, then we can say that W and V_+ have the same *dimension.* If γ_W is Fredholm, i.e., if it has finite-dimensional kernel and cokernel, then W and V_+ differ by some finite dimension. This motivates us to define the *Grassmannian* in this algebraic setting.

Definition 3.2. The Grassmannian $Gr(\mu)$ of *index* μ is the set of all vector subspaces W of V such that the natural map γ_W is Fredholm of index μ:

$$\dim\ \mathrm{Ker}\ \gamma_W - \dim\ \mathrm{Coker}\ \gamma_W = \mu .$$

The big-cell of the index zero Grassmannian $Gr(0)$ is defined by

$$Gr^+(0) = \{W \subset V \mid \text{Ker } \gamma_W = \text{Coker } \gamma_W = 0\} \ .$$

The natural projection

$$\rho : E \longrightarrow V = E/Ex$$

induces a map

$$SG^+ \ni J \longmapsto \rho(J) \in Gr^+(0) \tag{3.5}$$

in an obvious way. The following theorem is due to Sato [48] [49].

Theorem 3.3. *The natural map (3.5) is a bijection between the Sato Grassmannian and the big-cell of the index zero Grassmannian. Moreover, there is a bijection between the group G_- of monic zeroth order pseudo-differential operators and the Sato Grassmannian given by*

$$\sigma : G_- \ni S \longmapsto S^{-1}D = J \in SG^+.$$

For a proof, we refer to Appendix of [35]. If we choose a zeroth order operator $T \in E$ for every given $P \in D$ such that $T^{-1} \cdot P \cdot T \in \mathbb{C}((\partial^{-1}))$, then we have a pair (A_P, W_T) of vector subspaces of V that are defined by

$$\begin{cases} A_P = T^{-1} \cdot B_P \cdot T \\ W_T = \rho(T^{-1}D) = T^{-1}\rho(D) = T^{-1}V_+ \ . \end{cases} \tag{3.6}$$

This pair satisfies the following conditions: W_T satisfies the Fredholm condition; i.e., there is a $\mu \in \mathbb{Z}$ such that

$$W_T \in Gr(\mu) \ . \tag{3.7}$$

A_P is a $\mathbb{C}$-subalgebra of $\mathbb{C}((z))$ stabilizing W_T; i.e.,

$$A_P \cdot W_T \subset W_T \ . \tag{3.8}$$

A_P is nontrivial in the sense that

$$A_P \setminus \mathbb{C} \neq \emptyset \ . \tag{3.9}$$

From the construction of (3.6), W_T is actually a point of the big-cell of the Grassmannian of index zero. We call a pair (A, W) satisfying the above three conditions a *Schur pair.* The pair (A_P, W_T) is not uniquely determined by the operator P. It really depends on the choice of T. But it is easy to analyze the ambiguity we have here. Let us call an invertible zeroth order operator $T \in E$ *admissible* if it preserves operators with constant coefficients $\mathbb{C}((\partial^{-1}))$ in E under conjugation; i.e., if

$$T \cdot \partial \cdot T^{-1} \in \mathbb{C}((\partial^{-1})) \ . \tag{3.10}$$

We denote by G_a the group of all admissible operators. This group acts on the set of all Schur pairs by

$$T \cdot (A, W) = (T \cdot A \cdot T^{-1}, TW) \ , \quad G_a \ni T \ .$$

We say that (A, W) and (A', W') are *isomorphic* if there is an admissible operator $T \in G_a$ such that $T \cdot (A, W) = (A', W')$. From these definitions, it is obvious that a monic linear differential operator P determines a unique isomorphism class of Schur pairs.

For a Schur pair (A, W), W is an A-module because of condition (3.8). Since it is torsion-free, we can define the *rank* of W as an A-module. We call it the rank of (A, W). It can be computed from A alone by

$$\operatorname{rank}(A, W) = G.C.D.\{\operatorname{ord} a \mid a \in A\} ,$$

where the order of an element $a \in A \subset \mathbb{C}((\partial^{-1}))$ means the order of a as a pseudo-differential operator, or equivalently, the pole order of the corresponding formal Laurent series $a = a(z) \in \mathbb{C}((z))$ at the origin $z = 0$.

Definition 3.4. The set of objects of the category $\mathcal{S}$ of Schur pairs consists of all pairs (A, W) satisfying the conditions (3.7)–(3.9). A morphism $T : (A, W) \to (A', W')$ is a pair of twisted inclusions

$$\begin{cases} T \cdot A \cdot T^{-1} \hookrightarrow A' \\ T^{-1} W' \hookrightarrow W \end{cases}$$

defined by an admissible operator $T \in G_a$.

Our task is to find a category of geometric data and to construct a faithful functor from the category $\mathcal{S}$ to this geometric category. Since a differential operator P gives rise to an isomorphism class of Schur pairs, it corresponds to an isomorphism class of the geometric data through this functor.

Definition 3.5. The set of objects of the category $\mathcal{Q}$ of quintuples consists of geometric data $(C, p, \pi, \mathcal{F}, \phi)$ of rank r for an arbitrary positive integer r, where

(1) C is a reduced irreducible complete algebraic curve, which may be singular.
(2) $p \in C$ is a smooth point of C.
(3) $\pi : U_o \to U_p$ is a local r-sheeted covering ramified at p, where U_o is an open disk of $\mathbb{C}$ centered at the origin 0 and U_p is an open neighborhood of C around p. To be precise, U_o is the formal completion of the affine line $\mathbb{A}^1_{\mathbb{C}}$ at the origin and U_p is the formal completion of the curve C along the divisor p.
(4) $\mathcal{F}$ is a torsion-free sheaf of $\mathcal{O}_C$-modules on C of rank r satisfying

$$\dim H^0(C, \mathcal{F}) - \dim H^1(C, \mathcal{F}) = \mu$$

for some integer μ, which we call the *index* of the quintuple. If C is a nonsingular curve of genus g, then this simply means that $\mathcal{F}$ is a vector bundle on C of rank r and degree $\mu + r(g-1)$.
(5) $\phi : \mathcal{F}|_{U_p} \xrightarrow{\sim} \pi_* \mathcal{O}_{U_o}(-1)$ is an $\mathcal{O}_{U_p}$-module isomorphism between the restricted vector bundle $\mathcal{F}|_{U_p}$ of $\mathcal{F}$ on U_p and the direct image $\pi_* \mathcal{O}_{U_o}(-1)$ of the twisted structure sheaf $\mathcal{O}_{U_o}(-1)$ of the disk U_o via the holomorphic map π. Since π is an r-sheeted covering, $\pi_* \mathcal{O}_{U_o}(-1)$ is the trivial vector bundle of rank r on U_p. Therefore, ϕ gives a local trivialization of the vector bundle $\mathcal{F}$ near the distinguished point p.

A morphism between quintuples

$$(\alpha, h) : (C', p', \pi', \mathcal{F}', \phi') \longrightarrow (C, p, \pi, \mathcal{F}, \phi)$$

is a pair (α, h) consisting of a holomorphic mapping $\alpha : C' \longrightarrow C$ of degree $n \in \mathbb{N}$ and a sheaf homomorphism $h : \alpha_* \mathcal{F}' \longrightarrow \mathcal{F}$ such that $\alpha^{-1}(p) = n \cdot p'$, $\pi = \alpha \circ \pi'$

and

$$\begin{array}{ccc} \alpha_*\mathcal{F}'|_{U_{p'}} & \xrightarrow{h} & \mathcal{F}|_{U_p} \\ {\scriptstyle \alpha(\phi')}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi} \\ \alpha_*\pi'_*\mathcal{O}_{U_o}(-1) & = & \pi_*\mathcal{O}_{U_o}(-1) \end{array}$$

locally around $p \in C$.

Remark. We allow arbitrary algebraic curves and torsion-free sheaves on them as objects of the category. However, the morphism $\alpha : C' \longrightarrow C$ of algebraic curves in a morphism of quintuples (α, h) is not arbitrary. We are requiring that there be a point $p \in C$ such that $\alpha^{-1}(p) = n \cdot p'$ for some $p' \in C'$. This is indeed a strong restriction on α. This asymmetry motivates us to generalize the entire theory in Section 6.

Theorem 3.6 ([35])**.** *There is a fully faithful contravariant functor*

$$\chi : \mathcal{Q} \xrightarrow{\sim} \mathcal{S}$$

which makes these categories anti-equivalent. Under this functor, a quintuple of rank r and index μ corresponds to a Schur pair of the same rank and index.

Since the functor χ recovers the original construction of Krichever [23] for the generic rank one case, we call it the *Krichever Functor.* The functor is described as follows: let z be a coordinate of U_o. We can represent any meromorphic function on C as a Laurent series in z by pulling it back to U_o via the holomorphic map π. Similarly, the local trivialization ϕ translates a meromorphic section of $\mathcal{F}$ into a Laurent series in z. For example, let us assume that the map π is given by $y = z^r$ in terms of a local coordinate y of U_p. Every meromorphic function f on C has a local expansion $f = f(y)$ and hence it gives a Laurent series

$$\pi^* f = \pi^*\left(f(y)\right) = f(z^r)$$

in z. For a meromorphic section s of $\mathcal{F}$, we can choose a basis for $\mathcal{F}|_{U_p}$ such that it has a local expansion $s = (s_1(y), \cdots, s_r(y))$ in terms of meromorphic functions on U_p. Let us choose ϕ as follows:

$$\begin{aligned} \pi_*(1) &= \phi(1, 0, 0, \cdots, 0) \\ \pi_*(z) &= \phi(0, 1, 0, \cdots, 0) \\ &\cdots \\ \pi_*(z^{r-1}) &= \phi(0, 0, \cdots, 0, 1) \ . \end{aligned}$$

Then s becomes a Laurent series in z by

$$\phi(s) = \phi\left(s_1(y), \cdots, s_r(y)\right) = \pi_*\left(s_1(z^r) + s_2(z^r)z + \cdots + s_r(z^r)z^{r-1}\right) \ .$$

So we can define a Schur pair (A, W) from a quintuple $(C, p, \pi, \mathcal{F}, \phi)$ by setting

(3.11)

$$\begin{cases} A = \pi^*\left(\{\text{holomorphic functions on } C \setminus \{p\}\ \}\right) = \pi^*\big(H^0(C \setminus \{p\}, \mathcal{O}_C)\big) \\ W = \phi\left(\{\text{holomorphic sections of } \mathcal{F} \text{ on } C \setminus \{p\}\ \}\right) = \phi\big(H^0(C \setminus \{p\}, \mathcal{F})\big) \ . \end{cases}$$

Now our functor χ is defined by

$$\chi(C, p, \pi, \mathcal{F}, \phi) = (A, W) \ ,$$

which is essentially the zeroth cohomology functor $H^0(C \setminus \{p\}, \bullet)$. Note that both A and W of (3.11) are subspaces of $V = \mathbb{C}((z))$. Since there are more holomorphic functions than the constants on a punctured curve $C \setminus \{p\}$, A satisfies (3.9). The condition (3.8) simply says that the product of a holomorphic function and a holomorphic section of a vector bundle is another holomorphic section of the bundle. Thus we have $A \cdot W \subset W$. The Fredholm condition of W comes from the following:

Lemma 3.7 ([35]). *Under the correspondence of (3.11), we have canonical isomorphisms*

$$\begin{cases} \text{Ker } \gamma_W \cong H^0(C, \mathcal{F}) \\ \text{Coker } \gamma_W \cong H^1(C, \mathcal{F}) \ . \end{cases}$$

Since C is a complete curve, the cohomology groups have finite dimension, hence W satisfies (3.7). To see Lemma 3.7, we use the Čech cohomology associated with the Stein covering (to be precise, the *formal* covering)

$$C = (C \setminus \{p\}) \cup U_p$$

of the curve C. Then

$$\text{Ker } \gamma_W = W \cap V_-$$

is the set of holomorphic sections of $\mathcal{F}$ on $C \setminus \{p\}$ that are also holomorphic on U_p, because of the identification

$$\begin{aligned} \phi\left(H^0(U_p, \mathcal{F})\right) &\cong H^0(U_p, \pi_* \mathcal{O}_{U_o}(-1)) \\ &\cong H^0(U_o, \mathcal{O}_{U_o}(-1)) \\ &= \mathbb{C}[[z]]z \\ &= V_- \ . \end{aligned}$$

Thus we have Ker $\gamma_W \cong H^0(C, \mathcal{F})$. Similarly, let $U \subset C$ be an affine open subset of C containing p such that $\mathcal{F}$ is locally free on U. Then we have (see [35])

$$\begin{aligned} H^1(C, \mathcal{F}) &\simeq H^0(U \setminus \{p\}, \mathcal{F}) \big/ \big(H^0(C \setminus \{p\}, \mathcal{F}) + H^0(U, \mathcal{F})\big) \\ &\simeq H^0(U_p \setminus \{p\}, \mathcal{F}_{U_p}) \big/ \big(H^0(C \setminus \{p\}, \mathcal{F}) + H^0(U_p, \mathcal{F}_{U_p})\big) \\ &\simeq \phi\big(H^0(U_p \setminus \{p\}, \mathcal{F}_{U_p})\big) \big/ \phi\big(H^0(C \setminus \{p\}, \mathcal{F}) + H^0(U_p, \mathcal{F}_{U_p})\big) \\ &= \mathbb{C}((z)) \big/ (W + V_-\big) \\ &\simeq \text{Coker}\, \gamma_W \ . \end{aligned}$$

The construction of a geometric quintuple from a Schur pair is more technical. So we refer to [35] for details. Roughly speaking, we define C as the one-point completion of Spec(A) and $\mathcal{F}$ as an extension of the vector bundle $W^\sim$ on Spec(A) to the entire curve C. This is done by using a natural filtration on $V = \mathbb{C}((z))$. The inclusion maps $A \subset V$ and $W \subset V$ determine π and ϕ, respectively.

As an application of the equivalence of the categories, one can establish the complete solution to the classification problem of all commutative algebras of ordinary differential operators, which has been open since the time of Wallenberg and Schur:

Theorem 3.8 ([35])**.** *There is a bijective correspondence between commutative algebras consisting of ordinary differential operators (containing a monic operator) and isomorphism classes of geometric quintuples* $(C, p, \pi, \mathcal{F}, \phi)$ *such that*

$$H^0(C, \mathcal{F}) = H^1(C, \mathcal{F}) = 0 \ .$$

Remark. In this correspondence, we identify a commutative algebra $B \subset D$ and fBf^{-1} for an arbitrary invertible function f.

Proof. Let B be a commutative subalgebra of D with a monic element P of order $n > 0$. Take an operator T of Theorem 2.2. Then (3.6) gives a Schur pair (A_P, W_T). A different choice of $P \in B$ results in an isomorphic Schur pair. Since W_T is a point of the big-cell, the quintuple $(C, p, \pi, \mathcal{F}, \phi)$ corresponding to (A_P, W_T) satisfies the cohomology vanishing condition.

Conversely, let us start with a quintuple with vanishing cohomology groups. It corresponds to a Schur pair (A, W) with $W \in Gr^+(0)$. Thus by Theorem 3.3, there is a monic zeroth order operator S such that

$$W = S^{-1}V_+ \ .$$

Consider A as a subring of $\mathbb{C}((\partial^{-1}))$ and let $B = S \cdot A \cdot S^{-1}$. Since the Schur pair satisfies $A \cdot W \subset W$, we have

$$B \cdot V_+ = S \cdot A \cdot S^{-1} \cdot S \cdot W = S \cdot A \cdot W \subset S \cdot W = V_+ \ .$$

Then B is a commutative subalgebra of D because of the following: □

Lemma 3.9 (Sato)**.** *A pseudo-differential operator* $P \in E$ *is a differential operator if and only if it preserves the base point* V_+ *of the Grassmannian; i.e.,*

$$P \cdot V_+ \subset V_+ \ .$$

For a proof, see Appendix of [35].

We started with an operator $P \in D$. It defines a commutative subalgebra $B_P \subset D$. The above theorem tells us that we can recover the algebra B_P completely from its *geometric spectral data* $(C, p, \pi, \mathcal{F}, \phi)$.

To pinpoint the initial operator P from the geometric data, we need to consider a morphism between quintuples. The operator P determines another commutative algebra $\mathbb{C}[P]$, which corresponds to a quintuple $(\mathbb{P}^1, \infty, \pi', \mathcal{F}', \phi')$ through the inverse of the Krichever functor. The natural inclusion

$$\mathbb{C}[P] \hookrightarrow B_P$$

defines a morphism

$$(C, p, \pi, \mathcal{F}, \phi) \longrightarrow (\mathbb{P}^1, \infty, \pi', \mathcal{F}', \phi') \ ,$$

which is essentially the resolution of the multiplicity of the eigenvalues of P we have discussed in Section 2. Thus a particular operator in a commutative subalgebra $B \subset D$ corresponds to a choice of a holomorphic map of C onto $\mathbb{P}^1$. This is the geometric information that a single ordinary differential operator has—it is far richer than what we ever expected!

In the above theory, which we call *geometric inverse scattering theory*, we do not need any genericity assumption on the curve C that was necessary in [24]. This is one of the advantages of using a functorial method rather than looking for only a set-theoretical bijection.

4. Iso-spectral deformations and the KP system

In this section, we define iso-spectral deformations of arbitrary ordinary differential operators and introduce the KP system as a defining equation for a universal family of iso-spectral deformations. We then establish the unique solvability of the KP system by using infinite-dimensional geometric techniques.

Since we are not imposing any boundary conditions on the eigenvalue problem $P\psi = \lambda\psi$, P has continuous spectrum. Therefore, we have to be more specific when we say deformations 'preserving' the spectrum. Consider an analytic family

$$\{P(t) \mid t \in M\}$$

of operators, where the parameter space M is an open domain of $\mathbb{C}^N$ and $P(t)$ is an ordinary differential operator of the form

$$P(t) = \partial^n + a_1(x,t)\partial^{n-1} + \cdots + a_n(x,t)$$

depending on both $x \in \mathbb{C}$ and $t = (t_1, t_2, \cdots, t_N) \in M \subset \mathbb{C}^N$ analytically.

Definition 4.1. We say $\{P(t) \mid t \in M\}$ is a family of iso-spectral deformations if there exist ordinary differential operators $Q_1(t)$, $Q_2(t)$, $\cdots$, $Q_N(t)$ depending on the parameter $t \in M$ analytically such that the following system of equations has a nontrivial solution $\psi(x,t;\lambda)$ for every eigenvalue λ of $P(t)$:

$$\begin{cases} P(t)\psi(x,t;\lambda) = \lambda\psi(x,t;\lambda) \\ \frac{\partial}{\partial t_1}\psi(x,t;\lambda) = Q_1(t)\psi(x,t;\lambda) \\ \frac{\partial}{\partial t_2}\psi(x,t;\lambda) = Q_2(t)\psi(x,t;\lambda) \\ \qquad\cdots \\ \frac{\partial}{\partial t_N}\psi(x,t;\lambda) = Q_N(t)\psi(x,t;\lambda)\ . \end{cases} \tag{4.1}$$

The point here is that the eigenvalue λ in the first equation does not depend on the parameter t, i.e., it is *preserved.*

Let us compute the compatibility conditions of the system (4.1):

$$\begin{aligned} 0 &= \frac{\partial}{\partial t_i}\left(P(t)\psi(x,t;\lambda) - \lambda\psi(x,t;\lambda)\right) \\ &= \frac{\partial}{\partial t_i}\left(P(t)\psi(x,t;\lambda)\right) - \lambda\frac{\partial}{\partial t_i}\psi(x,t;\lambda) \\ &= \frac{\partial}{\partial t_i}P(t)\cdot\psi(x,t;\lambda) + P(t)\frac{\partial}{\partial t_i}\psi(x,t;\lambda) - \lambda Q_i(t)\psi(x,t;\lambda) \\ &= \frac{\partial}{\partial t_i}P(t)\cdot\psi(x,t;\lambda) + P(t)Q_i(t)\psi(x,t;\lambda) - Q_i(t)\lambda\psi(x,t;\lambda) \\ &= \frac{\partial}{\partial t_i}P(t)\cdot\psi(x,t;\lambda) + P(t)Q_i(t)\psi(x,t;\lambda) - Q_i(t)P(t)\psi(x,t;\lambda) \\ &= \left(\frac{\partial}{\partial t_i}P(t) - [Q_i(t),\ P(t)]\right)\psi(x,t;\lambda)\ . \end{aligned}$$

For every fixed $t \in M$, the eigenfunctions $\psi(x,t;\lambda)$ are linearly independent for distinct eigenvalues $\lambda \in \mathbb{C}$. Since $\frac{\partial}{\partial t_i}P(t) - [Q_i(t),\ P(t)]$ is an ordinary differential operator of finite order, it has only finitely many independent solutions. Therefore,

$$\frac{\partial}{\partial t_i}P(t) = [Q_i(t),\ P(t)]\ . \tag{4.2}$$

Similarly, $\frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}\psi = \frac{\partial}{\partial t_j}\frac{\partial}{\partial t_i}\psi$ gives

$$\frac{\partial}{\partial t_i}Q_j(t) - \frac{\partial}{\partial t_j}Q_i(t) = [Q_i(t),\ Q_j(t)]\ . \tag{4.3}$$

Eq.(4.2) is called the *Lax equation* [26]. The system of equations (4.2) and (4.3) is equivalent to the condition that Eq.(4.1) has a nontrivial solution for every $\lambda \in \mathbb{C}$. Therefore, finding a family $P(t)$ of iso-spectral deformations of a given operator $P(0)$ is equivalent to finding a solution of the Lax equation (4.2) for differential operators $Q_i(t)$ satisfying (4.3) together with the initial condition $P(t)|_{t=0} = P(0)$.

The simplest example of an iso-spectral deformation is the spatial translation $P(x, t_1) = P(x + t_1)$. Since

$$\frac{\partial}{\partial t_1}P(x+t_1) = [\partial,\ P(x+t_1)]\ ,$$

we have $Q_1(t) = \partial$ in this case.

For simplicity, let us assume that $P(t)$ is normalized from now on. (It is well known that every ordinary differential operator can be normalized by a suitable coordinate change and conjugation by an invertible function.) We have noted in Section 2 that $P(t)$ has a normalized n-th root

$$L(t) = \partial + u_2(x,t)\partial^{-1} + u_3(x,t)\partial^{-2} + \cdots\ . \tag{4.4}$$

Since $L(t)^n = P(t)$, Eq.(4.2) is equivalent to

$$\frac{\partial}{\partial t_i}L(t) = [Q_i(t),\ L(t)]\ . \tag{4.5}$$

Here the left hand side of (4.5) is a pseudo-differential operator of order at most -1. Therefore, the differential operator $Q_i(t)$ must satisfy

$$[Q_i(t),\ L(t)] \in E^{(-1)}\ .$$

Lemma 4.2 ([12]). *Let*

$$L = \partial + u_2(x)\partial^{-1} + u_3(x)\partial^{-2} + \cdots$$

be an arbitrary normalized pseudo-differential operator of order 1. *Then*

$$F_L = \left\{Q \in D \ \middle|\ [Q, L] \in E^{(-1)}\right\}$$

coincides with the $\mathbb{C}$*-linear space generated by the operators* $(L^m)_+$, $m \geq 0$. *Here, we decompose* L^m *into the sum*

$$L^m = (L^m)_+ + (L^m)_-$$

of a differential operator $(L^m)_+ \in D$ *and an integral operator* $(L^m)_- \in E^{(-1)}$ *according to the direct sum decomposition of (3.1).*

Since $[L^m,\ L] = \left[L^m_+ + L^m_-,\ L\right] = 0$, we have

$$\left[L^m_+,\ L\right] = -\left[L^m_-,\ L\right] \in E^{(-1)}\ .$$

Conversely, let $Q \in F_L$ be an element of order m. The condition $[Q, L] \in E^{(-1)}$ implies that the leading coefficient of Q is a constant, say $c \in \mathbb{C}$. Since L^m_+ is monic, the linear combination $Q - cL^m_+$ has order less than m. Since $[Q - cL^m_+, L] \in E^{(-1)}$, the lemma follows by induction on m.

Definition 4.3. The following system of nonlinear partial differential equations on the coefficients of an operator

$$L(t) = \partial + u_2(x,t)\partial^{-1} + u_3(x,t)\partial^{-2} + \cdots$$

is called the total hierarchy of the *Kadomtsev-Petviashvili equations*, or simply the *KP system*:

$$\frac{\partial}{\partial t_i} L(t) = \left[L^i(t)_+,\ L(t)\right], \quad i = 1,\ 2,\ 3,\ \cdots .$$

Note that the KP system corresponds to the first condition (4.2) for the compatibility of the system (4.1). In order for $L(t)$ to give an iso-spectral deformation, we have to check the other condition (4.3). Remarkably, this follows automatically from the KP system:

Lemma 4.4 ([60]). *If $L(t)$ satisfies the KP system (Definition 4.3), then $L^i(t)_+$ and $L^i(t)_-$ satisfy*

$$\begin{cases} \frac{\partial}{\partial t_i} L^j(t)_+ - \frac{\partial}{\partial t_j} L^i(t)_+ = \left[L^i(t)_+,\ L^j(t)_+\right] \\ \frac{\partial}{\partial t_i} L^j(t)_- - \frac{\partial}{\partial t_j} L^i(t)_- = -\left[L^i(t)_-,\ L^j(t)_-\right] \end{cases} .$$

These equations are called the Zakharov-Shabat system.

To prove this, we note that the KP system gives

$$\frac{\partial}{\partial t_i} L^j(t) = \left[L^i(t)_+,\ L^j(t)\right] = -\left[L^i(t)_-,\ L^j(t)\right]$$

for every i and j. Decompose $L^j = L^j_+ + L^j_-$ to obtain:

$$\begin{aligned} \frac{\partial L^j}{\partial t_i} &= [L^i_+, L^j_+] + [L^i_+, L^j_-] \\ &= -[L^i_-, L^j_+] - [L^i_-, L^j_-] \\ &= \frac{1}{2}[L^i_+, L^j_+] - \frac{1}{2}[L^i_-, L^j_-] + \frac{1}{2}\left([L^i_+, L^j_-] - [L^i_-, L^j_+]\right) . \end{aligned}$$

The third line is the average of the first two lines. Interchanging i and j, we have

$$\frac{\partial L^i}{\partial t_j} = \frac{1}{2}[L^j_+, L^i_+] - \frac{1}{2}[L^j_-, L^i_-] + \frac{1}{2}\left([L^j_+, L^i_-] - [L^j_-, L^i_+]\right) .$$

Decomposing these equations with respect to (3.1), we obtain Lemma 4.4.

Thus the KP system is the master equation for the largest possible family of iso-spectral deformations of arbitrary ordinary differential operators. If one wants to determine all possible iso-spectral deformations of a given n-th order differential operator $P \in D$, then first bring it to the normal form and compute its n-th root. Next, solve the KP system (Definition 4.3) with the initial datum

$$L(0) = \sqrt[n]{P} .$$

The operator $P(t) = L^n(t)$ then gives the desired largest family of iso-spectral deformations.

Of course we have to establish two things here: the one thing is to show the unique solvability of the Cauchy problem of the KP system, and the other is to show that $P(t)$ thus defined is a *differential* operator:

Theorem 4.5 ([30], [32])**.** *The KP system (Definition 4.3) is uniquely solvable for any initial datum $L(0)$. Moreover, if $L^n(0)$ is a differential operator, then so is $L^n(t)$ for the solution $L(t)$ of the KP system.*

This theorem is obtained as a corollary of a purely algebraic theorem [32] on the factorization of infinite-dimensional groups consisting of infinite-order pseudo-differential operators generalizing the celebrated Birkhoff decomposition of loop groups. Suppose we have a solution $L(t)$ of the KP system with initial datum $L(0) = L(t)|_{t=0}$. Let us introduce two 1-forms

$$Z_L^{\pm}(t) = \pm \sum_{n=1}^{\infty} L^n(t)_{\pm} \otimes dt_n \tag{4.6}$$

with values in E defined on the space

$$T = \lim_{N\to\infty} \mathbb{C}^N \tag{4.7}$$

of deformation parameters $t = (t_1, t_2, t_3, \cdots)$. We regard $Z_L^{\pm}(t)$ as connections on the trivial bundle $E \times T$ on which the Lie algebra E acts by the commutator. Then the KP system

$$dL(t) = \big[Z_L^+(t), L(t)\big] = \big[Z_L^-(t), L(t)\big] \tag{4.8}$$

implies that the solution $L(t)$ is a horizontal section of the Lie algebra bundle $E\times T$ with respect to the connections $Z_L^{\pm}(t)$, where d denotes exterior differentiation on T. The difficulty in solving the KP system lies in the fact that the connections $Z_L^{\pm}(t)$ depend on the solution $L(t)$. The Zakharov-Shabat equations

$$dZ_L^{\pm}(t) - \frac{1}{2}\big[Z_L^{\pm}(t), Z_L^{\pm}(t)\big] = 0 \tag{4.9}$$

mean that these connections have zero curvature. Since the flat connection $Z_L^-(t)$ on the trivial bundle $E \times T$ takes values in $E^{(-1)}$ and its Lie group is exactly the group

$$G_- = \exp\, E^{(-1)} = 1 + E^{(-1)}$$

of monic zeroth order pseudo-differential operators, there is a *gauge transformation*

$$S(t) \in \Gamma(T, G_- \times T)$$

that brings $Z_L^-(t)$ to the trivial connection:

$$S(t)^{-1} \cdot Z_L^-(t) \cdot S(t) - S(t)^{-1} \cdot dS(t) = 0\ . \tag{4.10}$$

Let us consider the trivial solution $L = \partial$ of the KP system. For this solution, we have $Z_\partial^-(t) = 0$ and

$$Z_\partial^+(t) = \sum_{n=1}^{\infty} \partial^n \otimes dt_n\ . \tag{4.11}$$

Our ∂ is indeed a solution because

$$d(\partial) = \big[Z_\partial^+, \partial\big] = \big[Z_\partial^-, \partial\big] = 0\ . \tag{4.12}$$

Observe that the connection (4.11) satisfies the zero curvature condition (4.9) trivially because

$$dZ_\partial^+(t) = [Z_\partial^+(t), Z_\partial^+(t)] = 0\ .$$

Since the gauge transformation $S(t)$ brings $Z_L^-(t)$ to the trivial connection $0 = Z_\partial^-(t)$, ∂ must be the gauge transform of the starting solution $L(t)$:

$$S(t)^{-1} \cdot L(t) \cdot S(t) = \partial \ .$$

Similarly, $Z_\partial^+(t)$ is the gauge transform of $Z_L^+(t)$ by $S(t)$:

$$\begin{aligned} Z_\partial^+(t) &= \sum_{n=1}^\infty \partial^n \otimes dt_n \\ &= S(t)^{-1} \cdot \left(\sum_{n=1}^\infty L^n(t) \otimes dt_n \right) \cdot S(t) \\ &= S(t)^{-1} \cdot \Big(Z_L^+(t) - Z_L^-(t) \Big) \cdot S(t) \\ &= S(t)^{-1} \cdot Z_L^+(t) \cdot S(t) - S(t)^{-1} \cdot dS(t) \ . \end{aligned}$$

Let us *pretend* for a moment that the Lie algebras D and E have corresponding Lie groups G_+ and G_E, respectively, as $E^{(-1)}$ does. Then the direct sum decomposition $E = D \oplus E^{(-1)}$ corresponds to a group factorization

$$\begin{cases} G_E = G_- \cdot G_+ \\ G_- \cap G_+ = \{1\} \end{cases} \tag{4.13}$$

that decomposes an element $U \in G_E$ uniquely as

$$U = S^{-1} \cdot Y, \qquad S \in G_-, \ Y \in G_+ \ .$$

Since $Z_L^+(t)$ is a flat connection on $E \times T$ with values in D, we can find a gauge transformation

$$\begin{cases} Y(t) \in \Gamma(T, G_D \times T) \\ Y(0) = 1 \end{cases}$$

such that $Y(t)$ brings $Z_L^+(t)$ to the 0-connection:

$$Y(t)^{-1} \cdot Z_L^+(t) \cdot Y(t) - Y(t)^{-1} \cdot dY(t) = 0 \ .$$

Therefore, the consecutive application of the gauge transformations $S(t)^{-1}$ and $Y(t)$ should change the connection $Z_\partial^+(t)$ to 0:

$$\big(S(t)^{-1}Y(t)\big)^{-1} \cdot Z_\partial^+(t) \cdot \big(S(t)^{-1}Y(t)\big) - \big(S(t)^{-1}Y(t)\big)^{-1} \cdot d\big(S(t)^{-1}Y(t)\big) = 0 \ .$$

Note that any gauge transformation $U(t) \in \Gamma(T, G_E \times T)$ that satisfies

$$U(t)^{-1} \cdot Z_\partial^+(t) \cdot U(t) - U(t)^{-1} \cdot dU(t) = 0 \ ,$$

i.e., $dU(t) = Z_\partial^+(t) \cdot U(t)$, is given by

$$U(t) = \exp\left(\sum_{n=1}^\infty t_n \partial^n \right) \cdot U(0) \ ,$$

because

$$Z_\partial^+(t) = d\left(\sum_{n=1}^\infty t_n \partial^n \right) \ .$$

Therefore, we have

$$S(t)^{-1} \cdot Y(t) = \exp\left(\sum_{n=1}^\infty t_n \partial^n \right) \cdot S(0)^{-1} \ .$$

Following the above argument backwards, we can solve the initial value problem of the KP system. We start with an initial datum $L(0)$. Compute an $S(0) \in G_-$ such that

$$S(0)^{-1} \cdot L(0) \cdot S(0) = \partial .$$

It will turn out shortly that the solution of the problem does not depend on the choice of $S(0)$. Let $U(t) \in \Gamma(T, G_E \times T)$ be defined by

$$U(t) = \exp\left(\sum_{n=1}^{\infty} t_n \partial^n\right) \cdot S(0)^{-1} \tag{4.14}$$

and

$$U(t) = S(t)^{-1} \cdot Y(t)$$

the decomposition according to (4.13). We choose the initial value $Y(0) = 1$. Using the G_--factor of $U(t)$ of (4.14), we define

$$\begin{cases} L(t) = S(t) \cdot \partial \cdot S(t)^{-1} \\ Z^+(t) = S(t) \cdot Z^+_\partial(t) \cdot S(t)^{-1} + dS(t) \cdot S(t)^{-1} \\ Z^-(t) = S(t) \cdot 0 \cdot S(t)^{-1} + dS(t) \cdot S(t)^{-1} \end{cases} \tag{4.15}$$

as the gauge transform of the trivial solution and the trivial connections. Note that $Z^\pm(t)$ satisfy the zero curvature condition automatically. It is obvious from (4.15) that $Z^-(t)$ has values in $E^{(-1)}$. Since

$$\begin{aligned} 0 &= U(t)^{-1} \cdot Z^+_\partial(t) \cdot U(t) - U(t)^{-1} \cdot dU(t) \\ &= Y(t)^{-1} \cdot \left(S(t) \cdot Z^+_\partial(t) \cdot S(t)^{-1} + dS(t) \cdot S(t)^{-1}\right) \cdot Y(t) - Y(t)^{-1} \cdot dY(t) , \end{aligned}$$

we have

$$Z^+(t) = Y(t) \cdot 0 \cdot Y(t)^{-1} + dY(t) \cdot Y(t)^{-1} ,$$

which has values in D. Therefore,

$$Z^+(t) - Z^-(t) = S(t) \cdot Z^+_\partial(t) \cdot S(t)^{-1} = \sum_{n=1}^{\infty} L^n(t) \otimes dt_n$$

coincides with our previous definition (4.9):

$$Z^\pm(t) = Z^\pm_L(t) = \pm \sum_{n=1}^{\infty} L^n(t)_\pm \otimes dt_n .$$

The KP equations (4.8) then follow from the trivial equation (4.12) and (4.15).

Thus the operator $S(t)$ plays an important role in establishing the solvability of the KP system. Then what role does $Y(t)$ play? Since

$$S(t)^{-1} \cdot Y(t) = \exp\left(\sum_{n=1}^{\infty} t_n \partial^n\right) \cdot S(0)^{-1} ,$$

we have

$$\begin{aligned} L(t) &= S(t) \cdot \partial \cdot S(t)^{-1} \\ &= Y(t) \cdot S(0)^{-1} \cdot \exp\left(-\sum_{n=1}^{\infty} t_n \partial^n\right) \cdot \partial \cdot \exp\left(\sum_{n=1}^{\infty} t_n \partial^n\right) \cdot S(0) \cdot Y(t)^{-1} \\ &= Y(t) \cdot S(0)^{-1} \cdot \partial \cdot S(0) \cdot Y(t)^{-1} \\ &= Y(t) \cdot L(0) \cdot Y(t)^{-1} . \end{aligned}$$

Therefore, the gauge transformation $Y(t)$ is the propagation operator

$$L(0) \longmapsto L(t) = Y(t) \cdot L(0) \cdot Y(t)^{-1} \; . \tag{4.16}$$

This also shows that the solution $L(t)$ does not depend of the choice of $S(0)$ satisfying $L(0) = S(0) \cdot \partial \cdot S(0)^{-1}$. Finally, (4.16) implies that if $L^n(0) = P$ is a differential operator, then

$$P(t) = L^n(t) = Y(t) \cdot P \cdot Y(t)^{-1}$$

is also a differential operator, because $Y(t)$ has no negative order terms.

The above considerations reduce the proof of Theorem 4.5 to establishing (4.13) with the right definition of the infinite-dimensional groups G_D and G_E. This was done in the papers [30] and [32]. Since things become more technical, we refer to the original papers.

Let us compute the first non-trivial equation of the KP system as a nonlinear partial differential equation. The first two terms of the KP system (Definition 4.3) for $i = 2$ give

$$\begin{aligned} \frac{\partial u_2}{\partial t_2} &= u_{2,xx} + u_{3,x} \\ \frac{\partial u_3}{\partial t_3} &= u_{3,xx} + 2u_{4,x} + 2u_2 u_{2,x} \end{aligned}$$

and the initial term for $i = 3$ gives

$$\frac{\partial u_2}{\partial t_2} = u_{2,xxx} + 3u_{3,xx} + 3u_{4,x} + 6u_2 u_{2,x} \; .$$

Eliminating u_3 and u_4 from the above, we recover the KP equation (1.1) for $u = u_2$, $y = t_2$ and $t = t_3$.

The KdV system is the set of nonlinear partial differential equations for a single unknown function $u = u(x, t_3, t_5, t_7, \cdots)$ defined as follows:

$$\frac{\partial u}{\partial t_{2n+1}} = \frac{\partial}{\partial x} \operatorname{Res}(L^{2n+1}) \; , \tag{4.17}$$

where

$$L = \sqrt{\partial^2 + 2u} \; ,$$

and Res P denotes the coefficient of ∂^{-1} in the expansion of a pseudo-differential operator P. By definition, the square of our pseudo-differential operator L is a differential operator. Hence

$$\frac{\partial L}{\partial t_{2n}} = \big[L^{2n}, \; L\big] = 0$$

for all $n \geq 1$ in the KP system. The nontrivial KP equations become

$$\begin{aligned} 2 \, \frac{\partial u}{\partial t_{2n+1}} &= \frac{\partial L^2}{\partial t_{2n+1}} \\ &= \big[(L^{2n+1})_+, \; L^2\big] \\ &= - \; \big[(L^{2n+1})_-, \; L^2\big] \\ &= - \; \big[\operatorname{Res}(L^{2n+1})\partial^{-1}, \; \partial^2\big] \\ &= 2 \, \frac{\partial}{\partial x} \operatorname{Res}(L^{2n+1}) \; , \end{aligned}$$

because the quantity represented in each line of the above expression is a function rather than an operator. The case $n = 1$ in (4.17) is precisely the original KdV equation (1.2). For the differential operator $P = \partial^2 + 2u$, we have always

$$B_P = \mathbb{C}\,[P,\ Q]$$

for some operator Q of order $2m+1$ provided that B_P is larger than $\mathbb{C}\,[P]$. Certainly P and Q satisfy a polynomial equation of the form

$$Q^2 = P^{2m+1} + \text{lower degree terms}$$

and this is why the KdV system is related to hyperelliptic curves. On the other hand, we have always $B_P = \mathbb{C}[P]$ for a generic potential u. Algebraic geometry does not help much in studying such generic operators.

In this section, we derived the KP system as an equation for the largest possible compatible family of iso-spectral deformations of an arbitrary differential operator. Since we used the Gelfand-Dickey Lemma to define the KP system, our equations depend on a coordinate. We used this coordinate to establish its unique solvability. Once we know the solvability, it is better to have a coordinate-free formulation of the KP system. In the next section, we will give a geometric definition of the KP system on the Grassmannian $Gr(0)$.

5. Jacobian varieties as moduli of iso-spectral deformations

We identified the spectrum of a linear ordinary differential operator $P \in D$ with the spectral curve C_P in Section 2. In Section 3, we went further to obtain all geometric spectral information that P has, and saw how to recover the original operator from the geometric data. The iso-spectral deformation defined in Section 4 is a deformation of an operator P that preserves its spectrum. Since it preserves the spectral curve C_P, an iso-spectral deformation naturally corresponds to a deformation of the vector bundle $\mathcal{F}$ under the Krichever functor. In this section, we see how these deformations are interrelated, and how simply they can be described in the language of the Grassmannian. Using this language, we establish that the moduli space of iso-spectral deformations is canonically isomorphic to a Jacobian variety. As a consequence, a characterization of Jacobian varieties follows.

In Section 2, we have seen that the key idea of the connection between differential operators and algebraic curves is the commutant $B_P = \{Q \in D \mid [Q,\ P] = 0\}$ of a given normalized operator P of order n, say. Note that B_P is a subset of

$$F_P = F_L = \left\{Q \in D \;\middle|\; [Q, L] \in E^{(-1)}\right\}\,,$$

where $L = \sqrt[n]{P}$ as in Section 4. From the Lax equation (4.2), B_P corresponds to the *trivial* deformations of P. Since F_P represents all possible deformations by Lemma 4.2, the quotient space F_P/B_P must represent the tangent space to the *moduli space* of the iso-spectral family of P. Usually, the tangent space of any kind of moduli space is given by a cohomology group. Moreover, every cohomology group is defined by

$$\text{cohomology} = \frac{\text{interesting objects}}{\text{trivial objects}}\,, \tag{5.1}$$

assuming that 'trivial' is not really the antonym of 'interesting.' Therefore, our F_P/B_P must be a cohomology.

From Lemma 4.2, we have a basis $\big\{(L^m)_+\big\}_{0\le m}$ for F_P. Assigning ∂^m to $(L^m)_+$ for every $m \ge 0$, we have a $\mathbb{C}$-linear isomorphism

$$F_P \cong \mathbb{C}[\partial] \cong \frac{\mathbb{C}((\partial^{-1}))}{\mathbb{C}[[\partial^{-1}]]\cdot\partial^{-1}} \ .$$

On the other hand, the Theorem of Schur (Theorem 2.2) tells us that there is a monic zeroth order operator $S \in G_-$ such that

$$A_P = S^{-1}\cdot B_P \cdot S \subset \mathbb{C}((\partial^{-1})) \ .$$

Therefore, as a vector space, we have

$$F_P/B_P \cong \frac{\mathbb{C}((\partial^{-1}))}{A_P \oplus \mathbb{C}[[\partial^{-1}]]\cdot\partial^{-1}} = \frac{\mathbb{C}((z))}{A_P + \mathbb{C}[[z]]} \ , \tag{5.2}$$

where we have used the variable z of (2.10) and the relation (2.11). Recall that we defined the spectral curve C_P in (2.12) as

$$C_P = \text{Spec } A_P \cup \text{Spec } \mathbb{C}[[z]] = \text{Spec } A_P \cup \{z = 0\} \ .$$

From now on, let us assume that the rank of B_P is one, i.e., the greatest common divisor of the orders of elements of B_P is equal to 1. This exactly means that the quotient space F_P/B_P has finite dimension. Consequently, the parameter z gives a local coordinate of the curve C_P at p. (For the rank r case, z^r gives a local coordinate, instead.) Then the Čech cohomology associated with the covering (2.12) gives a canonical isomorphism

$$F_P/B_P \cong \frac{\mathbb{C}((z))}{A_P + \mathbb{C}[[z]]} = H^1(C_P, \mathcal{O}_{C_P}) \ , \tag{5.3}$$

which justifies (5.1)!

The above consideration gives only the local structure of solutions of the KP system. In order to study the global behavior of solutions, we have to use the language of the Grassmannian. In the previous section, we rewrote the KP system in terms of a monic operator S of order 1 as in (4.10):

$$S(t)^{-1}\cdot Z_L^-(t)\cdot S(t) - S(t)^{-1}\cdot dS(t) = 0 \ .$$

This equation is equivalent to the system of partial differential equations

$$\frac{\partial S(t)}{\partial t_m}\cdot S(t)^{-1} = -\big(S(t)\cdot\partial^m\cdot S(t)^{-1}\big)_- \tag{5.4}$$

for all $m \ge 0$. Since the operator $S(t) \in G_-$ gives a point $W(t) = S(t)^{-1}\cdot V_+ \in Gr(0)$ for every $t \in T$ by Theorem 3.3, we can interpret the KP system as a dynamical system on the Grassmannian. Recall that our Grassmannian $Gr(0)$ is a set of vector subspaces W of $V = \mathbb{C}((z))$. Every element v of V acts on V by multiplication. Thus it induces an element of $\text{Hom}(W, V/W)$ by

$$W \hookrightarrow V \xrightarrow{v\times} V \to V/W \ .$$

Since this homomorphism group is canonically identified with the tangent space of the Grassmannian at the point W,

$$\text{Hom}(W, V/W) = T_W Gr(0) \ ,$$

every $v \in V$ induces a vector field on the Grassmannian. We use the notation $\Psi(v)$ for this vector field, and denote the tangent vector at W by $\Psi_W(v)$.

Theorem 5.1. *The KP system is equivalent to the collection of commuting vector fields* $\Psi\big(\mathbb{C}[z^{-1}]\big)$ *defined on the Grassmannian.*

Proof. The identification $\partial^m = z^{-m}$ gives a tangent vector $\Psi_W(z^{-m})$ at each $W \in Gr(0)$. Pulling it back to the base point $\mathbb{C}[z^{-1}] = V_+$ of the Grassmannian via the action of $S(t)^{-1}$, we obtain a tangent vector $S(t) \cdot z^{-m} \cdot S(t)^{-1}$ at V_+:

$$\begin{tikzcd} \mathbb{C}[z^{-1}] \arrow[r] \arrow[d, "S^{-1}"'] & V \arrow[r, "S\cdot z^{-m}\cdot S^{-1}"] \arrow[d, "S^{-1}"'] & V \arrow[r] \arrow[d, "S^{-1}"] & V\big/\mathbb{C}[z^{-1}] \arrow[d, "S^{-1}"] \\ W(t) \arrow[r] & V \arrow[r, "z^{-m}"'] & V \arrow[r] & V/W(t)\ , \end{tikzcd} \tag{5.5}$$

where $S = S(t)$. Since differential operators stabilize V_+ by Lemma 3.9, the tangent vector $S(t) \cdot z^{-m} \cdot S(t)^{-1}$ at V_+ is indeed equal to

$$\begin{aligned} \big(S(t) \cdot \partial^m \cdot S(t)^{-1}\big)_- &= -\frac{\partial S(t)}{\partial t_m} \cdot S(t)^{-1} \\ &= S(t) \cdot \frac{\partial S(t)^{-1}}{\partial t_m} \\ &\in E^{-1} \end{aligned}$$

by (5.4). Going back to $W(t)$ following (5.5), the KP system gives a tangent vector

$$\frac{\partial W(t)}{\partial t_m} = \frac{\partial S(t)^{-1}}{\partial t_m}$$

at $W(t)$. Since this is equal to the action of z^{-m} at $W(t)$, we obtain the KP system in terms of the Grassmannian language:

$$\frac{\partial W(t)}{\partial t_m} = z^{-m} \cdot W(t)\ . \tag{5.6}$$

Thus the KP system is the commutative Lie algebra $\Psi\big(\mathbb{C}[z^{-1}]\big)$ of vector fields on the Grassmannian $Gr(0)$. ☐

The *formal* integration of (5.6) is given by

$$W(t) = \exp\left(\sum_{m=1}^{\infty} t_m z^{-m}\right) \cdot W(0)\ . \tag{5.7}$$

It is of course no coincidence that (5.7) has exactly the same form as (4.14). Let us denote the exponential factor by

$$e(t) = \exp\left(\sum_{m=1}^{\infty} t_m z^{-m}\right)\ .$$

Although (5.7) does not make sense as a point of $Gr(0)$ because our Grassmannian is modeled on $\mathbb{C}((z))$ which does not contain $e(t)$, it makes perfect sense in algebraic geometry. To see this, let (A_P, W) be the Schur pair defined by

$$\begin{cases} A_P = S(0)^{-1} \cdot B_P \cdot S(0) \\ W = S(0)^{-1} \cdot \mathbb{C}[z^{-1}] \in Gr(0) \end{cases}$$

and $(C, p, \pi, \mathcal{F}, \phi)$ the corresponding quintuple. Since we are assuming that B_P has rank one, the rank of the Schur pair (A_P, W) is also 1. Thus the local morphism π

is the identity map and $\mathcal{F}$ is a line bundle, by which we mean a torsion-free rank one sheaf over C. (If C is nonsingular, then $\mathcal{F}$ is indeed locally free.) We use the exact sequence

$$0 \longrightarrow 2\pi\sqrt{-1}\mathbb{Z} \longrightarrow \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \longrightarrow 0 \, .$$

Note that every z^{-m} gives an element of $H^1(C, \mathcal{O}_C)$ by (5.3). Therefore, we have

$$\exp : H^1(C, \mathcal{O}_C) \ni \left[\sum_{m=1}^{\infty} t_m z^{-m}\right] \longmapsto [e(t)] = \mathcal{L}(t) \in H^1(C, \mathcal{O}_C^*) \, , \tag{5.8}$$

where $\mathcal{L}(t)$ is the line bundle of degree 0 corresponding to the cohomology class $[e(t)]$. Therefore, we can interpret the Schur pair $\big(A_P, W(t)\big)$ as the algebraic counterpart of the quintuple

$$\big(C, p, \pi, \mathcal{F} \otimes \mathcal{L}(t), \phi \cdot e(t)\big) \, . \tag{5.9}$$

Thus the iso-spectral deformation $P \longmapsto P(t)$ corresponds to the deformation

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \mathcal{L}(t)$$

of line bundles defined on the fixed spectral curve C. Because of the Riemann-Roch formula

$$\dim H^0(C, \mathcal{F}) - \dim H^1(C, \mathcal{F}) = \deg \mathcal{F} - g + 1 = \operatorname{index} \gamma_W \, ,$$

the line bundle $\mathcal{F}$ has degree $g-1$, where g is the genus of C. Since we can recover the solution (i.e., the isospectral deformation)

$$P(t) = S(t) \cdot \partial^n \cdot S(t)^{-1}$$

starting at $P(0)$ from the quintuple (5.9) uniquely, and since the image of (5.8) is by definition the *Jacobian variety*

$$\operatorname{Jac}(C) = H^1(C, \mathcal{O}_C) \big/ H^1(C, \mathbb{Z})$$

of C, we conclude that the moduli space of iso-spectral families is indeed isomorphic to the Jacobian variety [31] [35]:

$$\begin{aligned} \{P(t) | t \in T\} &= \left\{ \big(C, p, \pi, \mathcal{F} \otimes \mathcal{L}(t), \phi \cdot e(t)\big) \middle| t \in T \right\} \\ &= \operatorname{Pic}^{g-1}(C) \\ &\cong \operatorname{Jac}(C) \, . \end{aligned}$$

In other words, every finite-dimensional orbit (= integral manifold) of the KP system defined on the Grassmannian is canonically isomorphic to the Jacobian variety of an algebraic curve. This is one of the main results established in [31] which led to a characterization of Jacobian varieties of arbitrary genera (a solution to the Schottky problem) in terms of the KP system (see also [2] [8] [51]).

Still, we do not know much about infinite-dimensional orbits of the KP system. Even though our functor χ of Theorem 3.6 is fully faithful, we need to add more geometric information, such as a connection, to a quintuple to study infinite-dimensional orbits. However, it is not our intention to go into this subject here.

6. Morphisms of curves, Prym varieties and commuting partial differential operators

The categorical equivalence Theorem 3.6 is the key step in establishing a geometric classification Theorem 3.8 of commutative algebras of ordinary differential operators. It is quite natural to ask if we can generalize the whole machinery to find a classification of commutative algebras consisting of partial differential operators. Unfortunately, it is clear that most of the techniques we have used in the classification of Theorem 3.8 do not generalize to the case of partial differential operators. Schur's Theorem (Theorem 2.2) depends essentially on properties of ordinary differential operators, such as the ellipticity of monic operators. The Grassmannians of Section 3 are also hard to generalize for partial differential operators.

I'm not sure if there can possibly be any general theory of commuting partial differential operators. At the least, we do not have enough examples to construct such a theory. Probably, it is better for us to examine what has already been given to us and how much has been done than to look for hopeless generalizations. Is the theory we now have 100% satisfactory? Is there any possibility for more natural generalizations? We enumerate some points from the previous sections.

(1) In the definition of the category of geometric quintuples (Definition 3.5), we allowed an arbitrary algebraic curve and a vector bundle on it as an object of the category; however, we did not allow arbitrary morphisms of algebraic curves to form a morphism between quintuples. There must be a larger category in which arbitrary morphisms of curves are allowed.
(2) The theory of Section 4 should have a natural extension to operators with matrix coefficients.
(3) The Grassmannian of Definition 3.2 should be generalized to a Grassmannian modeled on vector-valued functions $\mathbb{C}((z))^{\oplus n}$.

In this section, we give these natural generalizations. Surprisingly, as a byproduct we will encounter a large class of commuting partial differential operators, and more remarkably, we will establish a characterization of arbitrary Prym varieties. It will turn out that the three different directions of generalization listed above are in fact exactly the same.

Let

$$f : C \longrightarrow C_o$$

be an arbitrary morphism of smooth algebraic curves. For a point $p \in C_o$, $f^{-1}(p) \subset C$ is a divisor of C. Its degree, n, say, is called the *degree* of the morphism f. Take an arbitrary vector bundle $\mathcal{F}$ of rank r on C. The direct image sheaf $f_*\mathcal{F}$ is a vector bundle of rank $r \cdot n$ on C_o. Supplying a local covering

$$U_o \longrightarrow (C_o)_p$$

of degree $r \cdot n$ ramified at p and a local trivialization

$$(f_*\mathcal{F})|_{U_o} \xrightarrow{\sim} \pi_*\mathcal{O}_{U_o}(-1)$$

as in Definition 3.5, we have a geometric quintuple. We have seen in Section 3 that we can encode all geometric information of C_o and p in an algebra embedding

$$H^0\big(C_o \setminus \{p\}, \mathcal{O}_{C_o}\big) \cong A_o \subset \mathbb{C}((z)) .$$

This suggests that the algebraic information of C and f should be encoded in an embedding of the cohomology $H^0\big(C \setminus f^{-1}(p), \mathcal{O}_C\big)$ to somewhere—but where?

Let y be a local coordinate of C_o around p, and $f^{-1}(p) = \{p_1, \cdots, p_\ell\}$. We can choose a local coordinate y_i of C around p_i so that the morphism f is given by

$$f : y_i \longmapsto y = y_i^{n_i} \tag{6.1}$$

on a neighborhood of p_i. To be precise, we use the formal completion U_i of C at p_i and U_p of C_o at p. The morphism f induces a ramified covering

$$f_i : U_i \longrightarrow U_p$$

of degree n_i defined by (6.1). Since f has degree n,

$$n_1 + n_2 + \cdots + n_\ell = n \; . \tag{6.2}$$

Using these local coordinates, we can embed the cohomology

$$H^0\big(C \setminus f^{-1}(p), \mathcal{O}_C\big) \cong A \subset \bigoplus_{i=1}^{\ell} \mathbb{C}((y_i)) \; .$$

How can we represent the natural map

$$f^* : H^0\big(C_o \setminus \{p\}, \mathcal{O}_{C_o}\big) \longrightarrow H^0\big(C \setminus f^{-1}(p), \mathcal{O}_C\big)$$

in terms of these coordinates? Since there is no canonical embedding of $\mathbb{C}((y))$ into $\bigoplus_{i=1}^{\ell} \mathbb{C}((y_i))$, we have to construct one. Note that the relation of (6.1) is satisfied by defining

$$y_i = h_{n_i}(y) = \begin{pmatrix} 0 & & & & 0 & y \\ 1 & 0 & & & & 0 \\ & 1 & \ddots & & & \\ & & \ddots & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} , \tag{6.3}$$

which is a square matrix of size $n_i \times n_i$. Let $\mathbf{n} = (n_1, n_2, \cdots, n_\ell)$ be the integral vector of the ramification degree of f at the divisor $f^{-1}(p)$. We can represent the algebra

$$H_{\mathbf{n}}(y) \overset{\text{def}}{=} \bigoplus_{i=1}^{\ell} \mathbb{C}\big(\big(h_{n_i}(y)\big)\big) \cong \bigoplus_{i=1}^{\ell} \mathbb{C}((y^{1/n_i}))$$

as a commutative Lie subalgebra of $gl\big(n, \mathbb{C}((y))\big)$ by embedding each

$$H_{n_i} = \mathbb{C}\big(\big(h_{n_i}(y)\big)\big)$$

as a disjoint principal diagonal block:

$$H_{\mathbf{n}}(y) = \begin{pmatrix} H_{n_1} & & & \\ & H_{n_2} & & \\ & & \ddots & \\ & & & H_{n_\ell} \end{pmatrix} . \tag{6.4}$$

This algebra is known as the *maximal commutative subalgebra of type* $\mathbf{n}$ of the formal loop algebra $gl\big(n, \mathbb{C}((y))\big)$, which contains $\mathbb{C}((y))$ as the scalar diagonal subalgebra. Now, we have a desired commutative diagram of injective maps:

$$
\begin{array}{ccc}
H^0\big(C_o \setminus \{p\}, \mathcal{O}_{C_o}\big) & \xrightarrow{f^*} & H^0\big(C \setminus f^{-1}(p), \mathcal{O}_C\big) \\
\big\downarrow & & \big\downarrow \\
\mathbb{C}((y)) & \longrightarrow & H_{\mathbf{n}}(y) \ ,
\end{array}
$$

where the right column is an extension of the left column of degree n, and the inclusion map of the right column is defined by expanding meromorphic functions of C in the local parameter y_i at each p_i. In order to deal with vector bundles on C, we further need ℓ local coverings π_i at each p_i. This motivates us to give the following Definition 6.2. First we need:

Definition 6.1. A morphism Spec $\mathbb{C}[[z]] \longrightarrow$ Spec $\mathbb{C}[[y]]$ is said to be a cyclic covering of degree r if it is induced by a ring homomorphism

$$\mathbb{C}[[y]] \ni y \longmapsto z^r \in \mathbb{C}[[z]] \ .$$

The function ring of the domain of a cyclic covering is a degree r cyclic extension of the function ring of its image. This is why we call this morphism a *cyclic* covering. We need to supply cyclic coverings in order to represent sections of vector bundles in terms of formal Laurent series in one variable (see Section 3).

Definition 6.2. A set of *geometric data* of a covering morphism of algebraic curves of type $\mathbf{n}$, index μ and rank r is a collection

$$\Big\langle f : \big(C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi\big) \longrightarrow \big(C_0, p, \pi, f_*\mathcal{F}, \phi\big)\Big\rangle$$

consisting of the following objects:

1. $\mathbf{n} = (n_1, n_2, \cdots, n_\ell)$ is an integral vector of positive integers n_j such that $n = n_1 + n_2 + \cdots + n_\ell$.
2. $C_{\mathbf{n}}$ is a reduced algebraic curve, which is not necessarily irreducible, and $\Delta = \{p_1, p_2, \cdots, p_\ell\}$ is a set of ℓ smooth points of $C_{\mathbf{n}}$.
3. C_o is an irreducible curve with a smooth marked point p.
4. $f : C_{\mathbf{n}} \longrightarrow C_o$ is a finite morphism of degree n of $C_{\mathbf{n}}$ onto C_o such that $f^{-1}(p) = \{p_1, \cdots, p_\ell\}$ with ramification index n_j at each point p_j.
5. $\Pi = (\pi_1, \cdots, \pi_\ell)$ consists of a cyclic covering $\pi_j : U_{oj} \longrightarrow U_j$ of degree r which maps the formal completion $U_{oj} =$ Spec $\mathbb{C}[[z_j]]$ of the affine line $\mathbb{C}$ at the origin onto the formal completion U_j of the curve $C_{\mathbf{n}}$ at p_j.
6. $\mathcal{F}$ is a torsion-free sheaf of rank r defined over $C_{\mathbf{n}}$ whose Euler characteristic is
$$\mu = \dim\ H^0(C_{\mathbf{n}}, \mathcal{F}) - \dim\ H^1(C_{\mathbf{n}}, \mathcal{F}) \ .$$
7. $\Phi = (\phi_1, \cdots, \phi_\ell)$ consists of $\mathcal{O}_{U_j}$-module isomorphisms
$$\phi_j : \mathcal{F}_{U_j} \xrightarrow{\sim} \pi_{j*}\big(\mathcal{O}_{U_{oj}}(-1)\big) \ ,$$
where $\mathcal{F}_{U_j}$ is the formal completion of $\mathcal{F}$ at p_j.
8. $\pi : U_o \longrightarrow U_p$ is a cyclic covering of degree r which maps the formal completion $U_o =$ Spec $\mathbb{C}[[z]]$ of the affine line $\mathbb{C}$ at the origin onto the formal completion U_p of the curve C_o at p.

(9) $\pi_j : U_{oj} \longrightarrow U_j$ and the formal completion $f_j : U_j \longrightarrow U_p$ of f at p_j satisfy

$$\begin{array}{ccc} U_{oj} & \xrightarrow{\pi_j} & U_j \\ {\scriptstyle \psi_j}\downarrow & & \downarrow{\scriptstyle f_j} \\ U_o & \xrightarrow[\pi]{} & U_p, \end{array}$$

where $\psi_j : U_{oj} \longrightarrow U_o$ is a cyclic covering of degree n_j defined by $z_j \longmapsto z^{n_j}$.

(10) $\phi : (f_*\mathcal{F})_{U_p} \xrightarrow{\sim} \pi_*\left(\bigoplus_{j=1}^{\ell} \psi_{j*}\big(\mathcal{O}_{U_{oj}}(-1)\big)\right)$ is an $\big(f_*\mathcal{O}_{C_{\mathbf{n}}}\big)_{U_p}$-module isomorphism of sheaves on the formal scheme U_p which is compatible with the data Φ on $C_{\mathbf{n}}$.

In order to define the algebraic counterpart of the geometric data, let us introduce the Grassmannian $Gr_n(\mu)$ of vector-valued functions consisting of vector subspaces W of $\mathbb{C}((z))^{\oplus n}$ such that the natural map

$$\gamma_W : W \longrightarrow \frac{\mathbb{C}((z))^{\oplus n}}{\big(\mathbb{C}[[z]]z\big)^{\oplus n}} \cong \mathbb{C}[z^{-1}]^{\oplus n}$$

is Fredholm of index μ.

Definition 6.3. A set of *algebraic data*

$$\big\langle i : A_o \hookrightarrow A_{\mathbf{n}}, W \big\rangle$$

of type $\mathbf{n}$, index μ, and rank r is a collection of objects satisfying the following:

(1) W is a point of the Grassmannian $Gr_n(\mu)$ of index μ of the vector-valued functions of size n.

(2) The type $\mathbf{n}$ is an integral vector $(n_1, \cdots, n_\ell)$ consisting of positive integers such that $n = n_1 + \cdots + n_\ell$.

(3) There is an element

$$y = z^r + c_1 z^{r+1} + c_2 z^{r+2} + \cdots \in \mathbb{C}((z))$$

such that A_o is a $\mathbb{C}$-subalgebra of $\mathbb{C}((y))$ containing the identity $1 \in \mathbb{C}$.

(4) The cokernel of the projection $\gamma_{A_o} : A_o \longrightarrow \mathbb{C}((y))/\mathbb{C}[[y]]$ has finite dimension.

(5) $A_{\mathbf{n}}$ is a subalgebra of the maximal commutative algebra

$$H_{\mathbf{n}}(y) \subset gl\big(n, \mathbb{C}((y))\big)$$

of type $\mathbf{n}$ such that the projection

$$\gamma_{A_{\mathbf{n}}} : A_{\mathbf{n}} \longrightarrow \frac{H_{\mathbf{n}}(y)}{H_{\mathbf{n}}(y) \cap gl\big(n, \mathbb{C}[[y]]\big)}$$

has finite-dimensional cokernel.

(6) As an A_o-module (which is automatically torsion-free), $A_{\mathbf{n}}$ has rank n over A_o. The inclusion i is a restriction of the scalar diagonal embedding

$$\begin{array}{ccc} \mathbb{C}((y)) & \longrightarrow & gl\big(n, \mathbb{C}((y))\big) \\ \uparrow & & \uparrow \\ A_o & \xrightarrow[i]{} & A_{\mathbf{n}}\ . \end{array}$$

(7) The algebra $A_{\mathbf{n}} \subset gl\big(n, \mathbb{C}((y))\big)$ stabilizes W, i.e., $A_{\mathbf{n}} \cdot W \subset W$. Since $\mathbb{C}((y)) \subset \mathbb{C}((z))$, $gl\big(n, \mathbb{C}((y))\big)$ acts on $\mathbb{C}((z))^{\oplus n}$ by the matrix multiplication.

We can make the sets of Definition 6.2 and Definition 6.3 into categories by supplying suitable morphisms. It is established in [27] that these categories are anti-equivalent by a fully faithful contravariant functor generalizing the Krichever functor (see also [1]).

Recall that the KP system is the action of the commutative Lie algebra $\mathbb{C}[z^{-1}]$ on the Grassmannian $Gr(0)$. The natural map

$$\mathbb{C}[z^{-1}] \longrightarrow \frac{\mathbb{C}((z))}{A_W \oplus \mathbb{C}[[z]] \cdot z} \tag{6.5}$$

describes the tangent space of the orbit of the KP system at a point W of $Gr(0)$, where

$$A_W = \{a \in \mathbb{C}((z)) \mid a \cdot W \subset W\}$$

is the maximal commutative stabilizer of $W \in Gr(0)$. In Section 5, we have seen that if A_W has rank one, then the quotient module of (6.5) is canonically isomorphic to the cohomology $H^1(C, \mathcal{O}_C)$ of the algebraic curve

$$C = \mathrm{Proj}\big(gr(A_o)\big) .$$

On the new Grassmannian $Gr_n(\mu)$ of vector-valued functions, we have a natural action of

$$H_{\mathbf{n}}(z)_+ = H_{\mathbf{n}}(z) \cap gl\big(n, \mathbb{C}[z^{-1}]\big) .$$

Let us call this action the *Heisenberg KP system* of type $\mathbf{n}$. We use this name because a central extension of the commutative algebra $H_{\mathbf{n}}(z)$ is the Heisenberg algebra associated with the conjugacy class of the Weyl group of $gl(n, \mathbb{C})$ defined by the integral vector $\mathbf{n}$. Every finite-dimensional orbit of the KP system is a (generalized) Jacobian variety. What is a finite-dimensional orbit of the Heisenberg KP system? To investigate an orbit, let us define the *maximal commutative stabilizer algebra of type* $\mathbf{n}$ of $W \in Gr_n(\mu)$ by

$$(A_{\mathbf{n}})_W = \big\{a \in H_{\mathbf{n}}(z) \bigm| a \cdot W \subset W\big\} .$$

We also define

$$(A_o)_W = \big\{v \in \mathbb{C}((z)) \bigm| v \cdot W \subset W\big\} ,$$

where $v \in \mathbb{C}((z))$ acts on W as a scalar. The tangent space of the orbit of the Heisenberg KP system at W is described by

$$H_{\mathbf{n}}(z)_+ \longrightarrow \frac{H_{\mathbf{n}}(z)}{(A_{\mathbf{n}})_W + H_{\mathbf{n}}(z) \cap gl\big(n, \mathbb{C}[[z]]\big)} .$$

Now, let us assume that the above quotient module has finite dimension. Then the set $\big\langle i : (A_o)_W \hookrightarrow (A_{\mathbf{n}})_W, W\big\rangle$ of algebraic data satisfies the condition of Definition 6.3 with $y = z$ in the definition. Therefore, by the categorical equivalence of [27], it corresponds to a set of geometric data

$$\big\langle f : \big(C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi\big) \longrightarrow \big(C_0, p, \pi, f_*\mathcal{F}, \phi\big)\big\rangle$$

of rank $r = 1$. In particular, the sheaf $\mathcal{F}$ is a line bundle on $C_{\mathbf{n}}$. By an argument similar to that of Section 5, we can show that

$$\frac{H_{\mathbf{n}}(z)}{(A_{\mathbf{n}})_W + H_{\mathbf{n}}(z) \cap gl\big(n, \mathbb{C}[[z]]\big)} \cong H^1(C_{\mathbf{n}}, \mathcal{O}_{C_{\mathbf{n}}}) \ .$$

Moreover, the orbit is canonically isomorphic to the Jacobian variety $\mathrm{Jac}(C_{\mathbf{n}})$.

Let us define the *traceless Heisenberg flows* on the Grassmannian $Gr_n(\mu)$ by the actions of

$$H_{\mathbf{n}}(z)^o_+ = H_{\mathbf{n}}(z) \cap sl\big(n, \mathbb{C}[z^{-1}]\big) \ .$$

This is the traceless commutative Lie subalgebra of $H_{\mathbf{n}}(z)_+$. When we deform the line bundle $\mathcal{F}$ on $C_{\mathbf{n}}$ by the actions of $H_{\mathbf{n}}(z)_+$, the vector bundle $f_*\mathcal{F}$ on C_o is deformed by the action of the trace part of the algebra $H_{\mathbf{n}}(z)_+$. Therefore, the traceless Heisenberg flows keep the determinant $\det(f_*\mathcal{F})$ fixed. The line bundles on $C_{\mathbf{n}}$ that have a fixed determinant on C_o form the *Prym variety* associated with the morphism $f : C_{\mathbf{n}} \longrightarrow C_o$. Therefore, we have obtained the following:

Theorem 6.4 ([27])**.** *Every finite-dimensional orbit of the traceless Heisenberg KP system on the Grassmannian $Gr_n(\mu)$ of vector-valued functions is isomorphic to the Prym variety associated with a morphism of algebraic curves.*

Conversely, we can show by using the categorical equivalence that the Prym variety of an arbitrary morphism of algebraic curves appears as an orbit of the traceless Heisenberg flows (a generalization of the Krichever construction). Thus we have established a characterization of arbitrary Prym varieties in terms of integrable systems! We refer to [27] for more details.

As we have shown in that paper, every point W of the big-cell $Gr_n^+(0)$ of the vector-valued Grassmannian corresponds bijectively to an $n \times n$ matrix S of zeroth order pseudo-differential operators whose leading term is the identity matrix. The correspondence is exactly the same as before: we simply define

$$W = S^{-1} \cdot \mathbb{C}[z^{-1}]^{\oplus n} \ .$$

Let us use the identification $z = \partial^{-1}$ as in (2.10). For $1 \leq j \leq \ell$ and $i \geq 1$, we define an $n \times n$ matrix of pseudo-differential operators by

$$B_{ij} = S \cdot h_{n_j}(z)^{-i} \cdot S^{-1} \ ,$$

where we identify

$$h_{n_j}(z) = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & h_{n_j}(z) & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} .$$

We decompose it into the differential operator part $(B_{ij})_+$ and the integral operator part $(B_{ij})_-$, as before. Then the Heisenberg KP system is equivalent to a system of partial differential equations

$$\frac{\partial S(t)}{\partial t_{ij}} = -B_{ij}(t)_- \cdot S(t) \ . \tag{6.6}$$

It implies automatically the Zakharov-Shabat equation

$$\left[\frac{\partial}{\partial t_{ij}} - B_{ij}(t)_+,\ \frac{\partial}{\partial t_{\alpha\beta}} - B_{\alpha\beta}(t)_+\right] = 0\ , \tag{6.7}$$

as in Section 4. Note that we have

$$\sum_{j=1}^{\ell} h_{n_j}(z)^{n_j} = z \cdot I_n$$

from the definition (6.4) of $H_{\mathbf{n}}(z)$ and the matrix (6.3), where I_n is the identity matrix of size n. Since the above sum is a direct sum, we have

$$\partial = \frac{\partial}{\partial x} = z^{-1} = \sum_{j=1}^{\ell} h_{n_j}(z)^{-n_j}\ .$$

Using this formula, we can represent ∂ as a linear combination of $\partial/\partial t_{ij}$s. To do so, we need the Lax formalism. Eq.(6.6) has a corresponding Lax equation

$$\frac{\partial L(t)}{\partial t_{ij}} = \left[B_{ij}(t)_+,\ L(t)\right]\ ,$$

where

$$L(t) = S(t)\cdot\partial\cdot S(t)^{-1} = S(t)\cdot\sum_{j=1}^{\ell} h_{n_j}(z)^{-n_j}\cdot S(t)^{-1} = \sum_{j=1}^{\ell} B_{n_j j}(t)\ .$$

Since $L(t) = I_n\cdot\partial +$ negative order terms,

$$\sum_{j=1}^{\ell}\frac{\partial L(t)}{\partial t_{n_j j}} = \left[\sum_{j=1}^{\ell} B_{n_j j}(t)_+,\ L(t)\right] = \left[L(t)_+, L(t)\right] = \left[\partial, L(t)\right]\ .$$

Therefore, we have a natural identification

$$\begin{aligned}\partial &= \sum_{j=1}^{\ell}\frac{\partial}{\partial t_{n_j j}}\\ x &= \frac{1}{\ell}\sum_{j=1}^{\ell} t_{n_j j}\ .\end{aligned} \tag{6.8}$$

Define a partial differential operator

$$P_{ij} = \frac{\partial}{\partial t_{ij}} - B_{ij}(t)_+$$

in the t-variables only, by substituting the quantities identified in (6.8) for x and ∂ in the above expression.

Eq.(6.7) is a commutation relation for partial differential operators, but it does not give any commuting operators unless we actually solve it. So let $S(t)$ and $B_{ij}(t)$ be a solution of (6.6) and (6.7), and X the corresponding finite-dimensional orbit of the traceless Heisenberg flows. Since the t_{ij}s form a natural linear coordinate system on X, the restriction $P_{ij}|_X$ is an $n\times n$ matrix of partial differential operators which are globally defined on X. Let

$$\left\langle f : \left(C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi\right) \longrightarrow \left(C_0, p, \pi, f_*\mathcal{F}, \phi\right)\right\rangle$$

be the geometric data corresponding to our solution. The result of [27] shows that X is canonically isomorphic to the Prym variety of the morphism $f : C_{\mathbf{n}} \longrightarrow C_o$. Since those P_{ij}s that are linearly independent generate the orbit X, the number of linearly independent operators among the $P_{ij}|_X$s is exactly equal to the dimension of X. The definition of Prym varieties tells us

$$\dim\ X = g(C_{\mathbf{n}}) - g(C_o)\ ,$$

where g denotes the arithmetic genus of the curve [27]. It is also obvious from the construction that the operators $P_{ij}|_X$ satisfy no algebraic relations. Therefore, the associative algebra $\mathcal{R}$ generated by these $P_{ij}|_X$s over $\mathbb{C}$ is isomorphic to the polynomial ring in ($\dim\ X$)-variables. Thus for every morphism of algebraic curves of degree n, we can construct a commutative algebra of $n \times n$ matrices of partial differential operators which are globally defined on the Prym variety associated with the morphism.

Theorem 6.5. *Let*

$$\big\langle f : \big(C_{\mathbf{n}}, \Delta, \Pi, \mathcal{F}, \Phi\big) \longrightarrow \big(C_0, p, \pi, f_*\mathcal{F}, \phi\big)\big\rangle$$

be a set of geometric data of type $\mathbf{n}$*, index* 0 *and rank* 1 *such that the corresponding algebraic data*

$$\big\langle i : A_o \hookrightarrow A_{\mathbf{n}}, W\big\rangle$$

have a point W *of the big-cell* $Gr_n^+(0)$*. This is equivalent to requiring that the line bundle* $\mathcal{F}$ *on* $C_{\mathbf{n}}$ *satisfies*

$$H^0(C_{\mathbf{n}}, \mathcal{F}) = H^1(C_{\mathbf{n}}, \mathcal{F}) = 0\ .$$

Then the Zakharov-Shabat equation (6.7) gives a commutative algebra consisting of $n\times n$ *matrices of partial differential operators which are globally defined on the Prym variety associated with the morphism* $f : C_{\mathbf{n}} \longrightarrow C_o$*. This algebra is isomorphic to the polynomial ring of* $\big(g(C_{\mathbf{n}}) - g(C_o)\big)$*-variables over* $\mathbb{C}$ *as an abstract algebra.*

Hitchin [14] and Beilinson, Drinfeld and Ginzburg (in an unpublished paper) have constructed commuting matrix partial differential operators on moduli spaces of vector bundles on algebraic curves with a fixed determinant in connection with conformal field theory and the geometric Langlands correspondence. Since every such moduli space admits a dominant finite morphism from a Prym variety of the same dimension, and since the structure of the algebra of Beilinson-Drinfeld-Ginzburg is isomorphic to the polynomial ring, we conjecture that our construction is a pull-back of their construction on the moduli space via a dominant morphism. It will be an interesting problem to look into this conjecture. We note that these examples are not obtained by the method of [39].

7. The τ-function and infinite-determinants

The notion of τ-functions was introduced by R. Hirota in the early 1970s in order to find (i.e., to compute) exact solutions of soliton equations. He used in [13] a new notation, which we call *Hirota's bilinear form*, for differentiation of products of functions. Let $t = (t_1, t_2, \cdots, t_n)$, $s = (s_1, s_2, \cdots, s_n)$, $\partial_s = (\partial_{s_1}, \partial_{s_2}, \cdots, \partial_{s_n})$ and $D = (D_1, D_2, \cdots, D_n)$.

Definition 7.1. For differentiable functions $f(t)$ and $g(t)$ in $t \in \mathbb{C}^n$ and a polynomial $P(D)$ in D, we write

$$P(D) f \cdot g = P(\partial_s)\big(f(t+s)g(t-s)\big)\big|_{s=0} .$$

When $n = 1$ for example, we have

$$D^N f \cdot g = \sum_{r=0}^{N} (-1)^{N-r} \binom{N}{r} f^{(r)} g^{(N-r)} .$$

Consider a function $\tau(x, t)$ in two variables $x \in \mathbb{C}$ and $t \in \mathbb{C}$ and let

$$u = \partial_x^2 \log \tau . \tag{7.1}$$

In the new variable τ, the KdV equation (1.2) becomes

$$\left(D_x^4 - 4D_x D_t\right) \ \tau \cdot \tau = 0 . \tag{7.2}$$

Why did Hirota want to rewrite the equation in this strange form? The answer is this: by using the new form, one can find *exact* solutions of the KdV equation by finite iterated approximation in the form of polynomials in simple exponential functions. The method he used, now known as *Hirota's direct method*, is therefore something like the Padé approximation. In order to see how it works, let us define

$$\tau = 1 + \tau_1 + \tau_2 + \tau_3 + \cdots ,$$
$$\tau_1 = \sum_{i=1}^{N} a_i e^{2(k_i x + \omega_i t)} ,$$

where τ_j is a polynomial of degree j in exponential functions of linear forms in x and t, such as $e^{2(kx+\omega t)}$. The simplest nontrivial case is

$$\tau = 1 + \tau_1 ,$$
$$\tau_1 = a e^{2(kx+\omega t)} .$$

Since the constant function and the exponential function of linear forms give the trivial solution of the KdV equation because of (7.1), the Hirota bilinear equation (7.2) for $\tau = 1 + \tau_1$ gives

$$\begin{aligned}
0 &= \left(D_x^4 - 4D_x D_t\right)(1 \cdot 1 + 1 \cdot \tau_1 + \tau_1 \cdot 1 + \tau_1 \cdot \tau_1) \\
&= \left(D_x^4 - 4D_x D_t\right)(1 \cdot \tau_1 + \tau_1 \cdot 1) \\
&= 2\left(\partial_x^4 - 4\partial_x \partial_t\right)\tau_1 \\
&= 2\left(2^4 k^4 - 4 \cdot 2k \cdot 2\omega\right)\tau_1 ,
\end{aligned}$$

which simply means $\omega = k^3$. Thus

$$u(x, t) = \partial_x^2 \log \left(1 + a e^{2(kx+k^3 t)}\right) = \frac{4k^2 a}{\left(e^{-kx-k^3 t} + a e^{kx+k^3 t}\right)^2}$$

is a solution. This is nothing but the one-soliton solution of the KdV equation, which recovers the solutions (1.9) and (1.10) of Section 1 by setting $k = \sqrt{c}$ and $a =$

± 1. More generally, Hirota found N-soliton solutions depending on $2N$ parameters $a_1, \cdots, a_N$ and $k_1, \cdots, k_N$ in the form of

$$
\begin{aligned}
\tau &= 1 + \tau_1 + \tau_2 + \cdots + \tau_N \ , \\
\tau_1 &= \sum_{n=1}^{N} a_n e^{2(k_n x + k_n^3 t)}, \\
\tau_2 &= \sum_{m<n}^{\binom{N}{2}\text{-terms}} c\left(k_m, k_n\right) a_m a_n e^{2(k_m x + k_m^3 t) + 2(k_n x + k_n^3 t)}, \\
\tau_3 &= \sum_{l<m<n}^{\binom{N}{3}\text{-terms}} c\left(k_l, k_m, k_n\right) a_l a_m a_n e^{2(k_l x + k_l^3 t) + 2(k_m x + k_m^3 t) + 2(k_n x + k_n^3 t)}, \\
&\vdots \\
\tau_N &= c\left(k_1, \cdots, k_N\right) a_1 \cdots a_N e^{2(k_1 x + k_1^3 t) + \cdots + 2(k_N x + k_N^3 t)},
\end{aligned}
$$

where

$$
\begin{aligned}
c\left(k_m, k_n\right) &= \left(\frac{k_m - k_n}{k_m + k_n}\right)^2 , \\
c\left(k_1, k_2, \cdots, k_r\right) &= \prod_{1 \le m < n \le r}^{\binom{r}{2}\text{-terms}} c\left(k_m, k_n\right) \ .
\end{aligned}
$$

One can certainly appreciate Hirota's method here because without it the N-soliton solution would be too complicated to write down explicitly.

During 1978–79, Sato and Hirota ran a joint seminar on soliton theory at RIMS, Kyoto. Learning the direct method of Hirota, Sato went on to recognize the deep mathematical nature behind it. The bilinear form of the KP equation (1.1) is given by

$$
\left(D_1^4 + 3D_2^2 - 4D_1 D_3\right) \ \tau \cdot \tau = 0 \ ,
$$

where $\tau = \tau\left(t_1, t_2, t_3\right)$ and the variables x, y and t of (1.1) are identified with t_1, t_2, and t_3, respectively. The relation between τ and u is the same as before:

$$
u\left(t_1, t_2, t_3\right) = \partial_1^2 \, \log \, \tau\left(t_1, t_2, t_3\right) \ . \tag{7.3}
$$

First, Sato gave a more general form of soliton solutions of the KP equation depending on N parameters k_1, k_2, $\cdots$, k_N and an $N \times m$ matrix $\xi = \left(\xi_{ij}\right)_{1 \le i \le N, 1 \le j \le m}$ as follows:

$$
\tau\left(t\right) = \sum_{1 \le n_1 < \cdots < n_m \le N}^{\binom{N}{m}\text{-terms}} \Delta\left(k_{n_1}, \cdots, k_{n_m}\right) \xi_{n_1 \cdots n_m} e^{\eta(t, k_{n_1}) + \cdots + \eta(t, k_{n_m})} \ , \tag{7.4}
$$

where

$$\Delta\left(k_{n_1},\cdots,k_{n_m}\right)=\prod_{1\le i<j\le m}^{\binom{m}{2}\text{-terms}}\left(k_{n_i}-k_{n_j}\right)\ ,$$

$$\xi_{n_1\cdots n_m}=\det\begin{pmatrix}\xi_{n_1 1} & \cdots & \xi_{n_1 m}\\ \vdots & & \vdots\\ \xi_{n_m 1} & \cdots & \xi_{n_m m}\end{pmatrix}\ ,$$

$$\eta\,(t,k)=t_1k+t_2k^2+t_3k^3\ .$$

Surprisingly, the same formula gives an exact solution to the entire KP system by simply replacing the above $\eta\,(t,k)$ by

$$\eta\,(t,k)=\sum_{i=1}^{\infty}t_ik^i\ .$$

One can recover Hirota's soliton solutions of the KdV equation by specializing the parameters k and ξ in (7.4). The next step due to Sato [48] is a giant leap of genius. He noticed that formula (7.4) is still a very special case of a far more general form of exact solutions. Let us consider infinite-size matrices

$$X=\left(x_{ij}\right)_{-\infty<i<\infty,-\infty<j\le 0}=\left(\begin{array}{cccc}\ddots & \vdots & \vdots & \vdots\\ \cdots & x_{-2-2} & x_{-2-1} & x_{-20}\\ \cdots & x_{-1-2} & x_{-1-1} & x_{-10}\\ \cdots & x_{0-2} & x_{0-1} & x_{00}\\ \hline \cdots & x_{1-2} & x_{1-1} & x_{10}\\ \cdots & x_{2-2} & x_{2-1} & x_{20}\\ \cdots & x_{3-2} & x_{3-1} & x_{30}\\ & \vdots & \vdots & \vdots\end{array}\right)=\left(\frac{X_+}{X_-}\right) \tag{7.5}$$

and

$$\Lambda=\left(\delta_{i+1,j}\right)_{-\infty<i,j<\infty}=\begin{pmatrix}\ddots & & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \\ & & & & & \ddots\end{pmatrix}\ .$$

Let $(X)_+=X_+$ denote the upper-half square part of the rectangular matrix X. Then

Theorem 7.2 ([48]). *For every rectangular matrix X of (7.5), the function (called Sato's τ-function)*

$$\tau\left(t_1,t_2,t_3,\cdots;X\right)=\det\left(e^{t_1\Lambda+t_2\Lambda^2+t_3\Lambda^3+\cdots}\cdot X\right)_+$$

gives a solution of the entire KP system. In particular, it solves the KP equation (1.1) with respect to the first three variables through (7.3).

Here, we use the definition of infinite-size determinant due to Fredholm: for an infinite square matrix of the form

$$I + A_+ = (\delta_{ij})_{-\infty<i,j\le 0} + (a_{ij})_{-\infty<i,j\le 0}\ ,$$

we define

$$\det(I + A_+) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{-\infty<\ell_1,\cdots,\ell_n\le 0} \det\left(a_{\ell_i\ell_j}\right)_{1\le i,j\le n}$$

$$= 1 + \sum_{\ell} a_{\ell\ell} + \frac{1}{2!}\sum_{\ell_1,\ell_2}\begin{vmatrix} a_{\ell_1\ell_1} & a_{\ell_1\ell_2}\\ a_{\ell_2\ell_1} & a_{\ell_2\ell_2}\end{vmatrix} + \frac{1}{3!}\sum_{\ell_1,\ell_2,\ell_3}\begin{vmatrix} a_{\ell_1\ell_1} & a_{\ell_1\ell_2} & a_{\ell_1\ell_3}\\ a_{\ell_2\ell_1} & a_{\ell_2\ell_2} & a_{\ell_2\ell_3}\\ a_{\ell_3\ell_1} & a_{\ell_3\ell_2} & a_{\ell_3\ell_3}\end{vmatrix} + \cdots .$$

Let us define deg $t_n = n$. We say a series in $t = (t_1,\ t_2,\ t_3,\ \cdots)$ is *well-defined* if the number of terms of degree less than n is finite for every integer n. One can give a necessary and sufficient condition for a matrix X so that the expression of Theorem 7.2 is well-defined as a formal power series in $t = (t_1,\ t_2,\ t_3,\ \cdots)$. For example, if the upper-half part X_+ of X is the identity matrix, then Sato's τ-function is well-defined. More generally, $\tau(t; X)$ is well-defined if X_+ differs from the identity matrix in finitely many rows. Let us call a matrix X *admissible* if it has maximal rank (i.e., all the column vectors are linearly independent) and X_+ differs from the identity matrix in finitely many rows.

Proposition 7.3. *Every point W of the index zero Grassmannian $Gr(0)$ of Definition 3.2 gives rise to an admissible matrix. Conversely, every admissible matrix determines a unique point of $Gr(0)$. An admissible matrix of the form*

$$\left(\frac{I}{X_-}\right)$$

corresponds to a point in the big-cell $Gr^+(0)$. The τ-zero-value $\tau(0; X) = \det(X_+)$ is nonzero if and only if the corresponding point W belongs to the big-cell.

Proof. Let $W \in Gr(0)$. This means that W is a vector subspace of $\mathbb{C}((z))$ such that the projection $\gamma_W : W \to \mathbb{C}[z^{-1}]$ is Fredholm of index zero. We say an element $v \in \mathbb{C}((z))$ has (pole) *order n* if its leading term is cz^{-n}, $c \ne 0$. Following Segal-Wilson [52], we call a basis $\{w_0,\ w_{-1},\ w_{-2},\ \cdots\}$ for W *admissible* if each w_j is monic and satisfies ord $w_0 <$ ord $w_{-1} <$ ord $w_{-2} < \cdots$. Because of the Fredholm condition, we can choose w_j as the inverse image of the canonical basis $\{z^j\}_{-\infty<j\le 0}$ of $\mathbb{C}[z^{-1}]$ via the projection γ_W except for a finite number of elements. Each of the basis elements has an expansion

$$w_j = \sum_{i=-\infty}^{\infty} x_{ij} z^i\ , \qquad -\infty < j \le 0$$

with $x_{ij} = 0$ for $i << 0$, and, except for a finite number of j, it has the form

$$w_j = z^j + \sum_{i=\nu}^{\infty} x_{ij} z^i$$

for some $\nu \le 0$ independent of j. Define a matrix

$$X = (x_{ij})_{-\infty<i<\infty,-\infty<j\le 0} = \left(\frac{X_+}{X_-}\right)$$

with these coefficients. Since it has maximal rank and since X_+ differs from the identity matrix in only finitely many rows, our X is an admissible matrix. The converse and the other assertions are obvious from the above construction. □

The matrix Λ acts on X from the left by shifting of rows, which corresponds exactly to the action of z^{-1} on W by multiplication. In Section 5 (5.6), we showed that the vector fields on the Grassmannian $Gr(0)$ corresponding to the KP system are given by the infinitesimal action of $t_1 z^{-1}$, $t_2 z^{-2}$, $t_3 z^{-3}$, $\cdots$. The integral manifolds of these vector fields are therefore spanned by the group action of an operator

$$e^{t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots}$$

on the Grassmannian, which is the matrix operator appearing in Sato's formula (Theorem 7.2) under the identification of $\Lambda = z^{-1}$.

In order to complete the story, we have to explain how to construct the operator $S(t)$ of (4.10) from the τ-function of Theorem 7.2. For this purpose, let us define polynomials

$$p_n(t) = \sum_{n = n_1 + 2n_2 + 3n_3 + \cdots}^{\text{finite sum}} \frac{t_1^{n_1} \cdot t_2^{n_2} \cdot t_3^{n_3} \cdots}{n_1! \cdot n_2! \cdot n_3! \cdots}, \qquad n = 0, 1, 2, \cdots .$$

Each $p_n(t)$ is a weighted homogeneous polynomial in $t_1, \cdots, t_n$ of degree n. The generating function of these polynomials is given by

$$e^{t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots} = \sum_{n=0}^{\infty} p_n(t) z^{-n} . \tag{7.6}$$

Let

$$\partial_t = \left(\partial_1, \ \frac{1}{2}\partial_2, \ \frac{1}{3}\partial_3, \cdots \right) ,$$

where $\partial_n = \partial / \partial t_n$.

Theorem 7.4 ([48]). *For every admissible matrix X, let $\tau(t; X)$ be the τ-function of Theorem 7.2. Then the pseudo-differential operator $S(t)$ defined by*

$$S(t) = \sum_{n=0}^{\infty} \left. \frac{p_n(\partial_t)\ \tau(t; X)}{\tau(t; X)} \right|_{t_1 = x} \cdot \left(\frac{d}{dx} \right)^{-n}$$

gives a solution $L(t) = S(t) \cdot \partial \cdot S(t)^{-1}$ of the KP system. In particular, every regular solution can be obtained from an admissible matrix X corresponding to a point of the big-cell of the Grassmannian in this way.

Sato derived Theorem 7.4 from the standard theory of Wronskian matrices and determinants. Indeed, after the substitution $t_1 \longmapsto x$, $\tau(t; X)$ is really an infinite-size Wronskian determinant and $S(t)$ is the operator corresponding to the Wronskian matrix

$$e^{x\Lambda} \cdot e^{t_2 \Lambda^2 + t_3 \Lambda^3 + \cdots} \cdot X .$$

The key point is that

$$\frac{d}{dx} e^{x\Lambda} \cdot e^{t_2 \Lambda^2 + t_3 \Lambda^3 + \cdots} \cdot X = \Lambda \cdot e^{x\Lambda} \cdot e^{t_2 \Lambda^2 + t_3 \Lambda^3 + \cdots} \cdot X ,$$

which is the same matrix shifted by a row.

Remark. One can give a condition for regularity of the operator $S(t)$ which corresponds to the entire Grassmannian $Gr(0)$ [49]. However, there are more general solutions of (4.10) which can be obtained by the method of the generalized Birkhoff decomposition of Section 4 but which have no corresponding points on our Grassmannian. The τ-functions associated with these more general solutions have essential singularities at the origin.

Now let us give the geometric meaning of the τ-function via the Krichever functor of Section 3. Since we are dealing with the space $V = \mathbb{C}((z))$ of formal Laurent series rather than a Hilbert space, we cannot define a complex manifold structure on $Gr(\mu)$. We have to compare our Grassmannian with finite-dimensional Grassmannians in order to introduce an algebraic structure. Choose a pair of positive integers (m, n) and consider the subspace

$$V_{m,n} = \mathbb{C} \cdot z^{-m+1} \oplus \mathbb{C} \cdot z^{-m+2} \oplus \cdots \oplus \mathbb{C} \cdot z^{n-1} \oplus \mathbb{C} \cdot z^{n} \subset V$$

of dimension $m + n$. We denote by $Gr_{m,n}$ the Grassmannian consisting of m-dimensional vector subspaces of $V_{m,n}$. For every element $W_{m,n} \in Gr_{m,n}$, we assign a vector subspace

$$W = W_{m,n} \oplus \mathbb{C}[z^{-1}] \cdot z^{-m} \subset V \ ,$$

which belongs to $Gr(0)$. This assignment defines an embedding

$$Gr_{m,n} \longrightarrow Gr(0)$$

for every pair (m, n).

Definition 7.5. The determinant line bundle DET^* on the Grassmannian $Gr(0)$ is a line bundle whose fiber at $W \in Gr(0)$ is the one-dimensional vector space

$$DET^*_W = \left(\bigwedge^{\max} \mathrm{Ker}\ \gamma_W \right)^* \bigotimes \bigwedge^{\max} \mathrm{Coker}\ \gamma_W \ .$$

By definition, the restriction of the determinant line bundle to the big-cell $Gr^+(0)$ is the trivial bundle. The restriction of DET^* to each finite-dimensional Grassmannian $Gr_{m,n} \subset Gr(0)$ is the very ample holomorphic line bundle $\mathcal{O}_{Gr_{m,n}}(1)$ on the complex algebraic manifold $Gr_{m,n}$. In this sense, we call DET^* a very ample *holomorphic* line bundle on $Gr(0)$. We denote by $\widetilde{Gr}(0)$ the total space of the $GL(1, \mathbb{C})$-principal fiber bundle associated with the dual of the determinant line bundle. The restriction of this principal bundle to each $Gr_{m,n}$ is denoted by $\widetilde{Gr}_{m,n}$.

Let W be a point of the Grassmannian $Gr(0)$ belonging to the embedded $Gr_{m,n}$; i.e., $W = W_{m,n} \oplus \mathbb{C}[z^{-1}] \cdot z^{-m}$ for some $W_{m,n} \in Gr_{m,n}$. We can define an admissible matrix $X = X_W$ associated with W following the proof of Proposition 7.3. It is easy to show that the τ-function $\tau(t; X_W)$ in this case is a polynomial (depending only on the first $m + n$ variables $t_1, \cdots, t_{m+n}$) known as the Schur polynomial. We have to note here that the correspondence $W \mapsto X_W$ is not unique unless we provide an ordered basis for $W_{m,n}$. Let $\{w_1, \cdots, w_m\}$ and $\{w'_1, \cdots, w'_m\}$ be two ordered bases of $W_{m,n}$ and let X and X' be the corresponding admissible matrices, respectively. Then the τ-functions associated with X and X' differ by a constant factor. More precisely, we have

$$\frac{\tau(t; X)}{\tau(t; X')} = \frac{w_1 \wedge \cdots \wedge w_m}{w'_1 \wedge \cdots \wedge w'_m} \ .$$

Since the ratio of two volume forms on $W_{m,n}$ is an element of $GL(1,\mathbb{C})$, we can conclude that the τ-function is a holomorphic function on the principal bundle $\widetilde{Gr}_{m,n}$. In this sense, we say that the τ-function $\tau(t;X)$ of Theorem 7.2 is a holomorphic function on $\widetilde{Gr}(0)$ depending holomorphically on X. The restriction of the τ-function to a generic fiber $GL(1,\mathbb{C})$ of $\widetilde{Gr}_{m,n}$ is a group endomorphism of the Abelian group $GL(1,\mathbb{C})$. Thus the logarithm of the τ-function $\log\tau(t;X)$ is a meromorphic function on the total space of the dual of the determinant line bundle which is linear on each fiber. In other words, $\log\tau(t;X)$ is a meromorphic section of the determinant line bundle DET^*.

Every point W determines a unique maximal Schur pair (A_W, W), where

$$A_W = \{v \in V \mid v \cdot W \subset W\} .$$

If A_W is nontrivial, then the pair corresponds to a quintuple $(C, p, \pi, \mathcal{F}, \phi)$ through the Krichever functor. Suppose that the point W is such that the corresponding curve C is nonsingular. Then the torsion-free sheaf $\mathcal{F}$ is actually a vector bundle on C. Let us choose a fiber metric h of $\mathcal{F}$ and consider the $\overline{\partial}$-complex

$$0 \longrightarrow C^\infty(\mathcal{F}) \xrightarrow{\overline{\partial}} C^\infty(\mathcal{F}) \otimes \wedge^{0,1}(C) \longrightarrow 0 , \tag{7.7}$$

where $C^\infty(\mathcal{F})$ denotes the set of all C^∞-sections of $\mathcal{F}$ and $\wedge^{0,1}(C)$ is the set of (0,1)-forms on C. By Dolbeault's theorem, we have the isomorphisms

$$\text{Ker } \overline{\partial} \cong H^0(C, \mathcal{F})$$
$$\text{Coker } \overline{\partial} \cong H^1(C, \mathcal{F})$$

depending on the metric h. Note that (3.4) defines a complex

$$0 \longrightarrow W \xrightarrow{\gamma_W} V/V_- \longrightarrow 0 \tag{7.8}$$

having the same cohomology groups

$$\text{Ker } \gamma_W \cong H^0(C, \mathcal{F})$$
$$\text{Coker } \gamma_W \cong H^1(C, \mathcal{F})$$

by Lemma 3.7. Thus complexes (7.7) and (7.8) are quasi-isomorphic. Recall that the determinant of a Fredholm operator

$$0 \longrightarrow V_1 \xrightarrow{\alpha} V_2 \longrightarrow 0$$

is defined to be an element

$$\det(\alpha) \in \left(\bigwedge^{\max} \text{Ker } \alpha \right)^* \bigotimes \bigwedge^{\max} \text{Coker } \alpha ,$$

which is nonzero if and only if α is an isomorphism. Therefore, we have an equality

$$\det\left(\overline{\partial}_{\mathcal{F}}\right) = c_h \cdot \det\left(\gamma_W\right) . \tag{7.9}$$

The nonzero factor c_h depends on the metric h.

Problem. Calculate the factor c_h of (7.9).

The vector bundle $\mathcal{F}$ is semi-stable if the corresponding point W belongs to the big-cell [35]. If the bundle is strictly stable, then it has a canonical fiber metric known as the Einstein-Hermitian metric [21]. It will be an interesting problem to determine c_h for this canonical metric (see Fay [9]).

As we have noted in Proposition 7.3, the τ-zero-value $\tau(0;X)$ is proportional to $\det(\gamma_W)$ if the matrix X corresponds to a point $W \in Gr(0)$. Thus we have

$$\tau(0;X_W) = c \cdot \det\left(\overline{\partial}_{\mathcal{F}}\right) , \tag{7.10}$$

where this time the factor depends on the choice of the fiber metric h of $\mathcal{F}$ and the volume element (the semi-infinite product of [10]) of the vector space W. In the sense of (7.10), we can say that the τ-function is essentially the *analytic torsion* of the vector bundle $\mathcal{F}$ on the curve C.

Let $W \in Gr(0)$ be a point such that the orbit of the KP flows has finite dimension. We have seen in Section 5 that the orbit is canonically isomorphic to the Jacobian variety $\mathrm{Pic}^{g-1}(C)$ of the curve C of genus g of the quintuple (C, p, π, $\mathcal{F}$, ϕ) corresponding to (A_W, W). It is easy to see that finite dimensionality of the orbit implies that $\mathcal{F}$ is a line bundle, i.e. (A_W, W) has rank one. Once we choose a matrix X_W for the point W, $\tau(t;X_W)$ becomes a holomorphic function in t which is proportional to the determinant of the Cauchy-Riemann operator of the line bundle $\mathcal{F}(t) = \mathcal{F} \otimes \mathcal{L}(t)$ whose transition function is given by

$$e(t) \cdot \psi = e^{t_1 z^{-1} + t_2 z^{-2} + t_3 z^{-3} + \cdots} \cdot \psi \ .$$

Here, $e(t)$ and $\mathcal{L}(t)$ are the cohomology classes of (5.8) and ψ is the transition function of the original line bundle $\mathcal{F}$ written in terms of the local coordinate $\pi(z)$ on the neighborhood U_p around $p \in C$ with respect to the Stein covering

$$C = (C \setminus \{p\}) \cup U_p$$

of the curve C. Since $\tau(t;X_W)$ vanishes if and only if

$$H^1(C, \mathcal{F}(t)) \neq 0 \ ,$$

it must be (by definition) the *theta* function associated with the Jacobian variety. Thus on the locus of the Grassmannian on which the KP flows produce finite-dimensional orbits, we have

$$\tau\text{-functions} = \text{theta functions} = \text{analytic torsion of line bundles}$$

up to nonzero factors. This explains the appearance of the elliptic functions, which are just theta functions of genus one, in the beginning (1.6) of this paper.

The τ-function is therefore a determinant in a two-fold way: by Theorem 7.2, which can be understood as a Wronskian determinant, and by (7.10) through the Krichever functor, which gives essentially the analytic torsion of arbitrary vector bundles defined on an algebraic curve. This is one of the mathematical structures hidden behind Hirota's ingenious substitution (7.1) some twenty years ago!

8. Hermitian matrix integrals as τ-functions

The Riemann theta functions associated with Jacobian varieties are τ-functions of the KP system. They correspond to finite-dimensional solutions of the KP system. Conversely, the results of [31] show that these solutions exhaust all finite-dimensional solutions if we take generalized Jacobian varieties into account. Quite recently, it has been discovered that some matrix integrals give examples of τ-functions corresponding to infinite-dimensional orbits. These solutions carry cohomological information on the topology of moduli spaces of algebraic curves of arbitrary genus [59] [16]. In this section, we study the simplest example of Hermitian matrix models. We first prove that the Hermitian matrix integral is itself an infinite-dimensional solution of the KP system with respect to the coupling constants. Amazingly, this solution gives the generating function of the Euler characteristics of the moduli spaces of algebraic curves of arbitrary genus together with marked points by substituting the coupling constants by powers of a single parameter.

The relation of our matrix integrals and the topology of the moduli spaces of curves is treated elsewhere [37]. In this article, we concentrate on more computational side of the theory. The standard references of the matrix integrals we deal with in this section are [4] and [42]. For deeply related, exiting new developments of the theory of random matrices, we refer to the fundamental paper by Tracy and Widom [55].

A finite-dimensional solution gives a commutative algebra of ordinary differential operators. Then what does an infinite-dimensional solution represent? Strikingly, Kontsevich discovered that the point of the Grassmannian corresponding to the solution coming from a matrix integral has a stabilizer algebra which is isomorphic to $sl(2,\mathbb{C})$. We can further prove that the stability by the $sl(2,\mathbb{C})$ algebra determines the point of the Grassmannian uniquely, and that this solution gives rise to an embedding of the universal enveloping algebra of $sl(2,\mathbb{C})$ into the ring D of ordinary differential operators.

We also give a new proof of an alternative formula for Sato's τ-function due to Kontsevich at the end of this section.

Let us consider the real line $\mathbb{R}^1$ with the Gaussian measure $\exp\big(-\frac{1}{2}k^2\big)dk$. The probability for a real number k to be in a subset K of $\mathbb{R}^1$ is then given by $Z_1(0;K)/Z_1(0;\mathbb{R}^1)$, where

$$Z_1(0;K) = \int_K e^{-\frac{1}{2}k^2}dk$$

is the relative probability. In the same spirit, the relative probability for a Hermitian matrix H to have its eigenvalues in $K \subset \mathbb{R}^1$ is given by

$$Z_n(0;K) = \int_{\mathcal{H}_K} e^{-\frac{1}{2}\text{trace } H^2}dH \ , \tag{8.1}$$

where $\mathcal{H}_K$ denotes the set of all Hermitian matrices whose eigenvalues are in K. Here we use the probability measure

$$\exp\left(-\frac{1}{2}\text{trace } H^2\right)dH$$

on the set of all $n \times n$ Hermitian matrices. It is well-known that (8.1) can also be written as

$$Z_n(0;K) = c\int_{K^n} \exp\left(-\frac{1}{2}\sum_{j=0}^{n-1}k_j^2\right)\Delta(k_0,\cdots,k_{n-1})^2 dk_0\cdots dk_{n-1}$$

for some constant c, where $\Delta(k_0,\cdots,k_{n-1})$ is the Vandermonde determinant as in Section 7.

If we regard (8.1) as the *free matrix model*, then we can likewise consider an *interacting matrix model*

(8.2)
$$
\begin{aligned}
&Z_n(t;K)\\
&= \int_{\mathcal{H}_K} \exp\left(\sum_{i=1}^{\infty} t_i \cdot \text{trace } H^i\right) \exp\left(-\frac{1}{2}\text{trace } H^2\right) dH\\
&= c\int_{K^n} \exp\left(\sum_{j=0}^{n-1} \eta(t,k_j)\right) \exp\left(-\frac{1}{2}\sum_{j=0}^{n-1} k_j^2\right) \Delta(k_0,\cdots,k_{n-1})^2 dk_0\cdots dk_{n-1}\ ,
\end{aligned}
$$

where

$$
\eta(t,k) = \sum_{i=1}^{\infty} t_i k^i
$$

is a general potential with coupling constants $t = (t_1, t_2, t_3, \cdots)$. Usually, we set $t_i = 0$ for all but finitely many even variables. However, the integral (8.2) always converges for arbitrary negative t_{2i} and purely imaginary t_{2i-1} so that it defines a complex holomorphic function in $t = (t_1, t_2, t_3, \cdots)$ on a wedge-shaped domain with the origin $t = 0$ as its vertex. Therefore, (8.2) makes sense as a formal power series in $t = (t_1, t_2, t_3, \cdots)$, which gives an asymptotic expansion of this analytic function defined on the complex domain.

Surprisingly, (8.2) is a τ-function of the KP system for every K with respect to the coupling constants! However, we emphasize here that the integral $Z_n(t;\mathbb{R})$ for $K = \mathbb{R}$ is *not* a τ-function in the sense of Segal-Wilson [52], but only in the sense of Theorem 7.2. In fact, the radius of convergence of (8.2) for $K = \mathbb{R}$ is 0 and the point of the Grassmannian $Gr(0)$ corresponding to this solution does not belong to the Grassmannian of [52] that is modeled on the Hilbert space of L^2-functions on a circle.

Maxim Kontsevich told me in 1991 that $Z_n(t;\mathbb{R})$ satisfies the KP system. The following proof I gave is applicable for an arbitrary $K \subset \mathbb{R}$. Many people have obtained similar results more or less independently.

Theorem 8.1. *Let $K \subset \mathbb{R}$ be a subset of positive measure. Then the formal power series $Z_n(t;K)$ in t is a τ-function of the KP system corresponding to a point $W_n(K)$ of the big-cell $Gr^+(0)$ defined by*

$$
W_n(K) = \bigoplus_{j=-n+1}^{0} \mathbb{C}\cdot w_j(K) \oplus \bigoplus_{m=n}^{\infty} \mathbb{C}\cdot z^{-m}\ ,
$$

where

$$
\begin{aligned}
w_j(K) &= \sum_{i=-n+1}^{\infty} \int_K \left(e^{-\frac{k^2}{2}} k^{i+j+2n-2} dk\right)\cdot z^i\\
&= \int_K e^{-\frac{k^2}{2}} \frac{k^{j+n-1} z^{-n+1}}{1-kz} dk\ .
\end{aligned}
$$

Proof. Our proof is based on the following formula for the Vandermonde determinant.

Lemma 8.2. *Let S_n denote the full symmetric group on n letters. Then*

$$\Delta(k_0, k_1, \cdots, k_{n-1})^2 = \sum_{\sigma \in S_n} \det \begin{pmatrix} 1 & k_{\sigma(1)} & k^2_{\sigma(2)} & \dots & k^{n-1}_{\sigma(n-1)} \\ k_{\sigma(0)} & k^2_{\sigma(1)} & k^3_{\sigma(2)} & \dots & k^{n}_{\sigma(n-1)} \\ k^2_{\sigma(0)} & k^3_{\sigma(1)} & k^4_{\sigma(2)} & \dots & k^{n+1}_{\sigma(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ k^{n-1}_{\sigma(0)} & k^{n}_{\sigma(1)} & k^{n+1}_{\sigma(2)} & \dots & k^{2n-2}_{\sigma(n-1)} \end{pmatrix}$$
$$= \sum_{\sigma \in S_n} \det \left(k^{\ell+m}_{\sigma(m)} \right)_{0 \le \ell, m \le n-1} .$$

This lemma can be shown by observing that the quantity is a homogeneous symmetric polynomial in $k_1, k_2, \cdots, k_{n-1}$ of degree $n(n-1)$ such that (i) it vanishes if $k_i = k_j$ for $i \neq j$, and (ii) the coefficient of $k_1^2 k_2^4 \cdots k_{n-1}^{2n-2}$ is equal to 1.

The matrix $X_n(K) = \big(x_{ij}(K)\big)_{-\infty < i < \infty, -\infty < j \le 0}$ of (7.5) corresponding to the point $W_n(K)$ can be easily calculated:

$$x_{ij}(K) = \begin{cases} \int_K e^{-\frac{k^2}{2}} k^{i+j+2n-2} dk & \text{for } -n+1 \le i, \ -n+1 \le j \le 0 \\ \delta_{ij} & \text{otherwise.} \end{cases} \tag{8.3}$$

Since the basis vectors for $W_n(K)$ differ from those for the base point $\mathbb{C}[z^{-1}] \in Gr^+(0)$ by finitely many vectors, the τ-function of Theorem 7.2 defined by the matrix (8.3) is a finite-size determinant. Let us rewrite our matrix (8.3) as

$$\xi_{\ell m} = \xi_{\ell m}(K) = x_{-n+1+\ell, -n+1+m}(K) = \int_K e^{-\frac{k^2}{2}} k^{\ell+m} dk ,$$

where $0 \le \ell$ and $0 \le m \le n-1$. Then the τ-function corresponding to the point $W_n(K)$ is given by

$$\tau\big(t; X_n(K)\big) =$$
$$\det \left(\left[\begin{array}{ccccc|cc} 1 & p_1(t) & p_2(t) & \dots & p_{n-1}(t) & p_n(t) & \dots \\ & 1 & p_1(t) & \dots & p_{n-2}(t) & p_{n-1}(t) & \dots \\ & & \ddots & & \vdots & \vdots & \\ & & & 1 & p_1(t) & p_2(t) & \dots \\ & & & & 1 & p_1(t) & \dots \end{array} \right] \left[\begin{array}{ccc} \xi_{0,0} & \cdots & \xi_{0,n-1} \\ \xi_{1,0} & \cdots & \xi_{1,n-1} \\ \vdots & & \vdots \\ \xi_{n-1,0} & \cdots & \xi_{n-1,n-1} \\ \hline \xi_{n,0} & \cdots & \xi_{n,n-1} \\ \xi_{n+1,0} & \cdots & \xi_{n+1,n-1} \\ \xi_{n+2,0} & \cdots & \xi_{n+2,n-1} \\ \vdots & & \vdots \end{array} \right] \right)$$
$$= \det \left(\sum_{\alpha=0}^{\infty} p_{-\ell+\alpha}(t) \cdot \xi_{\alpha m} \right)_{0 \le \ell, m \le n-1}$$
$$= \det \left(\int_K e^{-\frac{k^2}{2}} \sum_{\alpha=0}^{\infty} p_{-\ell+\alpha}(t) k^{\alpha+m} dk \right)_{0 \le \ell, m \le n-1}$$
$$= \det \left(\int_K e^{-\frac{k^2}{2}} e^{\eta(t,k)} k^{\ell+m} dk \right)_{0 \le \ell, m \le n-1} .$$

Since k is an integration variable, we can replace it by a new variable k_m for each m. Then

$$
\begin{aligned}
&\tau\big(t; X_n(K)\big)\\
&= \det\left(\int_K e^{-\frac{k_m^2}{2}} e^{\eta(t,k_m)} k_m^{\ell+m} dk_m\right)_{0\le \ell,m\le n-1}\\
&= \int_{K^n} \exp\left(\sum_{m=0}^{n-1}\eta(t,k_m)\right)\exp\left(-\frac{1}{2}\sum_{m=0}^{n-1}k_m^2\right)\cdot\\
&\qquad\qquad \det\left(k_m^{\ell+m}\right)_{0\le \ell,m\le n-1} dk_0 dk_1\cdots dk_{n-1}\\
&= \int_{K^n} \exp\left(\sum_{m=0}^{n-1}\eta(t,k_m)\right)\exp\left(-\frac{1}{2}\sum_{m=0}^{n-1}k_m^2\right)\cdot\\
&\qquad\qquad \frac{1}{n!}\sum_{\sigma\in S_n}\det\left(k_{\sigma(m)}^{\ell+m}\right)_{0\le \ell,m\le n-1} dk_0 dk_1\cdots dk_{n-1}\\
&= \frac{1}{n!}\int_{K^n} \exp\left(\sum_{m=0}^{n-1}\eta(t,k_m)\right)\exp\left(-\frac{1}{2}\sum_{m=0}^{n-1}k_m^2\right)\cdot\\
&\qquad\qquad \Delta\big(k_0,\cdots,k_{n-1}\big)^2 dk_0 dk_1\cdots dk_{n-1}\\
&= \frac{1}{cn!}\int_{\mathcal{H}_K} \exp\left(\sum_{i=1}^{\infty} t_i\cdot \text{trace } H^i\right)\exp\left(-\frac{1}{2}\text{trace } H^2\right) dH\\
&= \frac{1}{cn!} Z_n(t;K)\ .
\end{aligned}
$$

The above computation also shows that the point $W_n(K)$ belongs to the big-cell $Gr^+(0)$ if $K \subset \mathbb{R}$ has positive measure, since

$$
\begin{aligned}
\det\big(X_n(K)_+\big) &= \det\begin{pmatrix} \xi_{0,0} & \cdots & \xi_{0,n-1}\\ \xi_{1,0} & \cdots & \xi_{1,n-1}\\ \vdots & & \vdots\\ \xi_{n-1,0} & \cdots & \xi_{n-1,n-1}\end{pmatrix}\\
&= \tau\big(0; X_n(K)\big) = \frac{1}{cn!} Z_n(0;K) \neq 0\ .
\end{aligned}
$$

This completes the proof of Theorem 8.1. □

Let $W \in Gr(0)$ be a point of the Grassmannian. As we have seen in the previous sections, if the *maximal commutative stabilizer*

$$\mathbb{C} \subset A_W = \{a \in \mathbb{C}((z))\ | a\cdot W \subset W\}$$

contains more than constant elements (i.e., if $A_W \neq \mathbb{C}$), then the pair (A_W, W) is a Schur pair of Section 3, and hence it corresponds to a geometric quintuple $(C, p, \pi, \mathcal{F}, \phi)$ by the functor χ^{-1} of Theorem 3.6. For the point $W = W_n(\mathbb{R})$, we will show later that the maximal commutative stabilizer A_W is indeed equal to $\mathbb{C}$. Therefore, algebraic geometry cannot tell us anything about $W_n(\mathbb{R})$. However, $W_n(\mathbb{R})$ has a totally different kind of stabilizer algebra:

Theorem 8.3 (Kontsevich)**.** *The point $W_n(\mathbb{R})$ of the Grassmannian of Theorem 8.1 is stable under the action of the following three differential operators:*

$$\begin{cases} L_{-1}(n) = z^2\frac{d}{dz} + nz - z^{-1} \\ L_0(n) = z\frac{d}{dz} + \frac{3n-1}{2} - z^{-2} \\ L_1(n) = \frac{d}{dz} + (2n-1)z^{-1} - z^{-3}. \end{cases}$$

Remark. The operators $L_{-1}(n)$, $L_0(n)$ and $L_1(n)$ satisfy the commutation relation

$$\big[L_i(n), L_j(n)\big] = (i-j)L_{i+j}(n)\ .$$

Therefore, the Lie algebra generated by $L_{-1}(n)$, $L_0(n)$ and $L_1(n)$ is isomorphic to $sl(2,\mathbb{C})$. The above theorem asserts that the point $W_n(\mathbb{R})$ is stable under the action of the universal enveloping algebra of $sl(2,\mathbb{C})$.

We can extend our definition of the operators to

$$L_i(n) = z^{1-i}\frac{d}{dz} + \frac{3n-1+i(n-1)}{2}z^{-i} - z^{-i-2}$$

for all $i \in \mathbb{Z}$, which generate the Virasoro algebra without center. However, our point $W_n(\mathbb{R})$ is *not* stable under these Virasoro generators except for the generators of the $sl(2,\mathbb{C})$-subalgebra.

Proof. For our purpose, it is more convenient to rewrite the basis vectors of $W_n(\mathbb{R})$ as

$$w_m = \sum_{\ell=0}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\frac{k^2}{2}}k^{\ell+m}dk\right) z^{-n+1+\ell}$$

for $0 \le m \le n-1$. First of all, we note that the three operators $L_{-1}(n), L_0(n)$ and $L_1(n)$ stabilize the subspace of $W_n(\mathbb{R})$ spanned by z^{-j} for $j \ge n$. So let us compute the action of $L_i(n)$ on w_m. The trick we use is a formula

$$\ell\int_{-\infty}^{\infty} e^{-\frac{k^2}{2}}k^{\ell+m}dk = \int_{-\infty}^{\infty} e^{-\frac{k^2}{2}}k^{\ell+m+2}dk - (m+1)\int_{-\infty}^{\infty} e^{-\frac{k^2}{2}}k^{\ell+m}dk\ ,$$

which follows immediately from

$$\begin{aligned} 0 &= \int \frac{d}{dk}\left(e^{-\frac{k^2}{2}}k^{\ell+m+1}\right)dk \\ &= -\int e^{-\frac{k^2}{2}}k^{\ell+m+2}dk + (\ell+m+1)\int e^{-\frac{k^2}{2}}k^{\ell+m}dk\ . \end{aligned}$$

Now we have

$$
\begin{aligned}
&L_i(n)\cdot w_m\\
&=\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m}\Bigg((-n+1+\ell)z^{-n+1+\ell-i}+\frac{3n-1+i(n-1)}{2}z^{-n+1+\ell-i}\\
&\qquad -z^{-n+1+\ell-i-2}\Bigg)dk\\
&=\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m}\ell z^{-n+1+\ell-i}dk+\frac{n+1+i(n-1)}{2}\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m}z^{-n+1+\ell-i}dk\\
&\qquad -\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m}z^{-n+1+\ell-i-2}dk\\
&=\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m+2}z^{-n+1+\ell-i}dk\\
&\qquad +\frac{n+1+i(n-1)-2m-2}{2}\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m}z^{-n+1+\ell-i}dk\\
&\qquad -\sum_{\ell=0}^{\infty}\int e^{-\frac{k^2}{2}}k^{\ell+m+i+2}z^{-n+1+\ell}dk-\sum_{j=0}^{i+1}\int e^{-\frac{k^2}{2}}k^{-j+m+i+1}z^{-n-j}dk\ .
\end{aligned}
$$

If $i\geq 1$, then

$$
\begin{aligned}
&L_i(n)\cdot w_m\\
&=\sum_{j=0}^{i-1}\int e^{-\frac{k^2}{2}}k^{-j+m+i+1}z^{-n-j}dk-\\
&\quad\sum_{j=0}^{i+1}\int e^{-\frac{k^2}{2}}k^{-j+m+i+1}z^{-n-j}dk+(1-1)w_{m+i+2}\\
&\qquad +\frac{n+1+i(n-1)-2m-2}{2}\left(\sum_{j=0}^{i-1}\int e^{-\frac{k^2}{2}}k^{-j+m+i-1}z^{-n-j}dk+w_{m+i}\right)\\
&=-\int e^{-\frac{k^2}{2}}k^{m+1}z^{-n-i}dk-\int e^{-\frac{k^2}{2}}k^{m}z^{-n-i+1}dk\\
&\qquad +\frac{n+1+i(n-1)-2m-2}{2}\sum_{j=0}^{i-1}\int e^{-\frac{k^2}{2}}k^{-j+m+i-1}z^{-n-j}dk\\
&\qquad +\frac{n+1+i(n-1)-2m-2}{2}\cdot w_{m+i}\ .
\end{aligned}
$$

The last term is 0 for $(i,m)=(1,n-1)$. Therefore, $L_i(n)\cdot w_m$ is a linear combination of the basis vectors of $W_n(\mathbb{R})$ for $i=1$, but $L_i(n)$ for $i\geq 2$ does not stabilize $W_n(\mathbb{R})$. When $i=0$, we have

$$
L_0(n)\cdot w_m=\frac{n-2m-1}{2}\cdot w_m-\int e^{-\frac{k^2}{2}}k^{m+1}z^{-n}dk-\int e^{-\frac{k^2}{2}}k^{m}z^{-n-1}dk\in W_n(\mathbb{R})\ .
$$

Finally, for $i=-1$, we have

$$\begin{aligned}
&L_{-1}(n)\cdot w_m\\
&=-\int e^{-\frac{k^2}{2}}k^{m+1}z^{-n+1}dk-\int e^{-\frac{k^2}{2}}k^m z^{-n}dk+\\
&\quad m\int e^{-\frac{k^2}{2}}k^{m-1}z^{-n+1}dk-mw_{m-1}\\
&=\int\frac{d}{dk}\left(e^{-\frac{k^2}{2}}k^m z^{-n+1}\right)dk-\int e^{-\frac{k^2}{2}}k^m z^{-n}dk-mw_{m-1}\\
&=-\int e^{-\frac{k^2}{2}}k^m z^{-n}dk-mw_{m-1}\ ,
\end{aligned}$$

where the only possible problem comes from the term mw_{m-1} for $m=0$. But it does not occur because $m=0$ kills it. This completes the assertion of Kontsevich. $\square$

Theorem 8.4. *For an arbitrary complex number q, we define a formal power series $w(z)\in\mathbb{C}[[z]]$ by*

$$w(z)=\sum_{m=0}^{\infty}\frac{1}{2^m\cdot m!}\frac{\Gamma(q+2m)}{\Gamma(q)}\cdot z^{2m}=\sum_{m=0}^{\infty}(2m-1)!!\binom{q+2m-1}{q-1}\cdot z^{2m}\ ,$$

where

$$(2m-1)!!=\frac{(2m)!}{2^m\cdot m!}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{k^2}{2}}k^{2m}dk\ .$$

This is the unique regular solution of the differential equation

$$\left[z^3\left(\frac{d}{dz}\right)^2+(2q+2)z^2\frac{d}{dz}-\frac{d}{dz}+q(q+1)z\right]\cdot w(z)=0 \tag{8.4}$$

with $w(0)=1$. Let $W_q\in Gr^+(0)$ be a point of the big-cell defined by

$$W_q=\bigoplus_{\ell=0}^{\infty}\left(z^2\frac{d}{dz}+qz-z^{-1}\right)^{\ell}\cdot w(z)=\mathbb{C}\big[L_{-1}(q)\big]\cdot w(z)\ .$$

Then it satisfies the stability condition

$$L_i(q)\cdot W_q\subset W_q,\qquad i=-1,\ 0,\ 1, \tag{8.5}$$

where

$$\begin{cases}L_{-1}(q)=z^2\frac{d}{dz}+qz-z^{-1}\\ L_0(q)=z\frac{d}{dz}+\frac{3q-1}{2}-z^{-2}\\ L_1(q)=\frac{d}{dz}+(2q-1)z^{-1}-z^{-3}\end{cases}$$

are generators of the universal enveloping algebra of $sl(2,\mathbb{C})$. Moreover, the stability condition (8.5) uniquely determines the point $W_q\in Gr^+(0)$. In particular, when $q=n$, our point W_q is precisely equal to the point $W_n(\mathbb{R})$ defined by the matrix model.

Proof. Let $W\in Gr^+(0)$ be a point stable under $L_{-1}(q)$, $L_0(q)$ and $L_1(q)$. Since W is in the big-cell, it has a power series $w(z)\in\mathbb{C}[[z]]$ with $w(0)=1$. Then W has to be generated by $L_{-1}(q)^{\ell}\cdot w(z)$ for $\ell\geq 0$ because $L_{-1}(q)$ has a term z^{-1}. The stability under $L_0(q)$ is then equivalent to

$$-L_0(q)\cdot w=a_0w+a_1L_{-1}(q)\cdot w+a_2(L_{-1}(q))^2\cdot w \tag{8.6}$$

for some constants a_0, a_1, a_2. Similarly, the $L_1(q)$-stability gives

$$-L_1(q) \cdot w = b_0 w + b_1 L_{-1}(q) \cdot w + b_2 L_0(q) \cdot w + \tag{8.7}$$

One can immediately derive that it is necessary to have $a_0 = (q-1)/2$, $a_1 = 0$, $a_2 = 1$, $b_0 = 0$, $b_1 = (1-q)/2$, $b_2 = 0$ and $b_3 = 1$ in order for these equations to have a simultaneous regular solution $w(z) \in \mathbb{C}[[z]]$. Then both (8.6) and (8.7) coincide with (8.4). Although (8.4) has two independent solutions, only one has a regular power series expansion at $z = 0$. (The other solution contains negative powers of z such as z^{-q-m}, $m \geq 0$.) The Taylor series solution of (8.4) with $w(0) = 1$ is thus uniquely determined. Since $W_n(\mathbb{R})$ satisfies the same stability condition (8.5) for $q = n$, we have $W_n = W_n(\mathbb{R})$. This completes the proof. $\square$

From Theorem 8.4, we can prove that the maximal commutative stabilizer A_{W_n} is trivial. Indeed, if $a(z) \in \mathbb{C}((z))$ is a stabilizer of W_n of pole order $m > 0$, then

$$\big[L_{-1}(n), a(z)\big] = z^2 a'(z) \in A_{W_n}$$

has order $m-1$. Since

$$A_{W_n} \cap \mathbb{C}[[z]] = \mathbb{C} \ ,$$

we conclude that $a(z) \in A_{W_n} = \mathbb{C}[z^{-1}]$. But then

$$L_{-1}(q) + z^{-1} = z^2 \frac{d}{dz} + qz$$

should also stabilize W_q, which is impossible.

An unexpected byproduct of Theorem 8.4 is that we have constructed a one-parameter family of deformations of an embedding

$$U\big(sl(2,\mathbb{C})\big) \subset D$$

of the universal enveloping algebra of $sl(2,\mathbb{C})$ into the ring D of differential operators.

Corollary 8.5. *Let $S_q \in G_-$ be the operator of Theorem 3.3 corresponding to the point $W_q \in Gr^+(0)$ and let $\mathcal{A}_q$ denote the associative algebra generated by $L_{-1}(q), L_0(q)$ and $L_1(q)$, which is isomorphic to $U\big(sl(2,\mathbb{C})\big)$. Then*

$$\mathcal{B}_q = S_q \cdot \mathcal{A}_q \cdot S_q^{-1} \subset D \ .$$

Proof. The relation between S_q and W_q, i.e., $W_q = S_q^{-1} \cdot \mathbb{C}[z^{-1}]$, and the stability condition $\mathcal{A}_q \cdot W_q \subset W_q$ imply

$$S_q \cdot \mathcal{A}_q \cdot S_q^{-1} \cdot \mathbb{C}[z^{-1}] \subset \mathbb{C}[z^{-1}] \ .$$

Then by Lemma 3.9, we have the desired embedding

$$S_q \cdot \mathcal{A}_q \cdot S_q^{-1} \subset D \ .$$

$\square$

Let us determine the image of this embedding. We recall the relation $\partial = z^{-1}$ of (2.10). Since

$$[x, \partial] = \left[z^2 \frac{d}{dz},\ z^{-1}\right] = -1 \ ,$$

we can identify

$$x = z^2 \frac{d}{dz} \ .$$

Then we can rewrite the operators as

$$\begin{cases} L_{-1}(q) = -\partial + x + q\partial^{-1} \\ L_0(q) = -\partial^2 + \partial \cdot x + \frac{3q-1}{2} = -\partial^2 + x\partial + \frac{3q+1}{2} \\ L_1(q) = -\partial^3 + \partial^2 \cdot x + (2q-1)\partial = -\partial^3 + x\partial^2 + 2(q+1)\partial \ . \end{cases}$$

Since these operators uniquely determine the point W_q, the corresponding S_q is also unique. By a straightforward calculation, we obtain

$$\begin{cases} S_q \cdot L_{-1}(q) \cdot S_q^{-1} = -\partial + x \\ S_q \cdot L_0(q) \cdot S_q^{-1} = -\partial^2 + x\partial + \frac{1-q}{2} \\ S_q \cdot L_1(q) \cdot S_q^{-1} = -\partial^3 + x\partial^2 + (1-q)\partial \ . \end{cases} \tag{8.8}$$

How large is this class (8.8) among the $sl(2,\mathbb{C})$-subalgebras of the ring D of differential operators? The following proposition tells us that it is rather small.

Proposition 8.6. *Let L_{-1}, L_0 and L_1 be the generators of $sl(2,\mathbb{C})$. For an arbitrary function $f = f(x)$, the assignment*

$$\begin{cases} L_{-1} \longmapsto -\partial + f(x) \\ L_0 \longmapsto -\partial^2 + g_1(x)\partial + g_2(x) \\ L_1 \longmapsto -\partial^3 + h_1(x)\partial^2 + h_2(x)\partial + h_3(x) \end{cases}$$

gives an injective homomorphism of the universal enveloping algebra $U\big(sl(2,\mathbb{C})\big)$ into D, where

$$\begin{aligned} g_1 &= 2f - x + c_1 \ , \\ g_2 &= f' - f^2 + xf - c_1 f + c_2 \ , \\ h_1 &= 3f - 2x + 2c_1 \ , \\ h_2 &= 3f' - 3f^2 + 4xf - 4c_1 f - x^2 + 2c_1 x + 2c_2 - c_1^2 - 1 \ , \\ h_3 &= f'' - 3ff' + 2xf' - 2c_1 f' + f^3 - 2xf^2 + 2c_1 f^2 + x^2 f - 2c_1 xf \\ &\quad - (2c_2 - c_1^2 - 1)f + 2c_2 x - 2c_1 c_2 \ , \end{aligned}$$

and c_1 and c_2 are arbitrary constants.

The proof is straightforward. First of all, by conjugation by the function

$$\exp\left(\int^x f(x)dx\right) \ , \tag{8.9}$$

we can bring the operator assigned to L_{-1} to $-\partial$. Then the commutation relation dictates that the generators should be assigned to

$$\begin{cases} L_{-1} \longmapsto -\partial \\ L_0 \longmapsto -\partial^2 - (x - c_1)\partial + c_2 \\ L_1 \longmapsto -\partial^3 - 2(x-c_1)\partial^2 + \big(2c_2 - (x-c_1)^2 - 1\big)\partial + 2c_2(x - c_1) \ . \end{cases} \tag{8.10}$$

Applying the inverse of the conjugation by (8.9), we recover Proposition 8.6.

I do not know what are the points of the Grassmannian corresponding to these $sl(2,\mathbb{C})$-subalgebras of D of Proposition 8.6. Do they have any interesting meaning? I don't know this, either. Recalling that the classification of commutative subalgebras of D is a rich mathematical subject, one might expect that classifying

$sl(2,\mathbb{C})$-subalgebras of D is an interesting problem, too. In any classification, probably we should identify differential operators of Proposition 8.6 and (8.10) as we have done in the case of commutative algebras (see Theorem 3.8). We have not yet constructed enough examples to speculate on the fate of this problem. The unexpected appearance of the $sl(2,\mathbb{C})$-stabilizer algebras in the context of the Hermitian matrix model due to Kontsevich is still very mysterious.

Finally, let us give a new proof of the Kontsevich formula for the τ-function.

Theorem 8.7 (Lemma 4.2 of [16])**.** *Let $z_0, z_1, z_2, \cdots$ be an infinite set of variables of degree 1 which are 0 except for finitely many z_is. We define*

$$t_n = \frac{1}{n}\sum_{i=0}^{\infty} z_i^n, \qquad n = 1,\ 2,\ 3, \cdots ,$$

which is a finite sum for every n. Then for an arbitrary matrix

$$X = \big(x_{ij}\big)_{-\infty<i<\infty, -\infty<j\le 0}$$

of (7.5), the expression

$$\det\left(\begin{bmatrix} & \vdots & \vdots & \vdots \\ \cdots & z_2^{-2} & z_2^{-1} & 1 \\ \cdots & z_1^{-2} & z_1^{-1} & 1 \\ \cdots & z_0^{-2} & z_0^{-1} & 1\end{bmatrix}^{-1} \cdot \left[\begin{array}{cccc|ccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & z_2^{-2} & z_2^{-1} & 1 & z_2 & z_2^2 & \cdots \\ \cdots & z_1^{-2} & z_1^{-1} & 1 & z_1 & z_1^2 & \cdots \\ \cdots & z_0^{-2} & z_0^{-1} & 1 & z_0 & z_0^2 & \cdots \end{array}\right] \cdot X\right)$$

coincides with Sato's τ-function of Theorem 7.2.

Proof. This is a direct consequence of the following matrix formula:

$$\begin{aligned} &\begin{bmatrix} & \vdots & \vdots & \vdots \\ \cdots & z_2^{-2} & z_2^{-1} & 1 \\ \cdots & z_1^{-2} & z_1^{-1} & 1 \\ \cdots & z_0^{-2} & z_0^{-1} & 1\end{bmatrix}^{-1} \cdot \left[\begin{array}{cccc|ccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & z_2^{-2} & z_2^{-1} & 1 & z_2 & z_2^2 & \cdots \\ \cdots & z_1^{-2} & z_1^{-1} & 1 & z_1 & z_1^2 & \cdots \\ \cdots & z_0^{-2} & z_0^{-1} & 1 & z_0 & z_0^2 & \cdots \end{array}\right] \\ \equiv &\begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ & 1 & p_1(t) & p_2(t) \\ & & 1 & p_1(t) \\ & & & 1 \end{bmatrix}^{-1} \cdot \left[\begin{array}{cccc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & 1 & p_1(t) & p_2(t) & p_3(t) & p_4(t) & \cdots \\ & & 1 & p_1(t) & p_2(t) & p_3(t) & \cdots \\ & & & 1 & p_1(t) & p_2(t) & \cdots \end{array}\right] \\ &\mod \left(z_0^N, z_1^N, z_2^N, \cdots\right) \end{aligned}$$

for every $N > 0$. Let

$$A = \begin{bmatrix} & \vdots & \vdots & \vdots \\ \cdots & z_2^{-2} & z_2^{-1} & 1 \\ \cdots & z_1^{-2} & z_1^{-1} & 1 \\ \cdots & z_0^{-2} & z_0^{-1} & 1\end{bmatrix} = \left[z_{-i}^j\right]_{-\infty<i,j\le 0},$$

$$B = \begin{bmatrix} \vdots & \vdots & \\ z_2 & z_2^2 & \cdots \\ z_1 & z_1^2 & \cdots \\ z_0 & z_0^2 & \cdots \end{bmatrix} = \left[z_{-i}^j\right]_{-\infty<i\le 0, 0<j<\infty},$$

$$C = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ & 1 & p_1(t) & p_2(t) \\ & & 1 & p_1(t) \\ & & & 1 \end{bmatrix} = \left[p_{-i+j}(t)\right]_{-\infty<i,j\leq 0},$$

and

$$D = \begin{bmatrix} \vdots & \vdots & \\ p_3(t) & p_4(t) & \cdots \\ p_2(t) & p_3(t) & \cdots \\ p_1(t) & p_2(t) & \cdots \end{bmatrix} = \left[p_{-i+j}(t)\right]_{-\infty<i\leq 0, 0<j<\infty}.$$

We have to show $A^{-1} \cdot \left[A\middle|B\right] = C^{-1} \cdot \left[C\middle|D\right]$, i.e., $B = A \cdot C^{-1} \cdot D$. We note that (7.6) implies

$$C^{-1} = \left[p_{-i+j}(-t)\right]_{-\infty<i,j\leq 0}.$$

Then, as a matrix of formal series in $t = (t_1, t_2, t_3, \cdots)$ and $z = (z_0, z_1, z_2, \cdots)$, we have

$$\begin{aligned}
&A \cdot C^{-1} \cdot D \\
&= \left[\sum_{\ell,k\leq 0} z_{-i}^{\ell} \cdot p_{-\ell+k}(-t) \cdot p_{-k+j}(t)\right]_{i\leq 0, 0<j} \\
&= \left[\sum_{\ell,k\leq 0} z_{-i}^{-(-\ell+k)} \cdot p_{-\ell+k}(-t) \cdot z_{-i}^{-(-k+j)} \cdot p_{-k+j}(t) \cdot z_{-i}^{j}\right]_{i\leq 0, 0<j} \\
&= \left[\sum_{k\leq 0} \exp\bigl(\eta(-t, z_{-i}^{-1})\bigr) \cdot z_{-i}^{-(-k+j)} \cdot p_{-k+j}(t) \cdot z_{-i}^{j}\right]_{i\leq 0, 0<j} \\
&= \left[\exp\bigl(\eta(-t, z_{-i}^{-1})\bigr) \cdot \left(\exp\bigl(\eta(t, z_{-i}^{-1})\bigr) - \sum_{\ell=0}^{j-1} p_\ell(t) \cdot z_{-i}^{-\ell}\right) \cdot z_{-i}^{j}\right]_{i\leq 0, 0<j} \\
&= B - \left[\exp\bigl(\eta(-t, z_{-i}^{-1})\bigr) \cdot \sum_{\ell=0}^{j-1} p_\ell(t) \cdot z_{-i}^{j-\ell}\right]_{i\leq 0, 0<j}.
\end{aligned}$$

Substituting the relation

$$t_n = \frac{1}{n}\bigl(z_0^n + z_1^n + z_2^n + \cdots\bigr)$$

in the above formula, we have for an arbitrary $i \geq 0$,

$$\begin{aligned}
&\exp\bigl(\eta(-t, z_i^{-1})\bigr) \\
&= \exp\left(-\frac{z_0 + z_1 + z_2 + \cdots}{z_i} - \frac{z_0^2 + z_1^2 + z_2^2 + \cdots}{2z_i^2} - \frac{z_0^3 + z_1^3 + z_2^3 + \cdots}{3z_i^3} - \cdots\right) \\
&= \exp\left(-1 - \frac{1}{2} - \frac{1}{3} - \cdots\right) \cdot \\
&\qquad \exp\left(\text{formal power series in } \frac{z_j}{z_i}, j \neq i, \text{ without constant terms}\right) \\
&= 0.
\end{aligned}$$

□

This formula is one of the key steps in Kontsevich's solution to the Witten Conjecture.

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