MIRROR SYMMETRY FOR ORBIFOLD HURWITZ NUMBERS

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ABSTRACT. We study mirror symmetry for orbifold Hurwitz numbers. We show that the Laplace transform of orbifold Hurwitz numbers satisfy a differential recursion, which is then proved to be equivalent to the integral recursion of Eynard and Orantin with spectral curve given by the r-Lambert curve. We argue that the r-Lambert curve also arises in the infinite framing limit of orbifold Gromov-Witten theory of $[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$. Finally, we prove that the mirror model to orbifold Hurwitz numbers admits a quantum curve.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** In recent years, it has been found that many counting problems involving the moduli space $\overline{\mathcal{M}}_{g,n}$, such as Gromov-Witten invariants of toric target spaces and enumeration of various ramified coverings of $\mathbb{P}^1$, have a common feature: they have a "mirror symmetric" counterpart which is governed by a universal integral recursion formula due to Eynard and Orantin [22]. The key ingredient of the mirror theory is the existence of a *spectral curve*, which is a Lagrangian subvariety of the holomorphic symplectic surface $\mathbb{C}^* \times \mathbb{C}^*$.

Once the spectral curve mirror to a given counting problem is determined, the universal recursion calculates the generating functions of the corresponding enumerative invariants.

Simple Hurwitz numbers provide an interesting example of such a story. It was first conjectured in [8] that the generating functions for simple Hurwitz numbers should satisfy the integral recursion of Eynard and Orantin, with spectral curve given by the *Lambert curve*

$$(1.1) \qquad\qquad x = y e^{-y}.$$

The conjecture followed from the broader *remodeling conjecture* [6, 32], which claims that generating functions for Gromov-Witten invariants of toric Calabi-Yau threefolds/orbifolds should satisfy the integral recursion of Eynard and Orantin, with spectral curve given by the standard mirror curve of Hori and Vafa [28]. The conjecture for simple Hurwitz numbers is derived as the infinite framing limit of the simplest case of the remodeling conjecture, namely for Gromov-Witten invariants of $\mathbb{C}^3$.

The conjecture for simple Hurwitz numbers was solved in [21, 35]. There, it was shown that the generating functions of simple Hurwitz numbers defined in [8] are in fact the Laplace transform of the simple Hurwitz numbers $H_{g,n}(\vec{\mu})$ (defined below), and that the combinatorial equation known as the *cut-and-join* equation [24, 25, 37] automatically changes into the Eynard-Orantin integral recursion defined on the Lambert curve (1.1), after taking the Laplace transform, Galois averaging, and restricting to the principal part. In this way the simple Hurwitz number conjecture was solved.

Through the infinite framing limit, the mathematical solution of the simple Hurwitz number conjecture presents a strong evidence for the remodeling conjecture itself. Recently, there have been many developments towards a proof of the remodeling conjecture (see for example [5, 13, 40], and most notably, [23]). In its full generality, however, the remodeling conjecture is still open. In particular, there are no rigorous mathematical results for the cases of orbifold Gromov-Witten invariants.

In this paper we study mirror symmetry for Hurwitz numbers of the orbifold $\mathbb{P}^1[r]$ with one stack point $\left[0/(\mathbb{Z}/r\mathbb{Z})\right]$.

As a first step, we use the remodeling conjecture to argue that the generating functions of such orbifold Hurwitz numbers should satisfy the integral recursion of Eynard and Orantin. As for simple Hurwitz numbers, we show that generating functions for orbifold Hurwitz numbers can be obtained in the infinite framing limit of generating functions for Gromov-Witten theory; however, instead of considering Gromov-Witten theory of $\mathbb{C}^3$, we must now consider orbifold Gromov-Witten theory of $[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$. Via the remodeling conjecture, this implies that generating functions of orbifold Hurwitz numbers should satisfy the integral recursion, with spectral curve the infinite framing limit of the curve mirror to the orbifolds $[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$. We show that the resulting spectral curve for orbifold Hurwitz numbers is the *r-Lambert curve*:

$$(1.2) \qquad\qquad x^r = y e^{-ry}.$$

We then give a rigorous proof of the recursion formula, generalizing the result of [8, 21, 35] to the orbifold case. First, we prove that the r-Lambert curve is the correct spectral curve via Laplace transform. Then, we establish a system of recursive partial differential equations that uniquely determines the Laplace transform of the orbifold Hurwitz numbers for arbitrary genus and ramification profile at $\infty \in \mathbb{P}^1[r]$. These functions are called *free energies*. The Eynard-Orantin topological recursion is then established by taking the Galois average of the Laplace transform of the cut-and-join equation and restricting to the principal part of the free energies. Note that this result also provides strong evidence for the remodeling

conjecture in the context of orbifold Gromov-Witten theory of $[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$, which is still open.

We also study the appearance of a *quantum curve* for orbifold Hurwitz numbers. Quantum curves arise when the mirror symmetric side of a counting problem is governed by a complex analytic curve. Here, a quantum curve [1, 17, 15, 16] means a *holonomic system* that characterizes the *partition function* of the theory, the latter being defined in terms of the principal specialization of the free energies. In the context of orbifold Hurwitz numbers, we show that the partition function (which is the diagonal restriction of a KP τ-function) satisfies a stationary Schrödinger-type equation of [33], that is, a quantum curve exists. Surprisingly, this linear equation alone uniquely determines the free energies for arbitrary genus.

1.2. **Main results.** For a vector of n positive integers $\vec{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_+^n$, the *simple Hurwitz number* $H_{g,n}(\vec{\mu})$ counts the automorphism weighted number of the topological types of simple Hurwitz covers of $\mathbb{P}^1$ of type $(g, \vec{\mu})$. A holomorphic map $\varphi : C \longrightarrow \mathbb{P}^1$ is a *simple Hurwitz cover* of type $(g, \vec{\mu})$ if C is a complete nonsingular algebraic curve defined over $\mathbb{C}$ of genus g, φ has n labeled poles of orders $(\mu_1, \ldots, \mu_n)$, and all other critical points of φ are unlabeled simple ramification points.

In a similar way, we define the *orbifold Hurwitz number* $H_{g,n}^{(r)}(\vec{\mu})$ for every positive integer $r > 0$ to be the automorphism weighted count of the topological types of smooth orbifold morphisms $\varphi : C \longrightarrow \mathbb{P}^1[r]$ with the same pole structure as the simple Hurwitz number case. Here C is a connected 1-dimensional orbifold (or a *twisted* curve) modeled on a nonsingular curve of genus g with $(\mu_1 + \cdots + \mu_n)/r$ stack points of the type $[p/(\mathbb{Z}/r\mathbb{Z})]$. We impose that the inverse image of the morphism φ of the stack point $[0/(\mathbb{Z}/r\mathbb{Z})] \in \mathbb{P}^1[r]$ coincides with the set of stack points of C. When $r = 1$ we recover the simple Hurwitz number $H_{g,n}^{(1)}(\vec{\mu}) = H_{g,n}(\vec{\mu})$.

Consider $H_{g,n}^{(r)}(\vec{\mu})$ as a function in $\vec{\mu} \in \mathbb{Z}_+^n$. Following the recipe of [19, 21, 34], we define the *free energies* as the Laplace transform

$$(1.3) \qquad F_{g,n}^{(r)}(z_1, \ldots, z_n) = \sum_{\vec{\mu} \in \mathbb{Z}_+^n} H_{g,n}^{(r)}(\vec{\mu}) \, e^{-\langle \vec{w}, \vec{\mu} \rangle}.$$

Here $\vec{w} = (w_1, \ldots, w_n)$ is the vector of the Laplace dual coordinates of $\vec{\mu}$, $\langle \vec{w}, \vec{\mu} \rangle = w_1 \mu_1 + \cdots + w_n \mu_n$, and the function variable z_i and w_i for each i are related by the *r-Lambert function*

$$(1.4) \qquad e^{-w} = z e^{-z^r}.$$

It is often convenient to use a different variable

$$(1.5) \qquad x = e^{-w},$$

with which the r-Lambert curve is given by $x = ze^{-z^r}$. Then the free energies $F_{g,n}^{(r)}$ of (1.3) are generating functions of the orbifold Hurwitz numbers. We use the notation

$$(1.6) \qquad F_{g,n}^{(r)}[x_1, \ldots, x_n] = \sum_{\vec{\mu} \in \mathbb{Z}_+^n} H_{g,n}^{(r)}(\vec{\mu}) \prod_{i=1}^n x_i^{\mu_i}$$

to indicate the same function (1.3) in the different set of variables. For every (g, n) the power series (1.6) is convergent and defines an analytic function.

Our first result, Theorem 3.2, states that the generating functions (1.6) can be obtained in the infinite framing limit of generating functions for orbifold Gromov-Witten invariants of

$[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$. This follows by rewriting both generating functions in terms of Hurwitz-Hodge integrals. On one side, a ELSV-type [20] formula expressing orbifold Hurwitz numbers in terms of Hurwitz-Hodge integrals was established by Johnson-Pandharipande-Tseng [29], where orbifold Hurwitz numbers were considered as a special case of double Hurwitz numbers. On the other side, orbifold Gromov-Witten invariants can also be expressed in terms of Hurwitz-Hodge integrals, through the orbifold topological vertex [9, 36]. Using these expressions in terms of Hurwitz-Hodge integrals we establish the infinite framing correspondence for the generating functions.

Through the remodeling conjecture, it is expected that the generating functions for orbifold Gromov-Witten invariants of $[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$ should satisfy the integral recursion of Eynard and Orantin with spectral curve

$$(1.7) \qquad\qquad y^{s+rf}(1-y) - x^r = 0,$$

where $f \in \mathbb{Z}$ is a framing parameter and $s \in \mathbb{Z}$ determines the weight of the action of $\mathbb{Z}/r\mathbb{Z}$ on $\mathbb{C}^3$. By taking the limit of infinite framing, $f \to \infty$, after an appropriate coordinate change

$$(1.8) \qquad\qquad \begin{cases} x \longmapsto \frac{x}{f^{1/r}} \\ y \longmapsto 1 - \frac{y}{f} \end{cases}$$

we obtain the r-Lambert curve (1.2). Therefore, we expect the free energies (1.6) to satisfy the integral recursion of Eynard and Orantin, with spectral curve the r-Lambert curve.

Our next result is an explicit determination of all the free energies (1.3):

Theorem 1.1. *In terms of the z-variables, the free energies are calculated as follows.*

$$(1.9) \qquad\qquad F_{0,1}^{(r)}(z) = \frac{1}{r}z^r - \frac{1}{2}z^{2r},$$

$$(1.10) \qquad\qquad F_{0,2}^{(r)}(z_1, z_2) = \log\frac{z_1 - z_2}{x_1 - x_2} - (z_1^r + z_2^r),$$

where $x_i = z_i e^{-z_i^r}$. For (g,n) in the stable range, i.e., when $2g-2+n > 0$, the free energies satisfy the differential recursion equation

$$(1.11) \quad \left(2g - 2 + n + \frac{1}{r}\sum_{i=1}^n z_i \frac{\partial}{\partial z_i}\right) F_{g,n}^{(r)}(z_1, \ldots, z_n)$$

$$= \frac{1}{2}\sum_{i \neq j} \frac{z_i z_j}{z_i - z_j} \left[\frac{1}{(1-rz_i^r)^2}\frac{\partial}{\partial z_i}F_{g,n-1}^{(r)}(z_{[\hat{j}]}) - \frac{1}{(1-rz_j^r)^2}\frac{\partial}{\partial z_j}F_{g,n-1}^{(r)}(z_{[\hat{i}]})\right]$$

$$+ \frac{1}{2}\sum_{i=1}^n \frac{z_i^2}{(1-rz_i^r)^2}\frac{\partial^2}{\partial u_1 \partial u_2}F_{g-1,n+1}^{(r)}(u_1, u_2, z_{[\hat{i}]})\bigg|_{u_1=u_2=z_i}$$

$$+ \frac{1}{2}\sum_{i=1}^n \frac{z_i^2}{(1-rz_i^r)^2}\sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{i}]}}^{\text{stable}} \left(\frac{\partial}{\partial z_i}F_{g_1,|I|+1}^{(r)}(z_i, z_I)\right)\left(\frac{\partial}{\partial z_i}F_{g_2,|J|+1}^{(r)}(z_i, z_J)\right).$$

Here we use the following convention for indices. The index set is $[n] = \{1, 2, \ldots, n\}$, and for a subset $I \subset [n]$, $z_I = (z_i)_{i \in I}$. The hat symbol $\hat{i}$ means the omission of i from $[n]$. The final summation is over all non-negative integer partitions of g and set partitions of $[\hat{i}]$ subject to the stability conditions $2g_1 - 2 + |I| \geq 0$ and $2g_2 - 2 + |J| \geq 0$.

Remark 1.2. Since $F_{g,n}^{(r)}(z_1,\ldots,z_n)\big|_{z_i=0} = 0$ for every i, the differential recursion (1.11), which is a linear first order partial differential equation, uniquely determines $F_{g,n}^{(r)}$ one by one inductively for all (g,n) subject to $2g-2+n > 0$. This generalizes the result of [35] to the orbifold case.

The differential recursion of Theorem 1.1 is obtained by taking the Laplace transform of the cut-and-join equation for $H_{g,n}^{(r)}(\vec{\mu})$. The r-Lambert curve itself, (1.2), is obtained by computing the Laplace transform of $H_{0,1}^{(r)}(\mu)$.

Our third theorem concerns the existence of a quantum curve for orbifold Hurwitz numbers. Since the r-Lambert curve has a global parameter z, the algebraic K-theory condition required for the existence of the quantization (see for instance [27]) is automatically satisfied, and we have the following result.

Theorem 1.3. *The partition function of the orbifold Hurwitz numbers is given by*

$$(1.12) \qquad Z^{(r)}(z,\hbar) = \exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\hbar^{2g-2+n}F_{g,n}^{(r)}(z,z,\ldots,z)\right).$$

It satisfies the following system of (an infinite-order) linear differential equations.

$$(1.13) \qquad \left(\hbar D - e^{r\left(-w+\frac{r-1}{2}\hbar\right)}e^{r\hbar D}\right)Z^{(r)}(z,\hbar) = 0,$$

$$(1.14) \qquad \left(\frac{\hbar}{2}D^2 - \left(\frac{1}{r}+\frac{\hbar}{2}\right)D - \hbar\frac{\partial}{\partial\hbar}\right)Z^{(r)}(z,\hbar) = 0,$$

where

$$D = \frac{z}{1-rz^r}\frac{\partial}{\partial z} = x\frac{\partial}{\partial x} = -\frac{\partial}{\partial w}.$$

Let the differential operator of (1.13) (resp. (1.14) be denoted by P (resp. Q). Then we have the commutator relation

$$(1.15) \qquad [P,Q] = P.$$

The semi-classical limit of each of the equations (1.13) or (1.14) recovers the r-Lambert curve (1.4).

Remark 1.4. The Schrödinger equation (1.13) is established in [33].

Remark 1.5. The above theorem is a generalization of [34, Theorem 1.3] for an arbitrary $r > 0$. The restriction $r = 1$ reduces to the simple Hurwitz case.

Our final result establishes the prediction from the infinite framing limit that the free energies (1.3) should satisfy the integral recursion of Eynard and Orantin with spectral curve the r-Lambert curve (1.2). More precisely, it is the symmetric differential forms

$$(1.16) \qquad W_{g,n}^{(r)}(z_1,\ldots,z_n) := d_1 d_2\cdots d_n F_{g,n}^{(r)}(z_1,\ldots,z_n)$$

that should satisfy the Eynard-Orantin integral recursion on the r-Lambert curve. We establish this fact in the next theorem.[1]

Remark 1.6. The significance of the integral formalism is its universality. The differential equation (1.11) takes a different form depending on the counting problem, whereas the integral formula (1.17) depends only on the choice of the spectral curve.

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[1]We refer to [34] for the precise mathematical formulation of the Eynard-Orantin recursion formalism.

The Eynard-Orantin integral recursion requires a set of geometric data from the r-Lambert curve, given in parameteric form by $x(z) = ze^{-z^r}$, $y(z) = z^r$. The function $x(z)$ has r critical points at $1 - rz^r = 0$. Let $\{p_1, \ldots, p_r\}$ be the list of these critical points. Since $dx = 0$ has a simple zero at each p_j, the map $x(z)$ is locally a double-sheeted covering around $z = p_j$. We denote by s_j the deck transformation on a small neighborhood of p_j.

Theorem 1.7. *For the stable range $2g - 2 + n > 0$, the symmetric differential forms satisfy the following integral recursion formula.*

$$(1.17) \quad W_{g,n}^{(r)}(z_1, \ldots, z_n) = \frac{1}{2\pi i} \sum_{j=1}^{r} \oint_{\gamma_j} K_j(z, z_1) \Bigg[W_{g-1,n+1}^{(r)}(z, s_j(z), z_2, \ldots, z_n)$$

$$+ \sum_{i=2}^{n} \left(W_{0,2}^{(r)}(z, z_i) \otimes W_{g,n-1}^{(r)}(s_j(z), z_{[\hat{1},\hat{i}]}) + W_{0,2}^{(r)}(s_j(z), z_i) \otimes W_{g,n-1}^{(r)}(z, z_{[\hat{1},\hat{i}]}) \right)$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}}^{\text{stable}} W_{g_1,|I|+1}^{(r)}(z, z_I) \otimes W_{g_2,|J|+1}^{(r)}(s_j(z), z_J) \Bigg].$$

Here the integration is taken with respect to z along a small simple closed loop γ_j around p_j. The integration kernel is defined by

$$(1.18) \qquad K_j(z, z_1) = \frac{1}{2} \frac{1}{W_{0,1}^{(r)}(s_j(z_1)) - W_{0,1}^{(r)}(z_1)} \otimes \int_{z}^{s_j(z)} W_{0,2}^{(r)}(\,\cdot\,, z_1).$$

Remark 1.8. The proof is based on the idea of [21]. The notion of the *principal part* of meromorphic differentials plays a key role in converting the Laplace transform of the cut-and-join equation into a residue formula. We generalize this technique to a more suitable one that works for the current orbifold case.

Remark 1.9. When our manuscript was being finalized, we noticed an extremely interesting paper [18]. The authors of [18] derive the same spectral curve using a concrete graph counting argument, and establish Theorem 1.1 independently. They also claim to have proved our Theorem 1.7. Although they have the right strategy, their proof as written is in error. [18, Lemma 13] does not hold, while it is used in the key step of proving [18, Eqn.(22)].

1.3. **Outline.** The paper is organized as follows. Section 2 reviews the orbifold Hurwitz numbers. The key formulas we use in this paper are the ELSV-type formula (2.3) of [29] and the cut-and-join equation (2.4). Section 3 is devoted to the infinite framing relation between orbifold Hurwitz numbers and Gromov-Witten theory of $[\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$. We then calculate the Laplace transform of the orbifold Hurwitz numbers and prove Theorem 1.1 in Section 4. Section 5 lists some properties enjoyed by the free energies. The quantum curve of the r-Lambert curve is studied in Section 6. The proof of Theorem 1.7 is given in Section 7.

2. THE ORBIFOLD HURWITZ NUMBERS

The polynomial behavior of simple Hurwitz numbers $H_{g,n}(\vec{\mu})$ [25, 37] as a function in $\vec{\mu}$ has been a long mystery. The *polynomiality* has become manifest in the Ekedahl-Lando-Shapiro-Vainshtein formula [20] that relates simple Hurwitz numbers and the Hodge integrals on

the Deligne-Mumford moduli $\overline{\mathcal{M}}_{g,n}$. Another manifestation of the polynomiality is found in [35], where it is established that the Laplace transform

$$(2.1) \qquad F_{g,n}(t_1, \ldots, t_n) = \sum_{\vec{\mu} \in \mathbb{Z}_+^n} H_{g,n}(\vec{\mu}) e^{-\langle \vec{w}, \vec{\mu} \rangle}$$

is a polynomial of degree $3(2g - 2 + n)$ in t_i-variables. Here the variables are related by

$$e^{-w} = ze^{-z}, \qquad z = \frac{t-1}{t}.$$

The *orbifold Hurwitz numbers* $H_{g,n}^{(r)}(\vec{\mu})$ no longer exhibits the same polynomiality. But it shows a *piecewise* polynomial behavior. Indeed, we can define $H_{g,n}^{(r)}(\vec{\mu})$ as a double Hurwitz number, which is the automorphism weighted count of the topological types of double Hurwitz covers $\varphi : C \longrightarrow \mathbb{P}^1$. Here C is a connected nonsingular curve of genus g, and φ is a holomorphic map that has n labeled poles of orders $\vec{\mu}$, m *unlabeled* zeros of degree r, and all other critical points are unlabeled simple ramification points. The number of zeros is given by

$$(2.2) \qquad m = \frac{\mu_1 + \cdots + \mu_n}{r}.$$

This is a special case of the fully general double Hurwitz numbers $H_{g,m,n}(\vec{\mu}, \vec{\nu})$ of arbitrary zeros and poles and otherwise simply ramified. We refer to [12, 26] for further discussions on the piecewise polynomiality.

Reflecting the chamber structure of the polynomiality, the ELSV-type formula for orbifold Hurwitz numbers is more complicated than the original case. The following formula is established in Johnson-Pandharipande-Tseng [29].

Theorem 2.1 ([29])**.** *The orbifold Hurwitz number has an expression in terms of linear Hodge integrals as follows:*

$$(2.3) \qquad H_{g,n}^{(r)}(\mu_1, \ldots, \mu_n) = r^{1-g+\sum_{i=1}^n \langle \frac{\mu_i}{r} \rangle} \int_{\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)} \frac{\sum_{j \geq 0}(-r)^j \lambda_j}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \prod_{i=1}^n \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor!}.$$

Here, $G = \mathbb{Z}/r\mathbb{Z}$, and BG is the classifying space of G. The floor and the fractional part of $q \in \mathbb{Q}$ is given by $q = \lfloor q \rfloor + \langle q \rangle$. $\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)$ denotes the moduli space of stable morphisms to BG from a stable curve of genus g and n smooth points on it, with a prescribed monodromy data $-\vec{\mu}$. The vector $-\vec{\mu}$, as the monodromy data, is identified with the residue class

$$-\vec{\mu} \mod r = (-\mu_1 \mod r, \ldots, -\mu_n \mod r) \in G^r$$

at each marked point. We fix a character

$$G = \mathbb{Z}/r\mathbb{Z} \ni k \longmapsto e^{\frac{2\pi i k}{r}} \in \mathbb{C}^*.$$

This defines a line bundle on $[C, (p_1, \ldots, p_n)] \in \overline{\mathcal{M}}_{g,n}$, and the choice of the monodromy data $\vec{\mu} \in G^r$ determines a covering $\tilde{C} \longrightarrow C$ as a multi-section of this line bundle. All these data give a point of the moduli stack $\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)$, and the Hodge 'bundle' $\mathbb{E}$ is defined on it by assigning the fiber $H^0(\tilde{C}, K_{\tilde{C}})$ to this point, where $K_{\tilde{C}}$ is the canonical sheaf. We then define

$$\lambda_j = c_j(\mathbb{E}) \in H^{2j}(\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG), \mathbb{Q}).$$

The ψ-classes on $\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)$ are the pull-back of the standard tautological cotangent classes on $\overline{\mathcal{M}}_{g,n}$ via the natural forgetful morphism

$$\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG) \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

The cut-and-join equation of orbifold Hurwitz numbers $H_{g,n}^{(r)}(\mu_1, \ldots, \mu_n)$ is derived from the analysis of the geometric deformation of confluence of one of the simple ramification points with $\infty \in \mathbb{P}^1[r]$. In terms of the monodromy data, the deformation corresponds to multiplying a transposition to the product of n disjoint cycles of type $(\mu_1, \ldots, \mu_n)$ that determine the ramification profile above ∞. Therefore, the geometric situation in our orbifold context does not change from the usual simple Hurwitz number case. As a result, the exact same proof of the original case (see for example, [35] and [41]) applies to establish the following.

Theorem 2.2 (Cut-and-join equation)**.** *The orbifold Hurwitz numbers $H_{g,n}^{(r)}(\mu_1, \ldots, \mu_n)$ satisfy the following equation.*

$$(2.4) \quad s H_{g,n}^{(r)}(\mu_1, \ldots, \mu_n) = \frac{1}{2} \sum_{i \neq j} (\mu_i + \mu_j) H_{g,n-1}^{(r)} \left(\mu_i + \mu_j, \mu_{[\hat{i},\hat{j}]} \right)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{\alpha+\beta=\mu_i} \alpha\beta \left[H_{g-1,n+1}^{(r)} \left(\alpha, \beta, \mu_{[\hat{i}]} \right) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{i}]}} H_{g_1,|I|+1}^{(r)} \left(\alpha, \mu_I \right) H_{g_2,|J|+1}^{(r)} \left(\beta, \mu_J \right) \right] .$$

Here

$$(2.5) \qquad\qquad s = s(g, \vec{\mu}) = 2g - 2 + n + \frac{\mu_1 + \cdots + \mu_n}{r}$$

is the number of simple ramification point given by the Riemann-Hurwitz formula, and we use the convention that for any subset $I \subset [n] = \{1, 2, \ldots, n\}$, $\mu_I = (\mu_i)_{i \in I}$. The hat notation $\hat{i}$ indicates that the index i is removed. The last summation is over all partitions of g and set partitions of $[\hat{i}] = \{1, \ldots, i-1, i+1, \ldots, n\}$.

3. The infinite framing limit of the orbifold topological vertex

The realization that generating functions for simple Hurwitz numbers satisfy the Eynard-Orantin recursion for the Lambert curve (1.1) originated from topological string theory. More precisely, the argument put forward in [8] was that generating functions for simple Hurwitz numbers can be obtained in the infinite framing limit of the topological vertex generating functions in open Gromov-Witten theory. The remodeling conjecture of [6] then asserts that the topological vertex generating functions should satisfy the Eynard-Orantin recursion for the framed curve mirror to $\mathbb{C}^3$, whose infinite framing limit is precisely the Lambert curve. Hence, it follows from the remodeling conjecture that generating functions for simple Hurwitz numbers should also satisfy the Eynard-Orantin recursion for the limiting curve, that is, the Lambert curve (1.1). In the context of simple Hurwitz numbers, this conjecture has been proved in [4, 21], and the remodeling conjecture for the topological vertex has also been proved in [40] following similar methods.

In this section, we argue that there exists a similar story for orbifold Hurwitz numbers. We show that generating functions for orbifold Hurwitz numbers can be obtained in the infinite framing limit of the orbifold topological vertex generating functions in open orbifold Gromov-Witten theory. By the remodeling conjecture, the latter are expected to satisfy the Eynard-Orantin recursion for the curve mirror to the orbifolds. We show that the infinite framing limit of these curves reproduce the r-Lambert curve (1.2), therefore suggesting that generating functions for orbifold Hurwitz numbers should satisfy the Eynard-Orantin recursion for the r-Lambert curve. We will prove this result in section 7.

3.1. **Open orbifold Gromov-Witten theory.**

3.1.1. *The geometry.* We consider the toric Calabi-Yau orbifold $X = [\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$, where $\mathbb{Z}/r\mathbb{Z}$ acts on the three complex coordinates of $\mathbb{C}^3$ as:

$$(3.1) \qquad (z_1, z_2, z_3) \mapsto (\alpha z_1, \alpha^s z_2, \alpha^{-s-1} z_3), \qquad \alpha = \exp\left(\frac{2\pi i}{r}\right), \qquad s \in \mathbb{Z}.$$

The rays for the fan of X can be taken to be:

$$(3.2) \qquad\qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} r \\ -s \\ 1 \end{pmatrix}.$$

The *fan triangulation* of X is the intersection of its fan with the plane at $z = 1$, which is shown in red in figure 3.1. Its dual diagram is the *toric diagram* (or *web diagram*) of X, which is shown in blue. For a good pedagogical introduction to web diagrams and fan triangulations of toric Calabi-Yau orbifolds, see for instance Appendix B in [11].
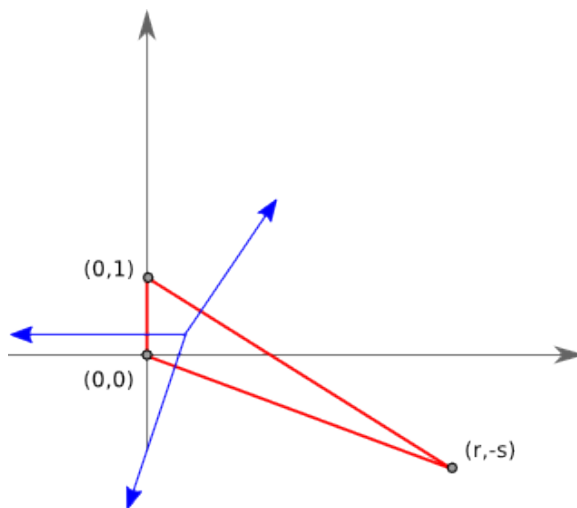

FIGURE 3.1. The fan triangulation (in red) and toric diagram (in blue) for $X = [\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$, with the action of $\mathbb{Z}/r\mathbb{Z}$ specified by (3.1). The fan triangulation is the Newton polygon for the curve mirror to X.

3.1.2. *Open orbifold Gromov-Witten invariants.* We are interested in open orbifold Gromov-Witten theory with target space X. Open Gromov-Witten invariants provide a virtual count of stable maps from Riemann surfaces with boundaries to a target space X. In addition to X, one must specify a Lagrangian submanifold $L \subset X$ where the boundary of the domain curve is required to lie. We choose our Lagrangian submanifold to be as constructed originally in [2, 3], intersecting the z_1 coordinate axis of X. In the language of [36], we are studying the "effective one-leg $\mathbb{Z}/r\mathbb{Z}$ orbifold topological vertex": one-leg because we consider only one Lagrangian submanifold for the boundary condition, and effective because our Lagrangian submanifold intersects the z_1 leg of X, where the action of $\mathbb{Z}/r\mathbb{Z}$ is effective. This is known as the orbifold topological vertex, because this type of geometry provides a building block that can be used to construct open/closed Gromov-Witten theory for any toric Calabi-Yau orbifolds, just as the original topological vertex of [2] is the building block to construct open/closed Gromov-Witten theory of toric Calabi-Yau manifolds.

We will not give a precise definition of open Gromov-Witten theory here; we refer the interested reader to [9, 30, 31, 36]. Roughly speaking, in [30], Katz and Liu were the first to construct a tangent/obstruction theory for the moduli space of open stable maps to toric Calabi-Yau manifolds. The construction was generalized to orbifolds in [9], and then in full generality by Ross in [36]. An important point is that the moduli theory is only defined via localization with respect to a torus action on the moduli space, induced from a torus action on the target space X. There is a choice of weights involved in the choice of torus action on the target space X, and it turns out that open Gromov-Witten invariants do depend on this choice of weight. More precisely, they depend on a residual integer $f \in \mathbb{Z}$, which is known as the *framing* of the open Gromov-Witten invariants (in fact, in the context of orbifolds, $f \in \frac{1}{r}\mathbb{Z}$). To make contact with the notation of [9, 36], here we choose the weights for the torus action with respect to which we localize to be

$$(3.3) \qquad \left(\frac{\hbar}{r}, f\hbar, -f\hbar - \frac{\hbar}{r} \right),$$

just as in [9].[2] In the non-equivariant limit in which we will evaluate Gromov-Witten invariants, we set $\hbar = 1$.

After localization, open Gromov-Witten theory becomes a theory of stable maps $\varphi : \Sigma \to X$, where Σ is a compact genus g Riemann surface with n disks attached at n (possibly k-twisted) distinct nodes. The map φ contracts the compact component to the origin of the target orbifold X, while the n boundaries of the disks are mapped to the Lagrangian submanifold L. Each disk is mapped with a given winding number $\mu_i \in \mathbb{Z}$, $i = 1, \ldots, n$. Thus, the data encoding a map φ is the genus g of the domain curve, a partition $\vec{\mu}$ of length $\ell(\mu) = n$ specifying the winding numbers of the disks, and a vector $\vec{k}$ of integers $0 < k_i \leq r$ specifying the twisting of the attachment points.

In fact, as shown in [9], for the theory to be k-twisted equivariant, the twisting vector $\vec{k}$ is not independent from the winding numbers $\vec{\mu}$: we must require that

$$(3.4) \qquad \mu_i \equiv k_i \bmod r,$$

which fully specifies $\vec{k}$ in terms of $\vec{\mu}$.

Remark 3.1. We remark here that we do not allow insertions, that is, stacky points on the compact components of the domain curves, aside from the attachment points of the disks. It would be interesting to study the Eynard-Orantin recursion for the orbifold topological vertex with insertions, and its infinite framing limit. We hope to report on that in the near future.

3.1.3. *The orbifold topological vertex.* Under the assumptions described above, we can construct the effective one-leg orbifold topological vertex $V_{g,n}^{(r,s)}(\vec{\mu}; f)$, which computes the open orbifold Gromov-Witten invariants of X from genus g domain curves with n disks with winding numbers specified by the partition $\vec{\mu}$. We form orbifold topological vertex generating functions:

$$(3.5) \qquad G_{g,n}^{(r,s)}[x_1, \ldots, x_n; f] = \sum_{\vec{\mu} \in \mathbb{Z}_+^n} V_{g,n}^{(r,s)}(\vec{\mu}; f) \prod_{i=1}^{n} x_i^{\mu_i}.$$

One of the main results of [9, 36] is that the orbifold topological vertex $V_{g,n}^{(r,s)}(\vec{\mu}; f)$ has an explicit formula in terms of Hurwitz-Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)$,

---

[2]Note that our f has a minus sign difference with [9], which is consistent with the framing that we will introduce for the mirror curve later on.

with $G = \mathbb{Z}/r\mathbb{Z}$. More precisely, in the non-equivariant limit, from the work of [9, 36, 39] we extract the following formula for the orbifold topological vertex described above:[3]

$$V_{g,n}^{(r,s)}(\vec{\mu}; f) = (-1)^{g-1+\sum_{i=1}^n \langle -\frac{\mu_i(s+1)}{r}\rangle} r^{n-\sum_{i=1}^n \delta_{\langle \frac{\mu_i}{r}\rangle,0}} (f)^{\sum_{i=1}^n \delta_{\langle \frac{\mu_i s}{r}\rangle,0}} \left(-f - \frac{1}{r}\right)^{\sum_{i=1}^n \delta_{\langle -\frac{(s+1)\mu_i}{r}\rangle,0}}$$

$$\times \prod_{i=1}^n \left(\frac{1}{\mu_i^{\lfloor \frac{\mu_i}{r} + \langle \frac{\mu_i s}{r}\rangle - \frac{1}{t_{\text{eff}}}\rfloor - \lfloor \frac{\mu_i}{r}\rfloor} \lfloor \frac{\mu_i}{r}\rfloor!} \prod_{j=1}^{\lfloor \frac{\mu_i}{r} + \langle \frac{\mu_i s}{r}\rangle - \frac{1}{t_{\text{eff}}}\rfloor} \left(f\mu_i - \langle \frac{\mu_i s}{r}\rangle + j\right) \right)$$

$$(3.6) \qquad \times \int_{\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)} \frac{\Lambda_g^{\vee,\alpha}\left(\frac{1}{r}\right) \Lambda_g^{\vee,\alpha^s}(f) \Lambda_g^{\vee,\alpha^{-s-1}}\left(-f - \frac{1}{r}\right)}{\prod_{i=1}^n (1 - \mu_i \psi_i)},$$

where we used the Kronecker delta symbol notation:

$$(3.7) \qquad \delta_{t,0} = \begin{cases} 0 \text{ if } t \neq 0, \\ 1 \text{ if } t = 0, \end{cases}$$

and we defined a rational number

$$(3.8) \qquad t_{\text{eff}} := \frac{r}{\gcd(\mu_i, r)}.$$

We also used the notation:

$$(3.9) \qquad \Lambda_g^{\vee,\alpha^k}(u) = u^{\text{rk}(\mathbb{E}_{\alpha^k})} \sum_{i=0}^{\text{rk}(\mathbb{E}_{\alpha^k})} \left(-\frac{1}{u}\right)^i \lambda_{i,\alpha^k},$$

where $\mathbb{E}_{\alpha^k}$ is the Hodge bundle corresponding to the representation of $\mathbb{Z}/r\mathbb{Z}$ given by

$$(3.10) \qquad \varphi_{\alpha^k} : \mathbb{Z}/r\mathbb{Z} \to \mathbb{C}^*, \qquad \varphi_{\alpha^k}(1) = \alpha^k = \exp\left(\frac{2\pi i k}{r}\right),$$

and

$$(3.11) \qquad \lambda_{i,\alpha^k} = c_i\left(\mathbb{E}_{\alpha^k}\right)$$

are its Chern classes.

3.2. **The infinite framing limit of the generating functions.** With this explicit formula for the orbifold topological vertex, we can study the limit of the generating functions (3.5) when the framing f goes to infinity. What we show is that the infinite framing limit of the orbifold topological vertex reproduces precisely the orbifold Hurwitz numbers defined previously.

Theorem 3.2. *Consider the generating functions $G_{g,n}^{(r,s)}[x_1, \ldots, x_n; f]$ defined in (3.5), with the orbifold topological vertex $V_{g,n}^{(r,s)}(\vec{\mu}; f)$ given by (3.6). Then:*

$$(3.12) \qquad \lim_{f\to\infty}\left((-1)^n f^{2-2g-n} G_{g,n}^{(r,s)}\left[\frac{x_1}{f^{1/r}}, \ldots, \frac{x_n}{f^{1/r}}; f\right]\right) = F_{g,n}^{(r)}[x_1, \ldots, x_n],$$

where $F_{g,n}^{(r)}[x_1, \ldots, x_n]$ is the generating functions for orbifold Hurwitz numbers defined in (1.6), with the orbifold Hurwitz numbers $H_{g,n}^{(r)}(\vec{\mu})$ satisfying (2.3).

---

[3]Note that the overall sign in $V_{g,n}^{(r,s)}(\vec{\mu}; f)$ differs from [9, 36]; as mentioned in [9], there are ambiguities with minus signs in open Gromov-Witten theory. Here, we fixed the overall minus sign such that it is consistent with the infinite framing limit that we will study. It would be interesting to investigate this issue of minus signs further.

Proof. Let us consider the leading order term in a large f expansion of the orbifold topological vertex $V_{g,n}^{(r,s)}(\vec{\mu}; f)$ in (3.6). Let us consider the Hurwitz-Hodge integral first. In the large f limit, it is easy to see that

$$(3.13) \qquad \Lambda_g^{\vee,\alpha^s}(f)\, \Lambda_g^{\vee,\alpha^{-s-1}}\left(-f - \frac{1}{r}\right) \simeq (f)^{\mathrm{rk}(\mathbb{E}_{\alpha^s})}(-f)^{\mathrm{rk}(\mathbb{E}_{\alpha^{-s-1}})},$$

since all other terms will be suppressed by powers of $1/f$. We can compute the rank of the Hodge bundles over $\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)$ using orbifold Riemann-Roch. We get that[4]

$$(3.14) \qquad \mathrm{rk}(\mathbb{E}_{\alpha^s}) = g - 1 + \sum_{i=1}^n \langle -\frac{\mu_i s}{r}\rangle,$$

$$(3.15) \qquad \mathrm{rk}(\mathbb{E}_{\alpha^{-s-1}}) = g - 1 + \sum_{i=1}^n \langle \frac{\mu_i(s+1)}{r}\rangle.$$

Moreover, we can write

$$(3.16) \qquad \Lambda_g^{\vee,\alpha}\left(\frac{1}{r}\right) = r^{-\mathrm{rk}(\mathbb{E}_\alpha)} \sum_{i=0}^{\mathrm{rk}(\mathbb{E}_\alpha)} (-r)^i \lambda_{i,\alpha},$$

and we compute

$$(3.17) \qquad \mathrm{rk}(\mathbb{E}_\alpha) = g - 1 + \sum_{i=1}^n \langle -\frac{\mu_i}{r}\rangle.$$

Thus, the Hurwitz-Hodge integral in the third line of (3.6) has the following leading order term in a large f expansion:

$$(3.18)$$
$$r^{1-g-\sum_{i=1}^n\langle -\frac{\mu_i}{r}\rangle}(-1)^{g-1+\sum_{i=1}^n\langle\frac{\mu_i(s+1)}{r}\rangle} f^{2g-2+\sum_{i=1}^n\left(\langle -\frac{\mu_i s}{r}\rangle + \langle\frac{\mu_i(s+1)}{r}\rangle\right)} \int_{\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)} \frac{\sum_{j\geq 0}(-r)^j\lambda_{j,\alpha}}{\prod_{i=1}^n(1-\mu_i\psi_i)}.$$

The first line of (3.6) has leading order term given by

$$(3.19) \qquad (-1)^{g-1+\sum_{i=1}^n\langle -\frac{\mu_i(s+1)}{r}\rangle} r^{n-\sum_{i=1}^n \delta_{\langle\frac{\mu_i}{r}\rangle,0}} f^{\sum_{i=1}^n \delta_{\langle\frac{\mu_i s}{r}\rangle,0}}(-f)^{\sum_{i=1}^n \delta_{\langle -\frac{(s+1)\mu_i}{r}\rangle,0}}.$$

As for the second line in (3.6), the leading order term is

$$(3.20) \qquad \prod_{i=1}^n \left(\frac{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}{\lfloor\frac{\mu_i}{r}\rfloor!} f^{\lfloor\frac{\mu_i}{r}+\langle\frac{\mu_i s}{r}\rangle - \frac{1}{t_{\mathrm{eff}}}\rfloor} \right)$$

To get our final answer we must combine these three lines together. For the exponent of the overall factor of f, we notice that

$$2g - 2\sum_{i=1}^n \left(\langle -\frac{\mu_i s}{r}\rangle + \langle\frac{\mu_i(s+1)}{r}\rangle + \delta_{\langle\frac{\mu_i s}{r}\rangle,0} + \delta_{\langle -\frac{(s+1)\mu_i}{r}\rangle,0} + \lfloor\frac{\mu_i}{r}+\langle\frac{\mu_i s}{r}\rangle - \frac{1}{t_{\mathrm{eff}}}\rfloor \right)$$

$$= 2g - 2 + n + \sum_{i=1}^n \left(1 - \langle\frac{\mu_i s}{r}\rangle - \langle\frac{-\mu_i(s+1)}{r}\rangle + \lfloor\frac{\mu_i}{r}+\langle\frac{\mu_i s}{r}\rangle - \frac{1}{t_{\mathrm{eff}}}\rfloor \right)$$

---

[4]To be precise, we should consider separately the cases when the moduli space has a component with trivial monodromy (see for instance [29]). But since the same formulae are valid in the end, for the sake of clarity we will not treat these cases separately.

$$=2g - 2 + n + \sum_{i=1}^{n} \left(1 + \frac{\mu_i}{r} + \lfloor -\frac{\mu_i(s+1)}{r} \rfloor + \lfloor \frac{\mu_i(s+1)}{r} - \frac{1}{t_{\text{eff}}} \rfloor \right)$$

$$(3.21) \quad =2g - 2 + n + \sum_{i=1}^{n} \frac{\mu_i}{r}.$$

The last equality follows because:

$$(3.22) \quad \lfloor -\frac{\mu_i(s+1)}{r} \rfloor + \lfloor \frac{\mu_i(s+1)}{r} - \frac{1}{t_{\text{eff}}} \rfloor = \begin{cases} \lfloor -\frac{1}{t_{\text{eff}}} \rfloor = \lfloor -\frac{\gcd(\mu_i,r)}{r} \rfloor = -1 & \text{for } \frac{\mu_i(s+1)}{r} \in \mathbb{Z}, \\ -1 + \lfloor \langle \frac{\alpha_i(s+1)}{t_{\text{eff}}} \rangle - \frac{1}{t_{\text{eff}}} \rfloor = -1 & \text{for } \frac{\mu_i(s+1)}{r} \notin \mathbb{Z}, \end{cases}$$

where $\alpha_i := \frac{\mu_i t_{\text{eff}}}{r} = \frac{\mu_i}{\gcd(\mu_i,r)} \in \mathbb{Z}$.

As for the exponent of the factor in r, we get

$$1 - g + n - \sum_{i=1}^{n} \left(\langle -\frac{\mu_i}{r} \rangle + \delta_{\langle \frac{\mu_i}{r} \rangle, 0} \right) = 1 - g + n - \sum_{i=1}^{n} \left(1 - \langle \frac{\mu_i}{r} \rangle \right)$$

$$(3.23) \qquad\qquad\qquad\qquad\qquad = 1 - g + \sum_{i=1}^{n} \langle \frac{\mu_i}{r} \rangle.$$

Finally, the overall minus sign has exponent:

$$(3.24) \qquad 2g - 2 + \sum_{i=1}^{n} \left(\langle -\frac{\mu_i(s+1)}{r} \rangle + \langle \frac{\mu_i(s+1)}{r} \rangle + \delta_{\langle -\frac{(s+1)\mu_i}{r} \rangle, 0} \right) = 2g - 2 + n.$$

Putting these together, we obtain that the leading term of (3.6) as f is large is

$$(3.25) \qquad\qquad (-1)^n f^{2g-2+n+\sum_{i=1}^{n} \frac{\mu_i}{r}} H_{g,n}^{(r)}(\vec{\mu}),$$

where the orbifold Hurwitz numbers $H_{g,n}^{(r)}(\vec{\mu})$ are defined in (2.3). Therefore, it follows that the generating functions satisfy

$$(3.26) \qquad \lim_{f \to \infty} \left((-1)^n f^{2-2g-n} G_{g,n}^{(r,s)} \left[\frac{x_1}{f^{1/r}}, \ldots, \frac{x_n}{f^{1/r}}; f \right] \right) = F_{g,n}^{(r)}[x_1, \ldots, x_n].$$

$\square$

3.3. The remodeling conjecture and the Eynard-Orantin recursion.

An interesting implication of the infinite framing limit studied in the previous subsection is the existence of a recursive structure for orbifold Hurwitz numbers, which follows from the remodeling conjecture of [6].

The remodeling conjecture asserts that the differentials $d_1 d_2 \cdots d_n G_{g,n}^{(r,s)}[x_1, \ldots, x_n; f]$ can be resummed as symmetric differential forms living on the complex curve mirror to the orbifold X, and that they satisfy the Eynard-Orantin recursion (which was defined in the introduction) for this particular spectral curve. Through the infinite framing limit of the generating functions, this conjecture implies that the generating functions for orbifold Hurwitz numbers should also satisfy the Eynard-Orantin recursion, with spectral curve given by the r-Lambert curve.

Recall that the mirror curve to the orbifold X can be read off directly from the fan triangulation of X. Indeed, the fan triangulation is the Newton polygon of the mirror curve. In the case of the orbifold $X = [\mathbb{C}^3/(\mathbb{Z}/r\mathbb{Z})]$ that we studied in this section, the fan

triangulation of X, shown in figure 3.1, has vertices $(0,0), (0,1)$ and $(r, -s)$, corresponding to the monomials 1, y and $x^r y^{-s}$. Therefore, the mirror curve can be written as:

$$(3.27) \qquad\qquad C : \{1 - y - x^r y^{-s} = 0\} \subset (\mathbb{C}^*)^2.$$

Note that in writing the mirror curve, we have chosen a particular parameterization (in the language of toric geometry, we chose a particular set of rays, (3.2), for the fan of X). We claim that this particular choice of parameterization should correspond to a Lagrangian submanifold intersecting the z_1 leg of the orbifold X (we refer the reader to [6, 7, 9, 10] for more on this).

To introduce framing for the mirror curve, we must reparameterize the curve by $(x, y) \mapsto (xy^{-f}, y)$ [6] . We then get the framed mirror curve:[5]

$$(3.28) \qquad\qquad C_f : \{y^{s+rf}(1 - y) - x^r = 0\} \subset (\mathbb{C}^*)^2.$$

Note that we denoted the framing by f, the same letter as in the previous subsection, but the two may not be precisely equal; they may be related via the addition of a constant (see for instance [9]). But this will not be important for us, since we are interested in the $f \to \infty$ limit.

The statement of the remodeling conjecture is that the differentials $d_1 d_2 \cdots d_n G_{g,n}^{(r,s)}[x_1, \ldots, x_n; f]$ are symmetric differential forms on C_f that satisfy the Eynard-Orantin recursion for the spectral curve C_f, with fundamental one-form

$$(3.29) \qquad\qquad dG_{0,1}^{(r,s)}[x; f] = \log y \frac{dx}{x},$$

where x and y are related by (3.28).

Now what happens in the infinite framing limit? First, we notice that if we define new variables

$$(3.30) \qquad\qquad x = \frac{\tilde{x}}{f^{1/r}}, \qquad y = 1 - \frac{\tilde{y}}{f},$$

the equation for the framed mirror curve C_f in (3.28) becomes

$$(3.31) \qquad\qquad \tilde{x}^r = \tilde{y}\left(1 - \frac{\tilde{y}}{f}\right)^{rf} \left(1 - \frac{\tilde{y}}{f}\right)^s.$$

Taking the limit $f \to \infty$, we obtain the curve

$$(3.32) \qquad\qquad \tilde{x}^r = \tilde{y}\mathrm{e}^{-r\tilde{y}},$$

which is precisely the equation of the r-Lambert curve (1.2)!

What does it mean for the recursion satisfied by the generating functions? The one-form that is fundamental for the recursion, (3.29), can be rewritten in terms of the new variables $\tilde{x}$ and $\tilde{y}$. It becomes

$$(3.33) \qquad\qquad dG_{0,1}^{(r,s)}[x; f] = \log y \frac{dx}{x} = \log\left(1 - \frac{\tilde{y}}{f}\right) \frac{d\tilde{x}}{\tilde{x}},$$

with x and y related by (3.28). If we send $f \to \infty$, the leading order term is

$$(3.34) \qquad\qquad -\frac{\tilde{y}}{f} \frac{d\tilde{x}}{\tilde{x}},$$

with $\tilde{x}$ and $\tilde{y}$ now related through (3.32).

---

[5]Notice that the framing transformation of [9] replaces f by $-f$, which is why our choice of torus weights in (3.3) had a minus sign difference with [9].

Looking at the explicit form of the recursion, the result of this analysis is that if we consider the infinite framing limit of the differentials

(3.35)
$$d_1 d_2 \cdots d_n F_{g,n}^{(r)}[\tilde{x}_1, \ldots, \tilde{x}_n] := \lim_{f \to \infty} \left((-1)^n f^{2-2g-n} d_1 d_2 \cdots d_n G_{g,n}^{(r,s)} \left[\frac{\tilde{x}_1}{f^{1/r}}, \ldots, \frac{\tilde{x}_n}{f^{1/r}}; f \right] \right),$$

then they should satisfy the Eynard-Orantin recursion with fundamental one-form

(3.36)
$$dF_{0,1}^{(r)}[\tilde{x}] = \tilde{y} \frac{d\tilde{x}}{\tilde{x}},$$

where $\tilde{x}$ and $\tilde{y}$ are related by (3.32). The $-1/f$ factor between (3.36) and (3.34) is precisely responsible for the $(-1)^n f^{2-2g-n}$ factor in front of the differentials constructed from the recursion.

But we know what these new objects $d_1 d_2 \cdots d_n F_{g,n}^{(r)}[\tilde{x}_1, \ldots, \tilde{x}_n]$ are: in the previous section, we showed that they are precisely the differentials of the generating functions of orbifold Hurwitz numbers! Therefore, if we believe the remodeling conjecture for orbifolds, then we are led to claim that the generating functions for orbifold Hurwitz numbers should also satisfy the Eynard-Orantin recursion, with spectral curve the r-Lambert curve (1.2), which can be written in parameteric form as

(3.37)
$$\tilde{x} = z e^{-z^r}, \qquad \tilde{y} = z^r,$$

with fundamental one-form

(3.38)
$$dF_{0,1}^{(r)}[\tilde{x}] = \tilde{y} \frac{d\tilde{x}}{\tilde{x}} = z^{r-1}(1 - rz^r) dz.$$

We will prove this statement in section 7.

4. THE LAPLACE TRANSFORM OF THE ORBIFOLD HURWITZ NUMBERS

In this section we prove Theorem 1.1. From the remodeling conjecture point of view presented in the previous section, we see that the mirror theory to orbifold Hurwitz numbers should be built on the r-Lambert curve (1.4). To launch the Eynard-Orantin topological recursion [22, 34] for the r-Lambert curve as its spectral curve, we need to find the Lagrangian immersion

$$\iota : \Sigma \longrightarrow T^*\mathbb{C}^*$$
$$\downarrow \pi$$
$$\mathbb{C}^*$$

of the open Riemann surface $\Sigma = \mathbb{C}^*$ given by

(4.1)
$$\begin{cases} x = z e^{-z^r} \\ y = f(z) \end{cases} \qquad z \in \Sigma,$$

where $y = f(z)$ is yet to be determined. We refer to [34] for a mathematical definition of the Eynard-Orantin topological recursion theory. The recipe of [34] tells us that the Laplace transform of the *disk amplitude* $H_{0,1}^{(r)}(\mu)$ should determine the Lagrangian immersion by the formula

(4.2)
$$W_{0,1}^{(r)}(z) \stackrel{\text{def}}{=} \iota^*(y d \log x) = dF_{0,1}^{(r)}(z),$$

where $\eta = y d \log x$ on $T^*\mathbb{C}^*$ is the tautological holomorphic 1-form on the cotangent bundle $T^*\mathbb{C}^*$. In this section we first identify the Lagrangian immersion (4.1) from the computation

of $F_{0,1}^{(r)}$. We learn from [29] that

$$H_{0,1}^{(r)}(\mu) = \frac{\mu^{\lfloor \frac{\mu}{r} \rfloor - 2}}{\lfloor \frac{\mu}{r} \rfloor!} \quad \text{if} \quad \mu \equiv 0 \mod r,$$

and $H_{0,1}^{(r)}(\mu) = 0$ otherwise. Therefore, the $(g, n) = (0, 1)$ free energy (1.3) is given by

$$F_{0,1}^{(r)} = \sum_{m=1}^{\infty} \frac{(rm)^{m-2}}{m!} x^{rm}.$$

We note that $F_{0,1}^{(r)} = 0$ when $x = 0$.

As the ELSV-type formula (2.3) indicates, the free energy computation requires that we need to find similar infinite sums. We thus introduce the following auxiliary functions:

$$\text{(4.3)} \qquad \begin{aligned} \xi_\ell^{r,k}(x) &= \sum_{m=0}^{\infty} \frac{(rm+k)^{m+\ell}}{m!} x^{rm+k}, \qquad k = 1, 2, \ldots, r-1, \\ \xi_\ell^{r,0}(x) &= \sum_{m=1}^{\infty} \frac{(rm)^{m+\ell}}{m!} x^{rm}. \end{aligned}$$

It is easy to see from Stirling's formula that the auxiliary functions are absolutely convergent with the radius of convergence $e^{-\frac{1}{r}}$. Since these functions do not have any constant terms, we have

$$\text{(4.4)} \qquad \xi_{\ell+1}^{r,k}(x) = x \frac{d}{dx} \xi_\ell^{r,k}(x), \qquad k = 0, 1, \ldots, r-1.$$

Therefore, all we need is to find the functions at $\ell = -1$. The standard procedure to compute (4.3) is to use the *Lambert function*. Let us define

$$\text{(4.5)} \qquad y(x) = \xi_{-1}^{1,0}(x) = \sum_{m=1}^{\infty} \frac{m^{m-1}}{m!} x^m.$$

Then its inverse is given by the Lambert function (1.1), which can be easily checked by the Lagrange inversion formula, and the following formula holds for every complex number $\alpha \in \mathbb{C}^*$ (see for example, [14]):

$$\text{(4.6)} \qquad \exp\left(\alpha y(x)\right) = \sum_{m=0}^{\infty} \frac{\alpha(m+\alpha)^{m-1}}{m!} x^m.$$

Therefore, the base case for (4.3) is computed by

$$\text{(4.7)} \qquad \begin{aligned} \xi_{-1}^{r,k}(x) &= \frac{1}{k} x^k \exp\left(\frac{k}{r} y(rx^r)\right), \qquad k \neq 0, \\ \xi_{-1}^{r,0}(x) &= \frac{1}{r} y(rx^r). \end{aligned}$$

We now *define* the variable z by

$$\text{(4.8)} \qquad z = z(x) = \left(\frac{1}{r} y(rx^r)\right)^{\frac{1}{r}},$$

so that its inverse function is given by the r-Lambert curve $x = ze^{-z^r}$ (1.4). In terms of z, the auxiliary functions (4.7) take much simpler form

(4.9)
$$\xi_{-1}^{r,k}(x) = \frac{1}{k}z^k, \qquad k \neq 0,$$
$$\xi_{-1}^{r,0}(x) = z^r.$$

The differential operator of (4.4) in z is

(4.10)
$$x\frac{d}{dx} = \frac{z}{1 - rz^r}\frac{d}{dz}.$$

Since $F_{0,1}^{(r)} = \xi_{-2}^{r,0}(x)$, we have

$$\frac{z}{1 - rz^r}\frac{d}{dz}F_{0,1}^{(r)} = \xi_{-1}^{r,0}(x) = z^r.$$

Therefore, considering the fact that $z = 0 \implies x = 0 \implies F_{0,1}^{(r)} = 0$, we find

$$F_{0,1}^{(r)}(z) = \frac{1}{r}z^r - \frac{1}{2}z^{2r},$$

which proves (1.9). Then from (4.2), we have

$$dF_{0,1}^{(r)}(z) = z^{r-1}(1 - rz^r)dz,$$
$$yd\log(x) = yz^{-1}e^{z^r}d\left(ze^{-z^r}\right) = yz^{-1}(1 - rz^r)dz.$$

Hence

$$y = f(z) = z^r.$$

We have thus determined the Lagrangian immersion

(4.11)
$$\iota : \Sigma = \mathbb{C}^* \longrightarrow T^*\mathbb{C}^*, \qquad \begin{cases} x = ze^{-z^r} \\ y = z^r, \end{cases} \qquad z \in \Sigma,$$

in agreement with (3.37). We note that (4.11) implies $rx^r = (ry)e^{-ry}$, hence $y(rx^r) = ry = rz^r$, which is consistent with (4.8).

Another important feature of the Eynard-Orantin theory [34] is the special relation between the Laplace transform $F_{0,2}^{(r)}(z_1, z_2)$ of the *annulus amplitude* $H_{0,2}^{(r)}(\mu_1, \mu_2)$ and the difference of the Riemann's prime forms of the x-projection $\pi : \Sigma \longrightarrow \mathbb{C}^*$ [22]. Again from [29], we know the annulus amplitude of the orbifold Hurwitz numbers:

$$H_{0,2}^{(r)}(\mu_1, \mu_2) = r^{\langle\frac{\mu_1}{r}\rangle + \langle\frac{\mu_2}{r}\rangle} \cdot \frac{1}{\mu_1 + \mu_2} \cdot \frac{\mu_1^{\lfloor\frac{\mu_1}{r}\rfloor}}{\lfloor\frac{\mu_1}{r}\rfloor!} \cdot \frac{\mu_2^{\lfloor\frac{\mu_2}{r}\rfloor}}{\lfloor\frac{\mu_2}{r}\rfloor!}, \qquad \text{if } \mu_1 + \mu_2 \equiv 0 \mod r,$$

and $H_{0,2}^{(r)}(\mu_1, \mu_2) = 0$ otherwise. Here $\langle q \rangle = q - \lfloor q \rfloor$ is the fractional part of $q \in \mathbb{Q}$.

Proof of (1.10). Write $\mu_i = rm_i + k_i$, $i = 1, 2$, with $0 \leq k_i \leq r - 1$. Then

$$\mu_1 + \mu_2 \equiv 0 \mod r \iff \begin{cases} k_1 = k_2 = 0 \quad \text{or} \\ k_1 + k_2 = r. \end{cases}$$

Therefore, we obtain a partial differential equation

$$\left(\frac{z_1}{1 - rz_1}\frac{\partial}{\partial z_1} + \frac{z_2}{1 - rz_2}\frac{\partial}{\partial z_2}\right)F_{0,2}^{(r)}(z_1, z_2)$$

$$= \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \sum_{(\mu_1,\mu_2)\in\mathbb{Z}_+^2} H_{0,2}^{(r)}(\mu_1,\mu_2) x_1^{\mu_1} x_2^{\mu_2}$$

$$= \sum_{(\mu_1,\mu_2)\in\mathbb{Z}_+^2} r^{\langle \frac{\mu_1}{r} \rangle + \langle \frac{\mu_2}{r} \rangle} \cdot \frac{\mu_1^{\lfloor \frac{\mu_1}{r} \rfloor}}{\lfloor \frac{\mu_1}{r} \rfloor!} \cdot \frac{\mu_2^{\lfloor \frac{\mu_2}{r} \rfloor}}{\lfloor \frac{\mu_2}{r} \rfloor!} x_1^{\mu_1} x_2^{\mu_2}$$

$$= \left(\sum_{m_1=1}^{\infty} \frac{(rm_1)^{m_1}}{m_1!} x_1^{rm_1} \right) \left(\sum_{m_1=2}^{\infty} \frac{(rm_2)^{m_2}}{m_2!} x_2^{rm_2} \right)$$

$$+ r \sum_{k=1}^{r-1} \left(\sum_{m_1=0}^{\infty} \frac{(rm_1+k)^{m_1}}{m_1!} x_1^{rm_1+k} \right) \left(\sum_{m_2=0}^{\infty} \frac{(rm_2+r-k)^{m_2}}{m_2!} x_2^{rm_2+r-k} \right)$$

$$= \xi_0^{r,0}(x_1)\xi_0^{r,0}(x_2) + r \sum_{k=1}^{r-1} \xi_0^{r,k}(x_1)\xi_0^{r,r-k}(x_2)$$

$$= \frac{rz_1^r}{1-rz_1} \cdot \frac{rz_2^r}{1-rz_2} + r \frac{1}{1-rz_1} \cdot \frac{1}{1-rz_2} \sum_{k=1}^{r-1} z_1^k z_2^{r-k}$$

$$= \frac{1}{(1-rz_1)(1-rz_2)} \left(r^2 z_1^r z_2^r + r \frac{z_1 z_2^r - z_2 z_1^r}{z_2 - z_1} \right),$$

where we have used (4.4) to find the auxiliary functions. It is easy to check that

$$F_{0,2}^{(r)}(z_1, z_2) = \log \frac{z_1 - z_2}{x_1 - x_2} - (z_1^r + z_2^r)$$

is a solution of this differential equation, where $x_i = z_i e^{-z_i^r}$. We note that as a convergent power series in (x_1, x_2), $F_{0,2}^{(r)}$ does not have any constant term. Since the convergent series eigenfunctions of the Euler differential operator $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ are homogeneous polynomials, its kernel consists of constants. Therefore, (1.10) is the only solution that satisfies the initial condition.   □

The main structural difference between the cut-and-join equation (2.4) and the differential recursion (1.11) is whether the unstable geometries are included in the right-hand side or not. While (1.11) is a genuine recursion for $F_{g,n}^{(r)}$ with respect to $2g - 2 + n$, (2.4) only gives a relation because $H_{g,n}^{(r)}$ appears on each side of the equation. In proving (1.11), we first calculate the Laplace transform of the cut-and-join equation, then use (1.9) and (1.10) to eliminate the unstable geometries from the right-hand side.

Lemma 4.1. *The straightforward Laplace transform of the cut-and-join equation* (2.4) *gives a differential equation*

$$(4.12) \quad \left(2g - 2 + n + \frac{1}{r} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) F_{g,n}^{(r)}[x_1, \ldots, x_n]$$

$$= \frac{1}{2} \sum_{i \neq j} \frac{1}{x_i - x_j} \left(x_i^2 \frac{\partial}{\partial x_i} F_{g,n-1}^{(r)}[x_{[\hat{j}]}] - x_j^2 \frac{\partial}{\partial x_j} F_{g,n-1}^{(r)}[x_{[\hat{i}]}] \right) - \sum_{i \neq j} x_i \frac{\partial}{\partial x_i} F_{g,n-1}^{(r)}[x_{[\hat{j}]}]$$

$$+ \frac{1}{2} \sum_{i=1}^{n} u_1 \frac{\partial}{\partial u_1} u_2 \frac{\partial}{\partial u_2} F_{g-1,n+1}^{(r)}[u_1, u_2, x_{[\hat{i}]}] \Big|_{u_1=u_2=x_i}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{i}]}} \left(x_i \frac{\partial}{\partial x_i} F^{(r)}_{g_1,|I|+1}[x_i, x_I] \right) \left(x_i \frac{\partial}{\partial x_i} F^{(r)}_{g_2,|J|+1}[x_i, x_J] \right),$$

where $F^{(r)}_{g,n}[x_1, \ldots, x_n]$ is defined by (1.6).

Proof. Since the cut-and-join equation (2.4) has the same structure as the simple Hurwitz number case of [35], the calculation of the Laplace transform goes exactly in parallel. Therefore, the left-hand side of (4.12) and the second and the third lines of the right-hand side are immediate from (1.6) and (2.4), noting how $x_i \frac{\partial}{\partial x_i}$ acts on x_i^α.

The trick we need is

$$\sum_{\mu_1, \mu_2 \geq 0} f(\mu_1 + \mu_2) x_1^{\mu_1} x_2^{\mu_2} = \sum_{k=0}^{\infty} f(k) \sum_{\mu_1 + \mu_2 = k} x_1^{\mu_1} x_2^{\mu_2} = \sum_{k=0}^{\infty} f(k) \left(\frac{x_1^{k+1} - x_2^{k+1}}{x_1 - x_2} \right),$$

which is valid if the series on the left-hand side is absolutely convergent. The ELSV formula (2.3) tells us that the power series $F^{(r)}_{g,n}[x_1, \ldots, x_n]$ is convergent on the polydisk

$$\left(|x_1| < e^{-\frac{1}{r}} \right) \times \cdots \times \left(|x_n| < e^{-\frac{1}{r}} \right).$$

Therefore, we can compute the Laplace transform of the first line of the right-hand side of (2.4) as follows.

$$\frac{1}{2} \sum_{\mu_1, \ldots, \mu_n \in \mathbb{Z}_+} \sum_{i \neq j} (\mu_i + \mu_j) H^{(r)}_{g,r-1} \left(\mu_i + \mu_j, \mu_{[\hat{i},\hat{j}]} \right) \prod_{i=1}^{n} x_i^{\mu_i}$$

$$= \frac{1}{2} \sum_{i \neq j} \sum_{\nu=0}^{\infty} \sum_{\vec{\mu}_{[\hat{i},\hat{j}]} \in \mathbb{Z}_+^{n-2}} \sum_{\mu_i + \mu_j = \nu} \nu H^{(r)}_{g,r-1} \left(\nu, \mu_{[\hat{i},\hat{j}]} \right) \prod_{i=1}^{n} x_i^{\mu_i}$$

$$- \frac{1}{2} \sum_{i \neq j} \sum_{\nu=0}^{\infty} \sum_{\vec{\mu}_{[\hat{i},\hat{j}]} \in \mathbb{Z}_+^{n-2}} \nu H^{(r)}_{g,r-1} \left(\nu, \mu_{[\hat{i},\hat{j}]} \right) (x_i^\nu + x_j^\nu) \prod_{k \neq i,j}^{n} x_k^{\mu_k}$$

$$= \frac{1}{2} \sum_{i \neq j} \sum_{\nu=0}^{\infty} \sum_{\vec{\mu}_{[\hat{i},\hat{j}]} \in \mathbb{Z}_+^{n-2}} \nu H^{(r)}_{g,r-1} \left(\nu, \mu_{[\hat{i},\hat{j}]} \right) \frac{x_i^{\nu+1} - x_j^{\nu+1}}{x_i - x_j} \prod_{k \neq i,j}^{n} x_k^{\mu_k}$$

$$- \sum_{i \neq j} x_i \frac{\partial}{\partial x_i} F^{(r)}_{g,r-1} \left[x_i, x_{[\hat{i},\hat{j}]} \right]$$

$$= \frac{1}{2} \sum_{i \neq j} \frac{1}{x_i - x_j} \left(x_i^2 \frac{\partial}{\partial x_i} F^{(r)}_{g,r-1} \left[x_i, x_{[\hat{i},\hat{j}]} \right] - x_j^2 \frac{\partial}{\partial x_j} F^{(r)}_{g,r-1} \left[x_j, x_{[\hat{i},\hat{j}]} \right] \right)$$

$$- \sum_{i \neq j} x_i \frac{\partial}{\partial x_i} F^{(r)}_{g,r-1} \left[x_{[\hat{j}]} \right].$$

This completes the proof. $\square$

Proof of Theorem 1.1. The conversion of (4.12) to the form (1.11) is now straightforward, using (4.10), and substituting the unstable geometries with the actual values (1.9) and (1.10) in the right-hand side.

The contribution from the terms of $g_1 = 0, I = \emptyset$ and $g_2 = 0, J = \emptyset$ in the third line of the right-hand side of (4.12) is

$$\sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} F_{0,1}^{(r)}[x_i] \right) \left(x_i \frac{\partial}{\partial x_i} F_{g,n}^{(r)} \left[x_i, x_{[\hat{i}]} \right] \right) = \sum_{i=1}^n z_i^r \frac{z_i}{1 - r z_i^r} \frac{\partial}{\partial z_i} F_{g,n}^{(r)}(z_1, \ldots, z_n).$$

If we bring this term to the left-hand side or (4.12), then we have

$$\left(2g - 2 + n + \frac{1}{r} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right) F_{g,n}^{(r)}(z_1, \ldots, z_n) - \sum_{i=1}^n z_i^r \frac{z_i}{1 - r z_i^r} \frac{\partial}{\partial z_i} F_{g,n}^{(r)}(z_1, \ldots, z_n)$$

$$= \left(2g - 2 + n + \frac{1}{r} \sum_{i=1}^n \frac{z_i}{1 - r z_i^r} (1 - r z_i^r) \frac{\partial}{\partial z_i} \right) F_{g,n}^{(r)}(z_1, \ldots, z_n),$$

which is the left-hand side of (1.11).

The other unstable terms come from $g_1 = 0, I = \{j\}$ and $g_2 = 0, J = \{j\}$. The contribution is

$$\sum_{i=1}^n \sum_{j \neq i} \left(x_i \frac{\partial}{\partial x_i} F_{0,2}^{(r)}[x_i, x_j] \right) \left(x_i \frac{\partial}{\partial x_i} F_{g,n-1}^{(r)} \left[x_i, x_{[\hat{i},\hat{j}]} \right] \right)$$

$$= \sum_{i \neq j} \left(\frac{z_i}{1 - r z_i^r} \frac{\partial}{\partial z_i} \left(\log(z_i - z_j) - (z_i^r + z_j^r) \right) - x_i \frac{\partial}{\partial x_i} \log(x_i - x_j) \right) \left(x_i \frac{\partial}{\partial x_i} F_{g,n-1}^{(r)} \left[x_{[\hat{j}]} \right] \right)$$

$$= \sum_{i \neq j} \left(\frac{z_i}{1 - r z_i^r} \left(\frac{1}{z_i - z_j} - r z_i^{r-1} \right) - \frac{x_i}{x_i - x_j} \right) \left(x_i \frac{\partial}{\partial x_i} F_{g,n-1}^{(r)} \left[x_{[\hat{j}]} \right] \right).$$

These terms and the first line of the right-hand side of (4.12) together yield

$$\sum_{i \neq j} \left(\frac{z_i}{1 - r z_i^r} \left(\frac{1}{z_i - z_j} - r z_i^{r-1} \right) - \frac{x_i}{x_i - x_j} + \frac{x_i}{x_i - x_j} - 1 \right) \left(x_i \frac{\partial}{\partial x_i} F_{g,n-1}^{(r)} \left[x_{[\hat{j}]} \right] \right)$$

$$= \sum_{i \neq j} \left(\frac{z_i}{1 - r z_i^r} \left(\frac{1}{z_i - z_j} - r z_i^{r-1} \right) - 1 \right) \left(\frac{z_i}{1 - r z_i^r} \frac{\partial}{\partial z_i} F_{g,n-1}^{(r)} \left(z_{[\hat{j}]} \right) \right)$$

$$= \sum_{i \neq j} \frac{z_i z_j}{z_i - z_j} \frac{1}{(1 - r z_i^r)^2} \frac{\partial}{\partial z_i} F_{g,n-1}^{(r)} \left(z_{[\hat{j}]} \right),$$

which is the same as the first line of the right-hand side of (1.11).

Converting the second line and the stable terms in the third line of the right-hand side of (4.12) is straightforward. This completes the proof of Theorem 1.1. $\square$

5. Some properties of the free energies

In this section we derive some properties of the free energies and compute a few examples. We also check our results with closed formulas obtained in [29].

A direct consequence of the ELSV formula (2.3) is the following.

Proposition 5.1. *The Laplace transform of* (2.3), *the free energy of type* (g, n), *is an element of the tensor algebra*

$$(5.1) \qquad\qquad F_{g,n}^{(r)}(z_1, \ldots, z_n) \in \mathrm{Sym}^{\otimes n} \left(\mathbb{C}(\mathbb{P}^1) \right),$$

except for $F_{0,2}^{(r)}(z_1, z_2)$. The poles $F_{g,n}^{(r)}(z_1, \ldots, z_n)$ are located at

$$D \times \left(\mathbb{P}^1\right)^{n-1} \cup \mathbb{P}^1 \times D \times \cdots \times \mathbb{P}^1 \cup \cdots \cup \left(\mathbb{P}^1\right)^{n-1} \times D,$$

where

$$D = \{z \in \mathbb{C} \mid 1 - rz^r = 0\}.$$

The highest total degree of poles of $F_{g,n}^{(r)}$ is $6g - 6 + 3n$.

Proof. This follows from the fact that the coefficient

$$r^{1-g+\sum_{i=1}^n \langle \frac{\mu_i}{r} \rangle} \int_{\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)} \prod_{i=1}^n \psi_i^{d_i} \sum_{j \geq 0} (-r)^j \lambda_j$$

of (2.3) depends only on $\vec{\mu} \mod r$, hence

(5.2) $\quad F_{g,n}^{(r)}(z_1, \ldots, z_n)$

$$= \sum_{\vec{\mu} \in \mathbb{Z}_+^n, \sum \mu_i \equiv 0 \, (r)} r^{1-g+\sum_{i=1}^n \langle \frac{\mu_i}{r} \rangle} \int_{\overline{\mathcal{M}}_{g,-\vec{\mu}}(BG)} \frac{\sum_{j \geq 0}(-r)^j \lambda_j}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \prod_{i=1}^n \frac{\mu_i^{\lfloor \frac{\mu_i}{r} \rfloor}}{\lfloor \frac{\mu_i}{r} \rfloor!} x_i^{\mu_i}$$

$$= \sum_{\substack{0 \leq k_1, \ldots, k_n < r, \sum k_i \equiv 0 \, (r) \\ d_1 + \cdots + d_n \leq 3g-3+n}} r^{1-g+\sum_{i=1}^n \frac{k_i}{r}} \left(\int_{\overline{\mathcal{M}}_{g,-\vec{k}}(BG)} \prod_{i=1}^n \psi_i^{d_i} \sum_{j \geq 0}(-r)^j \lambda_j \right) \prod_{i=1}^n \xi_{d_i}^{r,k_i}(x_i).$$

For $d_i \geq 0$, each $\xi_{d_i}^{r,k_i}(x_i)$ is a rational function in z_i with poles at $z_i \in D$ of degree $2d_i+1$ due to (4.4), (4.9), and (4.10). The highest degree poles occur when $d_1 + \cdots + d_n = 3g - 3 + n$, and then $F_{g,n}^{(r)}$ has poles of degree $6g - 6 + 3n$. $\qquad\square$

Using the same notation as in Theorem 2.1, let us denote

(5.3) $$\langle \tau_{2g-2+j} \lambda_{g-j} \rangle^{(r)} = \int_{\overline{\mathcal{M}}_{g,1}(BG)} \psi_1^{2g-2+j} \lambda_{g-j},$$

where $G = \mathbb{Z}/r\mathbb{Z}$. The generating function of these one-point intersection numbers is determined in [29]:

(5.4) $$\frac{1}{2r} \left(\frac{r\hbar/2}{\sin(r\hbar/2)} \right)^u \frac{1}{\sin(\hbar/2)} = \frac{1}{r\hbar} + \sum_{g=1}^\infty \left(\sum_{j=0}^g \langle \tau_{2g-2+j} \lambda_{g-j} \rangle^{(r)} u^j \right) \hbar^{2g-1}.$$

Note that from (2.3) and (4.3) we can calculate the one-point free energies:

(5.5) $$F_{g,1}^{(r)}(z) = \sum_{j=0}^g (-1)^{g-j} r^{1-j} \langle \tau_{2g-2+j} \lambda_{g-j} \rangle^{(r)} \xi_{2g-2+j}^{r,0}(x).$$

For example, in terms of

(5.6) $$t = \frac{1}{1 - rz^r},$$

we have

$$F_{1,1}^{(r)}(z) = \frac{1}{24}(r^2 t^3 - r^2 t^2 - t + 1),$$

$$F_{2,1}^{(r)}(z) = \frac{r^2}{5760} \Big(525 r^4 t^9 - 1575 r^4 t^8 + 10 r^2 (167 r^2 - 15) t^7 + 350 r^2 (-2r^2 + 1) t^6$$

$$+ (68r^4 - 260r^2 + 21)t^5 + (12r^4 + 60r^2 - 35)t^4 + 14t^3 \Big),$$

$$F_{3,1}^{(r)}(z) = \frac{r^4}{2903040} \Big(4729725r^6t^{15} - 23648625r^6t^{14} + 24255r^4(2012r^2 - 45)t^{13}$$

$$+ 35035r^4(-1516r^2 + 135)t^{12} + 35r^2(914912r^4 - 235116r^2 + 3969)t^{11}$$

$$+ 231r^2(-43156r^4 + 31430r^2 - 2205)t^{10}$$

$$+ 35(31016r^6 - 95340r^4 + 20580r^2 - 279)t^9$$

$$+ 7(15416r^6 + 100596r^4 - 69384r^2 + 4185)t^8$$

$$+ 12(-1128r^6 - 2646r^4 + 12789r^2 - 2635)t^7$$

$$+ 6(-320r^6 - 840r^4 - 2940r^2 + 2387)t^6 - 2232t^5 \Big).$$

In general,

Proposition 5.2. *The one-point free energy $F_{g,1}^{(r)}(z)$ of genus g is a polynomial of degree $6g - 3$ in $t = \frac{1}{1-rz^r}$.*

Proof. The expression (5.5) tells us that $F_{g,1}^{(r)}(z)$ is a function in z^r. More precisely, it is a ratio of a polynomial in z^r and a power of $1 - rz^r$. Therefore, it is a Laurent polynomial in t. The only auxiliary functions appearing in (5.5) are $\xi_\ell^{r,0}(x)$ for $\ell \geq 0$. From (4.4) and (4.9) we calculate

(5.7)
$$\xi_0^{r,0}(x) = \frac{rz^r}{1 - rz^r} = t - 1,$$

$$\xi_\ell^{r,0}(x) = \left(rt^2(t-1)\frac{d}{dt} \right)^\ell (t-1),$$

since

$$x\frac{d}{dx} = rt^2(t-1)\frac{d}{dt}.$$

Therefore, $F_{g,1}^{(r)}(z)$ is a polynomial of degree $2(3g-2)+1$ in t. The degree of the polynomial is the same as the degree of poles of Proposition 5.1 for $n = 1$.   □

The initial cases of the differential recursion (1.11) are $(g, n) = (1, 1)$ and $(0, 3)$. For the $g = n = 1$ case, the differential equation is

$$\left(1 + \frac{1}{r}z\frac{\partial}{\partial z} \right) F_{1,1}^{(r)}(z) = \frac{1}{2}\frac{z^2}{(1 - rz^r)^2}\frac{\partial^2}{\partial u_1 \partial u_2} F_{0,2}^{(r)}(u_1, u_2)\Big|_{u_1=u_2=z}.$$

The unique solution to this equation with the initial condition $F_{1,1}^{(r)}(0) = 0$ agrees with the above computation using the result of [29].

The free energy $F_{0,3}^{(r)}(z_1, z_2, z_3)$ can also be calculated from (2.3) since $\overline{\mathcal{M}}_{0,3}$ is a point. Thus the BG Hodge integral contribution in the formula is simply 1. We have

$$F_{0,3}^{(r)}(z_1, z_2, z_3) = \sum_{\substack{\vec{\mu} \in \mathbb{Z}_+^3 \\ \mu_1+\mu_2+\mu_3 \equiv 0}} r^{1+\langle\frac{\mu_1}{r}\rangle+\langle\frac{\mu_2}{r}\rangle+\langle\frac{\mu_3}{r}\rangle} \prod_{i=1}^3 \frac{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}{\lfloor\frac{\mu_i}{r}\rfloor!} x_i^{\mu_i}$$

$$= r\xi_0^{r,0}(x_1)\xi_0^{r,0}(x_2)\xi_0^{r,0}(x_3) + r^2 \sum_{\substack{k_1+k_2+k_3=r \\ 0\le k_i\le r-1}} \xi_0^{r,k_1}(x_1)\xi_0^{r,k_2}(x_2)\xi_0^{r,k_3}(x_3)$$

$$+ r^3 \sum_{\substack{k_1+k_2+k_3=2r \\ 0\le k_i\le r-1}} \xi_0^{r,k_1}(x_1)\xi_0^{r,k_2}(x_2)\xi_0^{r,k_3}(x_3).$$

More concretely,

$$F_{0,3}^{(2)}(z_1, z_2, z_3) = 8\frac{z_1z_2z_3(z_1+z_2+z_3+2z_1z_2z_3)}{(1-2z_1^2)(1-2z_2^2)(1-2z_3^2)},$$

$$F_{0,3}^{(3)}(z_1, z_2, z_3) = 9\frac{z_1z_2z_3\left(1+3z_1z_2z_3+3\sum_{i\ne j}z_i^2z_j+9z_1^2z_2^2z_3^2\right)}{(1-3z_1^3)(1-3z_2^3)(1-3z_3^3)}.$$

6. THE QUANTUM CURVE

Since the r-Lambert curve (1.4) has genus 0, we define the partition function $Z(z, \hbar)$ as in (1.12). In this section we prove Theorem 1.3.

Proposition 6.1. *The principal specialization $F_{g,n}^{(r)}(z, \ldots, z)$ for $2g-2+n > 0$ is a polynomial in t of degree $6g-6+3n$, where t is the variable introduced in (5.6).*

Proof. This is an immediate consequence of Proposition 5.1 and its proof. $\square$

For unstable geometries, we use the same argument of [34] to find

$$F_{0,1}^{(r)}(z) = \frac{1}{2r^2}\left(1 - \frac{1}{t^2}\right), \tag{6.1}$$

$$F_{0,2}^{(r)}(z, z) = \frac{1}{r}\left(1 - \frac{1}{t}\right) + \log t. \tag{6.2}$$

Proposition 6.2. *The 1-variable functions*

$$S_m^{(r)}(z) = \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}^{(r)}(z, \ldots, z), \qquad m = 0, 1, 2, \ldots, \tag{6.3}$$

satisfy the second order ordinary differential equation

$$\left(m + \frac{1}{r}\frac{d}{dz}\right)S_{m+1}^{(r)}(z) = \frac{1}{2}\left[\frac{d}{dz}\left(\frac{z}{1-rz^r}\right)^2 - \frac{2z}{(1-rz^r)^2}\right]\cdot\frac{d}{dz}S_m^{(r)}(z)$$

$$\tag{6.4}$$

$$+ \frac{1}{2}\frac{z^2}{(1-rz^r)^2}\left(\frac{d^2}{dz^2}S_m^{(r)}(z) + \sum_{\substack{a+b=m+1 \\ a,b\ge 2}}\frac{d}{dz}S_a^{(r)}(z)\frac{d}{dz}S_b^{(r)}(z)\right).$$

Proof. The principal specialization of the differential recursion (1.11) reduces to the following ordinary differential equation.

$$\left(2g-2+n+\frac{1}{r}z\frac{d}{dz}\right)F_{g,n}^{(r)}(z, \cdots, z) = \frac{n}{2}\frac{d}{dz}\left(\frac{z^2}{(1-rz^r)^2}\right)\frac{d}{dz}F_{g,n-1}^{(r)}(z, \cdots, z)$$

$$-n\frac{z}{(1-rz^r)^2}\frac{d}{dz}F_{g,n-1}^{(r)}(z, \cdots, z)$$

$$+ \frac{1}{2}n(n-1)\frac{z^2}{(1-rz^r)^2}\frac{\partial^2}{\partial u^2}\bigg|_{u=z}F_{g,n-1}^{(r)}(z, \cdots, z)$$

$$+ \frac{n}{2} \frac{z^2}{(1-rz^r)^2} \frac{\partial^2}{\partial u_1 \partial u_2}\bigg|_{u_1=u_2=z} F^{(r)}_{g-1,n+1}(u_1, u_2, z, \cdots, z)$$

$$+ \frac{n!}{2} \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n-1}}^{\text{stable}} \left[\frac{1}{(n_1+1)!} \frac{z}{1-rz^r} \frac{d}{dz} F^{(r)}_{g_1,n_1+1}(z,\cdots,z) \right.$$

$$\left. + \frac{1}{(n_2+1)!} \frac{z}{1-rz^r} \frac{d}{dz} F^{(r)}_{g_2,n_2+1}(z,\cdots,z) \right].$$

The summation (6.3) proves the proposition. □

Proof of Theorem 1.3. Note that we have

$$S_0^{(r)}(z) = z^r \left(\frac{1}{r} - \frac{1}{2} z^r \right), \quad S_1^{(r)}(z) = -\frac{1}{2} \log(1-rz^r) - \frac{1}{2} z^r.$$

If we include these unstable terms into (6.4), then we obtain

$$\left(m + \frac{1}{r} \frac{z}{1-rz^r} \frac{d}{dz} \right) S_{m+1}^{(r)}(z)$$

$$= \frac{1}{2} \left(\left(\frac{z}{1-rz^r} \frac{d}{dz} \right)^2 S_m^{(r)}(z) + \sum_{a+b=m+1} \frac{z}{1-rz^r} \frac{d}{dz} S_a^{(r)}(z) \cdot \frac{z}{1-rz^r} \frac{d}{dz} S_b^{(r)}(z) \right)$$

$$- \frac{1}{2} \frac{z}{1-rz^r} \frac{d}{dz} S_m^{(r)}(z).$$

In terms of the generating series

$$F^{(r)}(z, \hbar) = \sum_{m=0}^{\infty} S_m^{(r)} \hbar^{m-1},$$

the equation becomes

$$\hbar \frac{\partial}{\partial \hbar} F^{(r)}(z, \hbar) + \frac{1}{r} \frac{z}{1-rz^r} \frac{d}{dz} F^{(r)}(z, \hbar)$$

$$= \frac{\hbar}{2} \left[\left(\frac{z}{1-rz^r} \frac{d}{dz} \right)^2 F^{(r)}(z, \hbar) + \left(\frac{z}{1-rz^r} \frac{d}{dz} F^{(r)}(z, \hbar) \right)^2 \right]$$

$$- \frac{\hbar}{2} \frac{z}{1-rz^r} \frac{d}{dz} F^{(r)}(z, \hbar).$$

Since $Z^{(r)}(z, \hbar) = \exp F^r(z, \hbar)$ and $D = x \frac{d}{dx} = \frac{z}{1-rz^r} \frac{d}{dz}$, we have

(6.5) $$\left(\frac{\partial}{\partial \hbar} + \left(\frac{1}{r\hbar} + \frac{1}{2} \right) x \frac{d}{dx} - \frac{1}{2} \left(x \frac{d}{dx} \right)^2 \right) Z^{(r)}(z, \hbar) = 0,$$

which establishes (1.14).

Now define

(6.6) $$P = \hbar \frac{\partial}{\partial w} + e^{-\frac{r-1}{2} \hbar \frac{\partial}{\partial w}} e^{-rw} e^{\frac{r-1}{2} \hbar \frac{\partial}{\partial w}} e^{-r\hbar \frac{\partial}{\partial w}}$$

(6.7) $$Q = \frac{\hbar}{2} \frac{\partial^2}{\partial w^2} + \left(\frac{1}{r} + \frac{\hbar}{2} \right) \frac{\partial}{\partial w} - \hbar \frac{\partial}{\partial \hbar}.$$

It is proved in [33] that P annihilates the partition function $Z^{(r)}(z, \hbar)$:

$$(6.8) \qquad PZ^{(r)}(z, \hbar) = \left(\hbar \frac{\partial}{\partial w} + e^{-\frac{r-1}{2}\hbar\frac{\partial}{\partial w}} e^{-rw} e^{\frac{r-1}{2}\hbar\frac{\partial}{\partial w}} e^{-r\hbar\frac{\partial}{\partial w}} \right) Z^{(r)}(z, \hbar) = 0.$$

Since $e^{\frac{r-1}{2}\hbar\frac{\partial}{\partial w}}$ is a *shift* operator, with the multiplication operator by a function $f(w)$ it satisfies the relation

$$e^{\frac{r-1}{2}\hbar\frac{\partial}{\partial w}} \cdot f(w) = f(w + \frac{r-1}{2}\hbar) \cdot e^{\frac{r-1}{2}\hbar\frac{\partial}{\partial w}}.$$

Therefore, the operator P can also be written as

$$P = \hbar \frac{\partial}{\partial w} + e^{r(-w + \frac{r-1}{2}\hbar)} e^{-r\hbar\frac{\partial}{\partial w}}.$$

Now the commutator relation

$$[P, Q] = P$$

is straightforward.

The semi-classical limit calculations are the same as those in [34]. We have thus completed the proof of Theorem 1.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

7. The Eynard-Orantin topological recursion

In this section, we shall prove Theorem 1.7. For a mathematical definition of the Eynard-Orantin theory, we refer to [19, 34].

Because of the definition of the differentials

$$W_{g,n}^{(r)} = d_1 \cdots d_n F_{g,n}^{(r)},$$

we expect that the exterior differentiation of (1.11) should give the integral recursion (1.17). This naive idea does not work because of the specific reference to the local Galois conjugation s_j appearing in the integral recursion. The PDE (1.11) does not care about the x-projection of the spectral curve, while (1.17) heavily uses the local ramification structure of the spectral curve as a covering of the x-coordinate line. The integration kernel (1.18) shows that the residue calculation on the right-hand side of (1.17) is similar to the local Galois averaging. Yet evaluation of the free energies at any Galois conjugate point is no longer a rational function, since $s_j(z)$ is a very complicated holomorphic function in z.

The strategy we adopt in this section is to extract the *principal part* of the local Galois average, and then take the terms of the result that are the pull-back of a function in the x-coordinate. On the stable range $2g - 2 + n > 0$, the free energies are indeed functions in the x_i-variables, so the last step makes sense. And by taking the principal part of the Galois average, we maintain the finiteness (polynomial-like) structure of $W_{g,n}^{(r)}$ that represents the picewise polynomiality of the orbifold Hurwitz number $H_{g,n}^{(r)}(\vec{\mu})$.

Thus the simple residue operation of the right-hand side of (1.17) amounts to the combination of the algebraic operations listed in Subsection 7.5 and the projection to the principal part described in Definition 7.9.

7.1. **The spectral curve and the x-projection.** For the convenience of calculations we shall use the scaled coordinate $\eta = \sqrt[r]{r}z$ from now on. This change has no significance, but some formulas and statements become less cumbersome in the η-coordinate.

The r-Lambert curve (1.4) is now given by

$$x = \frac{1}{\sqrt[r]{r}} \eta\, e^{-\eta^r/r},$$

and the x-projection has r simple ramification points at the r-roots of unity $1 - \eta^r = 0$. We denote these ramification points by

$$\{\alpha_j \mid \alpha_j = e^{2(j-1)\pi i/r}, j = 1, 2, \cdots, r\}.$$

Around each critical point α_j, the x-projection is locally a double-sheeted covering. There is a neighborhood U_j of α_j such that when $\eta \in U_j$, there is another point $\widetilde{\eta}$ satisfying $x(\widetilde{\eta}) = x(\eta)$. This correspondence defines a local deck transformation (or local Galois conjugation) $s_j(\eta) := \widetilde{\eta}$ on U_j. Clearly, s_j is an involution: $s_j(s_j(\eta)) = \eta$.

Lemma 7.1. *For each $j = 1, \cdots, r$, the deck transformation $s_j(\eta)$ is a holomorphic function in η defined on U_j. Moreover, the function form of $s_j(\eta)$ in the variable η does not depend on the index j.*

Proof. Let us introduce notations $\Delta_j := 1 - s_j(\eta)^r$ and $\Delta := 1 - \eta^r$. The equation $x(s_j(\eta)) = x(\eta)$ then gives

$$\log(1 - \Delta_j) + \Delta_j = \log(1 - \Delta) + \Delta.$$

We make U_j smaller so that it lies in the region $|\Delta| < 1$. Then Δ_j has a power series expansion

$$\Delta_j = -\Delta - \frac{2}{3}\Delta^2 - \frac{4}{9}\Delta^3 - \frac{44}{135}\Delta^4 - \frac{104}{405}\Delta^5 - \frac{40}{189}\Delta^6 - \frac{7648}{42525}\Delta^7 - \frac{2848}{18225}\Delta^8 + O(\Delta^9),$$

which convergences for $|\Delta| < 1$. Therefore Δ_j is a holomorphic function of η defined on U_j whose function form in η does not depend on j. Since

$$s_j(\eta) = \eta \exp \frac{\Delta - \Delta_j}{r},$$

it is holomorphic in η on U_j, and the function form does not depend on j, either. $\square$

7.2. The free energies and the auxiliary functions in the η-coordinate.

By abuse of notation, we denote the auxiliary functions of (4.3) and (4.4) by the same notation and consider them as functions in η. Thus we re-define

$$(7.1) \qquad \xi_{-1}^{r,k}(\eta) = \begin{cases} \frac{1}{r}\eta^r & k = 0 \\ \frac{1}{kr^{k/r}}\eta^k & k > 0, \end{cases} \qquad \xi_{m+1}^{r,k}(\eta) = \frac{\eta}{1 - \eta^r}\frac{d}{d\eta}\xi_m^{r,k}(\eta), \quad m \geq -1.$$

Remark 7.2. It is easy to see that $\xi_m^{r,k}(\eta)$ is a proper rational function in η for $m \geq 0$, whose denominator is a constant times $(1 - \eta^r)^{2m+1}$. Thus $\xi_m^{r,k}(\eta)$ is meromorphic with poles only at α_j's.

A few examples of $\xi_m^{r,k}(\eta)$ are given in Table 1.

We denote the free energy $F_{g,n}^{(r)}(\eta_1, \ldots, \eta_n)$ as

$$(7.2) \qquad F_{g,n}^{(r)}(\eta_1, \ldots, \eta_n) = r^{1-g} \sum_{\substack{|\vec{k}| \equiv 0 \, (r) \\ |\vec{\ell}| \leq 3g-3+n}} r^{|\vec{k}|/r} \, \langle \tau_{\vec{\ell}} \Lambda \rangle^{(r),\vec{k}} \, \xi_{\vec{\ell}}^{r,\vec{k}}(\eta_1, \ldots, \eta_n),$$

where $\vec{\ell} := (\ell_1, \ldots, \ell_n)$ with $\ell_i \geq 0$, $\vec{k} := (k_1, \ldots, k_n)$ with $0 \leq k_i < r$, $|\vec{k}| := \sum_{i=1}^n k_i$, $\Lambda := \sum_{j \geq 0}(-r)^j \lambda_j$, and, $\xi_{\vec{\ell}}^{r,\vec{k}}(\eta_1, \ldots, \eta_n) := \prod_{i=1}^n \xi_{\ell_i}^{r,k_i}(\eta_i)$. The Hodge integrals are abbreviated as

$$(7.3) \qquad \langle \tau_{\vec{\ell}} \Lambda \rangle^{(r),\vec{k}} := \langle \tau_{\ell_1} \tau_{\ell_2} \ldots \tau_{\ell_n} \Lambda \rangle^{(r),\vec{k}} = \int_{\overline{\mathcal{M}}_{g,-\vec{k}}(BG)} \prod_{i=1}^n \psi_i^{\ell_i} \sum_{j \geq 0}(-r)^j \lambda_j.$$

	$k=0$	$k=1$	$k=2$
$m=-1$	$\dfrac{\eta^r}{r}$	$\dfrac{\eta}{r^{1/r}}$	$\dfrac{\eta^2}{2r^{2/r}}$
$m=0$	$\dfrac{\eta^r}{1-\eta^r}$	$\dfrac{\eta}{r^{1/r}(1-\eta^r)}$	$\dfrac{\eta^2}{r^{2/r}(1-\eta^r)}$
$m=1$	$\dfrac{r\eta^r}{(1-\eta^r)^3}$	$\dfrac{\eta\,[(r-1)\eta^r+1]}{r^{1/r}(1-\eta^r)^3}$	$\dfrac{\eta^2\,[(r-2)\eta^r+2]}{r^{2/r}(1-\eta^r)^3}$
$m=2$	$\dfrac{r^2\eta^r(2\eta^r+1)}{(1-\eta^r)^5}$	$\dfrac{\eta\,[(2r^2-3r+1)\eta^{2r}+(r^2+3r-2)\eta^r+1]}{r^{1/r}(1-\eta^r)^5}$	$\dfrac{\eta^2\,[2(r^2-3r+2)\eta^{2r}+(r^2+6r-8)\eta^r+4]}{r^{2/r}(1-\eta^r)^5}$

TABLE 1. $\xi_m^{r,k}(\eta)$ for $m=-1,\ldots,2$ and $k=0,1,2$.

In terms of the η-variables, the unstable free energies are given by

$$(7.4) \qquad F_{0,1}^{(r)}(\eta) = \frac{1}{r^2}\eta^r - \frac{1}{2r^2}\eta^{2r},$$

$$(7.5) \qquad F_{0,2}^{(r)}(\eta_1,\eta_2) = \log\frac{\eta_1-\eta_2}{x_1-x_2} - \frac{1}{r}(\log r + \eta_1^r + \eta_2^r).$$

For (g,n) in the stable range $2g-2+n>0$, (1.11) becomes

$$(7.6)\quad \left(2g-2+n+\frac{1}{r}\sum_{i=1}^{n}\eta_i\frac{\partial}{\partial\eta_i}\right)F_{g,n}^{(r)}(\eta_1,\ldots,\eta_n)$$

$$= \frac{1}{2}\sum_{i\neq j}\frac{\eta_i\eta_j}{\eta_i-\eta_j}\left[\frac{1}{(1-\eta_i^r)^2}\frac{\partial}{\partial\eta_i}F_{g,n-1}^{(r)}\big(\eta_{[\hat{j}]}\big) - \frac{1}{(1-\eta_j^r)^2}\frac{\partial}{\partial\eta_j}F_{g,n-1}^{(r)}\big(\eta_{[\hat{i}]}\big)\right]$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\frac{\eta_i^2}{(1-\eta_i^r)^2}\left.\frac{\partial^2}{\partial u_1\partial u_2}F_{g-1,n+1}^{(r)}\big(u_1,u_2,\eta_{[\hat{i}]}\big)\right|_{u_1=u_2=\eta_i}$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\frac{\eta_i^2}{(1-\eta_i^r)^2}\overset{\text{stable}}{\underset{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{i}]}}{\sum}}\left(\frac{\partial}{\partial\eta_i}F_{g_1,|I|+1}^{(r)}(\eta_i,\eta_I)\right)\left(\frac{\partial}{\partial\eta_i}F_{g_2,|J|+1}^{(r)}(\eta_i,\eta_J)\right).$$

7.3. The integration kernel for the Eynard-Orantin recursion. Using (7.4) and (7.5) we find

$$W_{0,1}^{(r)}(\eta) = \frac{1}{r}\eta^{r-1}(1-\eta^r)d\eta, \qquad W_{0,2}^{(r)}(\eta_1,\eta_2) = \frac{d\eta_1\otimes d\eta_2}{(\eta_1-\eta_2)^2} - \frac{dx_1\otimes dx_2}{(x_1-x_2)^2}.$$

We recall Lemma 7.1, which states that the local Galois conjugation $s_j(\eta)$, considered as a function in η, does not depend on the index j. Let us denote this function by $\widetilde{\eta}=\widetilde{\eta}(\eta)$. As a consequence, the integration kernel (1.18) has an expression independent of j as well, and is given by

$$(7.7)\qquad K_j(\eta,\eta_1) = \frac{r}{2}\frac{\eta}{(\widetilde{\eta}^r-\eta^r)(1-\eta^r)}\left(\frac{1}{\eta-\eta_1}-\frac{1}{\widetilde{\eta}-\eta_1}\right)\cdot d\eta_1\otimes\frac{1}{d\eta}.$$

Here we note that $W_{0,2}^{(r)}$ in the integral recursion (1.17) can be replaced by Riemann's normalized fundamental differential of the second kind

$$(7.8) \qquad B(\eta_1, \eta_2) = \frac{d\eta_1 \otimes d\eta_2}{(\eta_1 - \eta_2)^2}.$$

This is because the difference

$$W_{0,2}^{(r)}(\eta_1, \eta_2) - B(\eta_1, \eta_2) = -\frac{dx_1 \otimes dx_2}{(x_1 - x_2)^2}$$

does not contribute to the residue calculation of the second line of the right-hand side of (1.17).

7.4. The local analytic properties of the auxiliary functions. Let us denote

$$\Delta = 1 - \eta^r \quad \text{and} \quad \widetilde{\Delta} = 1 - \widetilde{\eta}^r$$

for η in each neighborhood U_j of the critical point α_j. When $\eta \to \alpha_i$, $\Delta(\eta)$ and $\widetilde{\Delta}(\eta)$ converge to 0. We therefore regard Δ and $\widetilde{\Delta}$ as small parameters, for example, $|\Delta| < 1$ and $|\widetilde{\Delta}| < 1$, for $\eta \in U_j$.

To analyze the r-Lambert curve locally around its critical point, let us introduce a local parameter u around α_j by

$$(7.9) \qquad e^{-u-1} = \eta^r e^{-\eta^r} = \widetilde{\eta}^r e^{-\widetilde{\eta}^r}.$$

When η is in any neighborhood U_j, we have

$$(7.10) \qquad u = -\Delta - \log(1 - \Delta) = -\widetilde{\Delta} - \log(1 - \widetilde{\Delta}).$$

Since the u-projection of the r-Lambert curve around α_j is a double-sheeted covering, let us define

$$(7.11) \qquad \frac{1}{2}v^2 = u.$$

This *Airy curve* equation describes the local behavior of our spectral curve around α_j. We choose the branch of v at α_j so that we have an expansion

$$(7.12) \qquad v = \Delta + \frac{1}{3}\Delta^2 + \frac{7}{36}\Delta^3 + \frac{73}{540}\Delta^4 + O(\Delta^5).$$

Here again we can see that v is a holomorphic function in η with the same expression at each neighborhood U_j of α_j, without any explicit dependence on the index j. From the definition and (7.10), we know that the other branch of the curve around α_j is given by

$$v(\widetilde{\eta}) = \widetilde{\Delta} + \frac{1}{3}\widetilde{\Delta}^2 + \frac{7}{36}\widetilde{\Delta}^3 + \frac{73}{540}\widetilde{\Delta}^4 + O(\widetilde{\Delta}^5).$$

Of course in terms of the v-coordinate the local Galois conjugate is given simply by

$$(7.13) \qquad v(\widetilde{\eta}) = -v(\eta),$$

while u is symmetric under the involution $\eta \mapsto \widetilde{\eta}$.

At each α_j, we can express Δ and $\widetilde{\Delta}$ as inverse series in v for sufficiently small v.

$$(7.14) \qquad \begin{aligned} \Delta &= \psi(v) := v - \frac{1}{3}v^2 + \frac{1}{36}v^3 + \frac{1}{270}v^4 + O(v^5), \\ \widetilde{\Delta} &= \psi(-v) = -v - \frac{1}{3}v^2 - \frac{1}{36}v^3 + \frac{1}{270}v^4 + O(v^5). \end{aligned}$$

To make the equations shorter, we use the following functions

$$(7.15) \qquad Y^k(\eta) = \frac{\eta^k + \widetilde{\eta}^k}{2},$$

$$(7.16) \qquad E^{r,k}(\eta) = \frac{\eta \frac{d}{d\eta} Y^k(\eta)}{(1 - \eta^r) Y^k(\eta)},$$

$$(7.17) \qquad \varphi_n^{r,k}(\eta) = \frac{\xi_n^{r,k}(\eta) - \xi_n^{r,k}(\widetilde{\eta})}{2\, Y^k(\eta)},$$

$$(7.18) \qquad h_n^{r,k}(\eta) = \frac{\xi_n^{r,k}(\eta) + \xi_n^{r,k}(\widetilde{\eta})}{2\, Y^k(\eta)}.$$

Here $\widetilde{\eta}$ is understood as $s_j(\eta)$ for any j.

Remark 7.3. Again thanks to Lemma 7.1, the above expressions are well defined and independent of which neighborhood U_j of the critical points the η-variable lies.

Proposition 7.4. *The functions* $Y^k(\eta)$, $E^{r,k}(\eta)$ *and* $h_n^{r,k}(\eta)$ *are symmetric under the involution* $\eta \mapsto \widetilde{\eta}$, *while* $\varphi_n^{r,k}(\eta)$ *is anti-symmetric. In terms of the local parameter* v, *we have*

(i) $E^{r,k}$ *is an even holomorphic function in* v, *or a holomorphic function in* u;

(ii) $\varphi_{-1}^{r,k}$ *is an odd holomorphic function in* v. *For* $n \geq 0$, $\varphi_n^{r,k}$ *is an odd meromorphic function in* v, *which has at most* $(2n+1)$-*th order pole at* $v = 0$, *and no other poles near* $v = 0$;

(iii) *For* $n \geq -1$, $h_n^{r,k}$ *is an even holomorphic function in* v, *or a holomorphic function in* u.

Proof. (i). By definition it is clear that $Y^k(\eta)$ and $h_n^{r,k}(\eta)$ are symmetric, and $\varphi_n^{r,k}$ is anti-symmetric, under the involution. Note that $E^{r,k}(\eta)$ has a local expression

$$E^{r,k}(\eta) = k \frac{\psi(-v) e^{\frac{k}{r} \psi(-v)} + \psi(v) e^{\frac{k}{r} \psi(v)}}{\psi(v)\psi(-v) \left(e^{\frac{k}{r} \psi(v)} + e^{\frac{k}{r} \psi(-v)} \right)} = k \frac{(\frac{k}{r} - \frac{1}{3}) + O(v)}{1 + O(v)}.$$

From the first equality we know that $E^{r,k}$ is a function in v, and symmetric under the involution. The second equality is due to the expansion (7.14) of $\psi(v)$, which indicates that $E^{r,k}(v)$ is holomorphic near $v = 0$. Thus near $v = 0$, $E^{r,k}$ expands into a power series containing only even powers of v. Hence $E^{r,k}$ is a power series in $u = \frac{1}{2} v^2$.

We now prove (ii) and (iii) by induction. (ii). For $n = -1$, when $\eta \in U_j$, we have

$$
\begin{aligned}
\varphi_{-1}^{r,k} &= \frac{1}{kr^{k/r}} \frac{\eta^k - \eta^k e^{\frac{k}{r}(\Delta - \widetilde{\Delta})}}{\eta^k + \eta^k\, e^{\frac{k}{r}(\Delta - \widetilde{\Delta})}} \\
(7.19) \qquad &= \frac{1}{kr^{k/r}} \tanh\left[\frac{k\,(\psi(-v) - \psi(v))}{2r} \right] = -\frac{1}{r^{1+k/r}} v + O(v^3) \quad \text{for} \quad k > 0, \\
\varphi_{-1}^{r,0} &= \frac{\psi(-v) - \psi(v)}{2r} = -\frac{v}{r} + O(v^3).
\end{aligned}
$$

These are odd holomorphic functions near $v = 0$. By induction, for $n \geq -1$, if $\varphi_n^{r,k}$ is an odd function in v, then we have:

$$2\, Y^k(\eta)\, \varphi_{n+1}^{r,k}(\eta) = \xi_{n+1}^{r,k}(\eta) - \xi_{n+1}^{r,k}(\widetilde{\eta}) = \frac{\eta}{1 - \eta^r} \frac{d}{d\eta} \left(\xi_n^{r,k}(\eta) - \xi_n^{r,k}(\widetilde{\eta}) \right)$$

$$= \frac{\eta}{1-\eta^r} \frac{d}{d\eta} \left(2\,Y^k(\eta)\, \varphi_n^{r,k}(v) \right) = 2\,Y^k(\eta) \left(E^{r,k}(u) - \frac{r}{v}\frac{d}{dv} \right) \varphi_n^{r,k}(v).$$

Here we used (7.16) and

(7.20) $$\frac{\eta}{1-\eta^r}\frac{d}{d\eta} = \frac{\widetilde{\eta}}{1-\widetilde{\eta}^r}\frac{d}{d\widetilde{\eta}} = -\frac{r}{v}\frac{d}{dv} = -r\frac{d}{du}.$$

Therefore we obtain a recursion formula

(7.21) $$\varphi_{n+1}^{r,k} = \left(E^{r,k}(u) - \frac{r}{v}\frac{d}{dv} \right) \varphi_n^{r,k}(v),$$

which proves that $\varphi_n^{r,k}$ for $n \geq 0$ are odd meromorphic functions of v, with poles of order at most $2n+1$ at $v = 0$ and no other poles near $v = 0$.

(iii). Note that $h_{-1}^{r,k} = \frac{1}{kr^{k/r}}$ is an even holomorphic function in v. For $n \geq -1$, suppose that $h_n^{r,k}$ is a function in u. Then the recursion

(7.22) $$h_{n+1}^{r,k} = \left(E^{r,k}(u) - r\frac{d}{du} \right) h_n^{r,k}(u)$$

shows that $h_{n+1}^{r,k}$ is again an even holomorphic function in v. This completes the proof. $\square$

Remark 7.5. Around each critical point α_j, we have

(7.23) $$\xi_n^{r,k}(\eta) = Y^k(\eta) \left(\varphi_n^{r,k}(v) + h_n^{r,k}(u) \right), \qquad \xi_n^{r,k}(\widetilde{\eta}) = Y^k(\eta) \left(-\varphi_n^{r,k}(v) + h_n^{r,k}(u) \right).$$

For $k = 0$, we have

(7.24) $$\varphi_{-1}^{r,0}(v) = \frac{\eta^r - \widetilde{\eta}^r}{2r} = \frac{\widetilde{\Delta} - \Delta}{2r},$$

(7.25) $$\Delta = 1 - r\varphi_{-1}^{r,0}(v) - rh_{-1}^{r,0}(u), \quad \widetilde{\Delta} = 1 + r\varphi_{-1}^{r,0}(v) - rh_{-1}^{r,0}(u),$$

and

$$\varphi_{n+1}^{r,0}(v) = -\frac{r}{v}\frac{d}{dv}\varphi_n^{r,0}(v), \qquad h_{n+1}^{r,0}(u) = -r\frac{d}{du}h_n^{r,0}(u).$$

7.5. **The local Galois averaging.** The shape of the Eynard-Orantin integral recursion (1.17), together with the integration kernel given by (7.7) and the local Galois conjugation of (7.13), suggests that the residue evaluation of the right-hand-side of (1.17) is equivalent to the local Galois averaging with respect to the single variable z. Since we already have a topological recursion in the form of the partial differential equation (1.11), it is natural to expect that the local Galois averaging of (1.11) should produce (1.17). In this subsection we apply the following three algebraic operations to the differential equation (1.11).

(1) Local Galois averaging with respect to the first variable η_1. This means that for a meromorphic function $f(\eta_1)$ defined on U_j, we apply

$$\pi_* : f(\eta_1) \mapsto \frac{f(\eta_1) + f(\widetilde{\eta}_1)}{2}.$$

(2) Extract the part of the function that is symmetric with respect to the local Galois conjugation (7.13). In this process we use Proposition 7.4 and (7.25).

(3) To make the matching of our formula with the integral recursion manifest, we then multiply by the factor

$$\frac{v_1\, dv_1}{r\varphi_{-1}^{r,0}(v_1)}.$$

Because of (7.14) and (7.24), $\varphi_{-1}^{r,0}(v_1) = O(v_1)$ near $v_1 = 0$. Therefore, the multiplication factor $\frac{v_1}{r\varphi_{-1}^{r,0}(v_1)}$ is a holomorphic function around each critical point α_j.

Our starting point is the following Laplace transform formula (7.6) of the cut-and-join equation (2.4), written in terms of the Hodge integrals (7.3) and the auxiliary functions (7.1) incorporating the ELSV-type formula (2.3) of [29].

$$(7.26) \qquad \sum_{\substack{|\vec{k}|\equiv 0 \\ |\vec{\ell}|\leq 3g-3+n}} r^{\frac{|\vec{k}|}{r}} \langle \tau_{\vec{\ell}}\Lambda \rangle^{r,\vec{k}} \left[(2g-2+n)\, \xi_{\vec{\ell}}^{r,\vec{k}}(\eta_{[n]}) + \frac{1}{r}\sum_{i=1}^{n}(1-\eta_i^r)\,\xi_{\ell_i+1}^{r,k_i}(\eta_i)\,\xi_{\ell_{[\hat{i}]}}^{r,k_{[\hat{i}]}}(\eta_{[\hat{i}]}) \right]$$

$$= \sum_{i<j} \sum_{\substack{a+\left|k_{[\hat{i},\hat{j}]}\right|\equiv 0 \\ m+\left|\ell_{[\hat{i},\hat{j}]}\right|\leq 3g-4+n}} r^{\frac{a+\left|k_{[\hat{i},\hat{j}]}\right|}{r}} \langle \tau_m\,\tau_{\ell_{[\hat{i},\hat{j}]}}\Lambda\rangle^{(r),(a,k_{[\hat{i},\hat{j}]})}\, \frac{1}{\eta_i - \eta_j}$$

$$\times \left[\frac{\eta_j\,\xi_{m+1}^{r,a}(\eta_i)}{1-\eta_i^r} - \frac{\eta_i\,\xi_{m+1}^{r,a}(\eta_j)}{1-\eta_j^r} \right] \xi_{\ell_{[\hat{i},\hat{j}]}}^{r,k_{[\hat{i},\hat{j}]}}(\eta_{[\hat{i},\hat{j}]})$$

$$+ \frac{r}{2}\sum_{i=1}^{n} \sum_{\substack{a+b+\left|k_{[\hat{i}]}\right|\equiv 0 \\ m+\ell+\left|\ell_{[\hat{i}]}\right|\leq 3g-5+n}} r^{\frac{a+b+\left|k_{[\hat{i}]}\right|}{r}} \langle \tau_m\,\tau_\ell\,\tau_{\ell_{[\hat{i}]}}\Lambda\rangle^{(r),(a,b,k_{[\hat{i}]})}\,\xi_{m+1}^{r,a}(\eta_i)\,\xi_{\ell+1}^{r,b}(\eta_i)\,\xi_{\ell_{[\hat{i}]}}^{r,k_{[\hat{i}]}}(\eta_{[\hat{i}]})$$

$$+ \frac{r}{2}\sum_{i=1}^{n} \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{i}]}}^{\text{stable}} \sum_{\substack{a+|k_I|\equiv 0 \\ b+|k_J|\equiv 0 \\ m+|\ell_I|\leq 3g_1-2+|I| \\ \ell+|\ell_J|\leq 3g_2-2+|J|}} r^{\frac{a+b+\left|k_{[\hat{i}]}\right|}{r}} \langle \tau_m\,\tau_{\ell_I}\Lambda\rangle^{r,(a,k_I)}\langle \tau_\ell\,\tau_{\ell_J}\Lambda\rangle^{r,(b,k_J)}$$

$$\times \xi_{m+1}^{r,a}(\eta_i)\,\xi_{\ell+1}^{r,b}(\eta_i)\,\xi_{\ell_{[\hat{i}]}}^{r,k_{[\hat{i}]}}(\eta_{[\hat{i}]}).$$

Here the bound of the summation indices are $0 \leq a,b < r$ and $m,\ell \geq 0$. Let us now apply the three algebraic operations listed above to (7.26).

The left-hand-side of (7.26) produces

$$(7.27) \qquad \sum_{\substack{|\vec{k}|\equiv 0 \\ |\vec{\ell}|\leq 3g-3+n}} r^{\frac{|\vec{k}|}{r}} \langle \tau_{\vec{\ell}}\Lambda\rangle^{(r),\vec{k}}\, Y^{k_1}(\eta_1) \left[-\frac{v_1}{r}\varphi_{\ell_1+1}^{r,k_1}(v_1)\,dv_1 + \mathcal{H}(v_1)\,dv_1 \right] \xi_{\ell_{[\hat{1}]}}^{r,k_{[\hat{1}]}}(\eta_{[\hat{1}]}),$$

where $\mathcal{H}(v_1)$ is a holomorphic function in v_1 near $v_1 = 0$ that comes from the third operation. We calculate, using, (7.21) and (7.23):

$$-\frac{v_1}{r}Y^{k_1}(\eta_1)\,\varphi_{\ell_1+1}^{r,k_1}(v_1)\,dv_1 = -\frac{v_1}{r}Y^{k_1}(\eta_1)\left[E^{r,k_1}(u_1) - \frac{r}{v_1}\frac{d}{dv_1} \right]\varphi_{\ell_1}^{r,k_1}(v_1)\,dv_1$$

$$= Y^{k_1}(\eta_1)\,d\varphi_{\ell_1}^{r,k_1}(v_1) - \frac{v_1}{r}Y^{k_1}(\eta_1)\,E^{r,k_1}(u_1)\,\varphi_{\ell_1}^{r,k_1}(v_1)\,dv_1$$

$$= d\xi_{\ell_1}^{r,k_1}(\eta_1) - d\left(Y^{k_1}(\eta_1)h_{\ell_1}^{r,k_1}(u_1) \right)$$

$$- \left(dY^{k_1}(\eta_1) \right) \varphi_{\ell_1}^{r,k_1}(v_1) - \frac{1}{r} Y^{k_1}(\eta_1) E^{r,k_1}(u_1) \varphi_{\ell_1}^{r,k_1}(v_1) \, v_1 \, dv_1.$$

The last two terms of the above formula cancel due to (7.16) and

$$(7.28) \qquad\qquad du = v \, dv = r \frac{\eta^r - 1}{\eta} d\eta = r \frac{\widetilde{\eta}^{\,r} - 1}{\widetilde{\eta}} d\widetilde{\eta}.$$

Notice that $Y^{k_1}(\eta_1)$, $\mathcal{H}(v_1)$, and $h_{\ell_1}^{r,k_1}(u_1)$ are holomorphic functions in $\eta_1 \in U$, where U is the union $\cup_{j=1}^r U_j$. Thus the left-hand side of (7.26) simply takes the form

$$(7.29) \qquad \sum_{\substack{|\vec{k}| \equiv 0 \\ |\vec{\ell}| \leq 3g-3+n}} r^{\frac{|\vec{k}|}{r}} \langle \tau_{\vec{\ell}} \Lambda \rangle^{(r),\vec{k}} \left[d\xi_{\ell_1}^{r,k_1}(\eta_1) + \mathcal{H}_1(\eta_1) \, d\eta_1 \right] \xi_{\ell_{[\hat{1}]}}^{r,k_{[\hat{1}]}}(\eta_{[\hat{1}]}),$$

with a holomorphic function $\mathcal{H}_1$ in η_1.

Again appealing to (7.28), we calculate the result of the three operations on the first term of the right-hand side of (7.26) as

$$(7.30) \quad \frac{1}{2} \sum_{1 < j} \sum_{\substack{a + |k_{[\hat{1},\hat{j}]}| \equiv 0 \\ m + |\ell_{[\hat{1},\hat{j}]}| \leq 3g-4+n}} r^{\frac{a + |k_{[\hat{1},\hat{j}]}|}{r}} \langle \tau_m \, \tau_{\ell_{[\hat{1},\hat{j}]}} \Lambda \rangle^{(r),(a,k_{[\hat{1},\hat{j}]})}$$

$$\times \left[\frac{-\xi_{m+1}^{r,a}(\eta_1) \, d\eta_1}{(\eta_1 - \eta_j) \, \varphi_{-1}^{r,0}(v_1)} + \frac{-\xi_{m+1}^{r,a}(\widetilde{\eta}_1) \, d\widetilde{\eta}_1}{(\widetilde{\eta}_1 - \eta_j) \, \varphi_{-1}^{r,0}(v_1)} + \Omega_{1,m+1}^{r,a}(\eta_1) + \Omega(\eta_1,\eta_j) \frac{\xi_{m+1}^{r,a}(\eta_j)}{1 - \eta_j^r} \right]$$

$$\times \xi_{\ell_{[\hat{1},\hat{j}]}}^{r,k_{[\hat{1},\hat{j}]}}(\eta_{[\hat{1},\hat{j}]}) + \mathcal{H}_2(\eta_1) \, d\eta_1 \, \mathcal{F}(\eta_{[\hat{1}]}).$$

Here $\mathcal{H}_2(\eta_1) \in \mathcal{O}(U)$. $\mathcal{F}(\eta_{[\hat{1}]})$ is a function in $\eta_2, \ldots, \eta_n$. $\Omega_{1,m+1}^{r,a}(\eta_1)$ is a meromorphic differential in η_1 defined by

$$\Omega_{1,m+1}^{r,a}(\eta_1) := \frac{\xi_{m+1}^{r,a}(\eta_1) \, d\eta_1}{\eta_1 \, \varphi_{-1}^{r,0}(v_1)} + \frac{\xi_{m+1}^{r,a}(\widetilde{\eta}_1) \, d\widetilde{\eta}_1}{\widetilde{\eta}_1 \, \varphi_{-1}^{r,0}(v_1)},$$

and $\Omega(\eta_1, \eta_j)$ is a holomorphic differential in η_1

$$\Omega(\eta_1, \eta_j) = \left[\frac{(1 - \eta_1^r) \, d\eta_1}{(\eta_1 - \eta_j) \, \varphi_{-1}^{r,0}(v_1)} + \frac{(1 - \widetilde{\eta}_1^r) \, d\widetilde{\eta}_1}{(\widetilde{\eta}_1 - \eta_j) \, \varphi_{-1}^{r,0}(v_1)} \right] \frac{\xi_{m+1}^{r,a}(\eta_j)}{1 - \eta_j^r}.$$

Without loss of generality, we can assume that $\eta_j \notin U$. Then $\Omega(\eta_1, \eta_j)$ is a holomorphic differential with respect to $\eta_1 \in U$, which follows from the local behavior of $\frac{(1-\eta_1^r)}{\varphi_{-1}^{r,0}(v_1)}$ and $\frac{(1-\widetilde{\eta}_1^r)}{\varphi_{-1}^{r,0}(v_1)}$ coming from (7.12) and (7.24).

The operation on the second and the third terms of the right-hand side of (7.26) produces

$$(7.31) \qquad \sum_{\substack{a + b + |k_{[\hat{1}]}| \equiv 0 \\ m + \ell + |\ell_{[\hat{1}]}| \leq 3g-5+n}} r^{\frac{a+b+|k_{[\hat{1}]}|}{r}} \langle \tau_m \, \tau_\ell \, \tau_{\ell_{[\hat{1}]}} \Lambda \rangle^{(r),(a,b,k_{[\hat{1}]})}$$

$$\times \left[\frac{Y^a(\eta_1) \, Y^b(\eta_1) \, \varphi_{m+1}^{r,a}(v_1) \, \varphi_{\ell+1}^{r,b}(v_1) \, v_1 \, dv_1}{2\varphi_{-1}^{r,0}(v_1)} \right] \xi_{\ell_{[\hat{1}]}}^{r,k_{[\hat{1}]}}(\eta_{[\hat{1}]}) + \mathcal{H}_3(\eta_1) \, d\eta_1 \, \mathcal{F}_1(\eta_{[\hat{1}]})$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J=[\hat{1}]}} \sum_{\substack{a+|k_I|\equiv 0 \\ b+|k_J|\equiv 0 \\ m+|\ell_I|\leq 3g_1-2+|I| \\ \ell+|\ell_J|\leq 3g_2-2+|J|}} r^{\frac{a+b+|k_{[\hat{1}]}|}{r}} \left\langle \tau_m \, \tau_{\ell_I} \, \Lambda \right\rangle^{(r),(a,k_I)} \left\langle \tau_\ell \, \tau_{\ell_J} \, \Lambda \right\rangle^{(r),(a,k_J)}$$

$$\times \left[\frac{Y^a(\eta_1) \, Y^b(\eta_1) \, \varphi_{m+1}^{r,a}(v_1) \, \varphi_{\ell+1}^{r,b}(v_1) \, v_1 \, dv_1}{2\varphi_{-1}^{r,0}(v_1)} \right] \xi_{\ell_{[\hat{1}]}}^{r,k_{[\hat{1}]}}(\eta_{[\hat{1}]}) + \mathcal{H}_4(\eta_1) \, d\eta_1 \, \mathcal{F}_2(\eta_{[\hat{1}]}).$$

Here $\mathcal{H}_3(\eta_1)$ and $\mathcal{H}_4(\eta_1)$ are in $\mathcal{O}(U)$, and $\mathcal{F}_1(\eta_{[\hat{1}]})$ and $\mathcal{F}_2(\eta_{[\hat{1}]})$ are functions in $\eta_2, \ldots, \eta_n$.

7.6. **The residue calculation.** Recall that the central idea of [21] to prove the Hurwitz number conjecture of [8] is to relate the *principal part* of the free energies with the residue calculation of the Eynard-Orantin integral recursion formula. Since the free energies in our case have r distinct poles, we need a more general notion of the principal part for a meromorphic function with many poles (see for example, [38]). In this subsection we derive the key formula (7.34) for the residue calculations we need.

Definition 7.6. Let us denote by $U = \cup_{j=1}^r U_j$ the union of the local neighborhood of the critical point α_j for all j. We define an $\mathcal{O}(U)$-module $\mathcal{M}_U$ by

$$\mathcal{M}_U = \left\{ m(\eta) \mid m(\eta) = \frac{h(\eta)}{(1-\eta^r)^k}, k \geq 0, h(\eta) \in \mathcal{O}(U). \right\}$$

Following [38], we define

Definition 7.7. Let $p(\eta) \in \mathbb{C}[\eta]$ be a non-constant polynomial, and $U \subset \mathbb{C}$ an open subset. Two functions $f, g \in \mathcal{O}(U)$ are said to be congruent modulo p (denoted by $f \equiv g \mod p$) if there is $q(\eta) \in \mathcal{O}(U)$ such that

$$f = g + p\,q.$$

Proposition 7.8 ([38]). *Under the same condition as above, suppose that $p(\eta)$ has all its zeros in U. Then for every holomorphic function $f(\eta) \in \mathcal{O}(U)$, there is a unique polynomial $r(\eta)$ such that*

$$f \equiv r \mod p \qquad and \qquad \deg r(\eta) < \deg p(\eta).$$

We denote this unique remainder polynomial by

(7.32) $$r(\eta) = \lfloor f(\eta) \rfloor_p.$$

This defines a natural $\mathbb{C}$-algebra homomorphism

$$\mathcal{O}(U) \ni f \longmapsto \lfloor f(\eta) \rfloor_p \in \mathbb{C}[\eta]/(p),$$

which is called the *reduction* of f modulo p.

Definition 7.9. Let $m(\eta) = h(\eta)/p(\eta)$ be a meromorphic function in $\mathcal{M}_U$, where $h(\eta) \in \mathcal{O}(U)$ and $p(\eta) = (1-\eta^r)^k$, $k \geq 0$. We define the following symbol

(7.33) $$\{m(\eta)\}_\eta := \frac{\lfloor h(\eta) \rfloor_p}{p(\eta)} \in \mathbb{C}(\eta).$$

Thus we have a linear map, which we simply call the projection to the *principal part*

$$\{\,\cdot\,\}_\eta : \mathcal{M}_U \longrightarrow \mathbb{C}(\eta).$$

The principal part $\{m(\eta)\}_\eta$ of a meromorphic function $m(\eta)$ is the "proper rational function part" of $m = h/p$. If $k = 0$, then we define the principal part to be 0.

Remark 7.10. From the definition it is obvious that for every $m(\eta) \in \mathcal{M}_U$, we have

$$m(\eta) - \{m(\eta)\}_\eta \in \mathcal{O}(U).$$

Thus $\{m(\eta)\}_\eta$ behaves much like the principal part of a meromorphic function at a pole. The image $\{m(\eta)\}_\eta \in \mathbb{C}(\eta)$ is always globally defined on $\mathbb{P}^1$, even though $m(\eta)$ is defined locally on U, and $\{m(\eta)\}_\eta$ has poles only at α_j.

The following lemma plays the key role in connecting the residue of the Eynard-Orantin recursion formula and taking the principal part.

Lemma 7.11. *For any element $m(\zeta) \in \mathcal{M}_U$ and $\eta \in \mathbb{C}$ such that $\eta \neq \alpha_j$, $j = 1, \ldots, r$, we have*

$$(7.34) \qquad \sum_{j=1}^r \mathrm{Res}_{\zeta=\alpha_j} \frac{m(\zeta)}{\zeta - \eta} = - \{m(\eta)\}_\eta.$$

Proof. Let γ_j be a small loop in U_j centered at α_j, γ_η a small loop around η, and Γ_R a large circle enclosing all α_j and η with radius $R \gg 1$ (see Figure 7.1). Then

$$\sum_{j=1}^r \mathrm{Res}_{\zeta=\alpha_j} \frac{m(\zeta)}{\zeta - \eta} = \sum_j \frac{1}{2\pi i} \oint_{\gamma_j} \frac{m(\zeta)}{\zeta - \eta} \, d\zeta$$

$$= \sum_j \frac{1}{2\pi i} \oint_{\gamma_j} \frac{\{m(\zeta)\}_\zeta}{\zeta - \eta} \, d\zeta + \sum_j \frac{1}{2\pi i} \oint_{\gamma_j} \frac{m(\zeta) - \{m(\zeta)\}_\zeta}{\zeta - \eta} \, d\zeta$$

$$= \sum_j \mathrm{Res}_{\zeta=\alpha_j} \frac{\{m(\zeta)\}_\zeta}{\zeta - \eta},$$

because $\frac{m(\zeta) - \{m(\zeta)\}_\zeta}{\zeta - \eta}$ does not have any pole in any of the U_j's. Noting that $\frac{\{m(\zeta)\}_\zeta}{\zeta - \eta}$ is a rational function with poles only at η and α_j, $j = 1, \ldots, r$, we calculate

$$0 = \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_R} \frac{\{m(\zeta)\}_\zeta}{\zeta - \eta} \, d\zeta$$

$$= \sum_j \mathrm{Res}_{\zeta=\alpha_j} \frac{\{m(\zeta)\}_\zeta}{\zeta - \eta} + \mathrm{Res}_{\zeta=\eta} \frac{\{m(\zeta)\}_\zeta}{\zeta - \eta}$$

$$= \sum_j \mathrm{Res}_{\zeta=\alpha_j} \frac{\{m(\zeta)\}_\zeta}{\zeta - \eta} + \{m(\eta)\}_\eta.$$

This completes the proof of (7.34). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.7. Proof of Theorem 1.7. We are now ready to complete the proof of Theorem 1.7. The operation we wish to apply to (7.29), (7.30), and (7.31) is

$$(d_{\eta_2} \cdots d_{\eta_n}) \circ \{\bullet\}_{\eta_1}.$$

This means we first calculate the principal part of the quantities with respect to η_1, and then apply the exterior differentiations with respect to $\eta_2, \ldots, \eta_n$. We obtain

$$(7.35) \qquad \sum_{\substack{|\vec{k}| \equiv 0 \\ |\vec{\ell}| \leq 3g-3+n}} r^{\frac{|\vec{k}|}{r}} \langle \tau_{\vec{\ell}} \Lambda \rangle^{(r),\vec{k}} \, d\xi_{\vec{\ell}}^{r,\vec{k}}(\eta_1, \ldots, \eta_n)$$
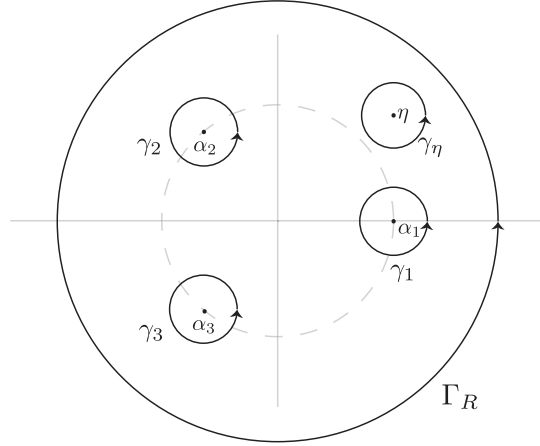
FIGURE 7.1. Integration contours.

$$= \sum_{1<j} \sum_{\substack{a+\left|k_{[\hat{1},\hat{j}]}\right|\equiv 0 \\ m+\left|\ell_{[\hat{1},\hat{j}]}\right|\leq 3g-4+n}} r^{\frac{a+\left|k_{[\hat{1},\hat{j}]}\right|}{r}} \left\langle \tau_m \, \tau_{\ell_{[\hat{1},\hat{j}]}} \, \Lambda \right\rangle^{(r),(a,k_{[\hat{1},\hat{j}]})} R_m^{r,a}(\eta_1,\eta_j) \otimes d\xi_{\ell_{[\hat{1},\hat{j}]}}^{r,k_{[\hat{1},\hat{j}]}}(\eta_{[\hat{1},\hat{j}]})$$

$$+ \sum_{\substack{a+b+\left|k_{[\hat{1}]}\right|\equiv 0 \\ m+\ell+\left|\ell_{[\hat{1}]}\right|\leq 3g-5+n}} r^{\frac{a+b+\left|k_{[\hat{1}]}\right|}{r}} \left\langle \tau_m \, \tau_\ell \, \tau_{\ell_{[\hat{1}]}} \, \Lambda \right\rangle^{(r),(a,b,k_{[\hat{1}]})} R_{m,\ell}^{r,a,b}(\eta_1) \otimes d\xi_{\ell_{[\hat{1}]}}^{r,k_{[\hat{1}]}}(\eta_{[\hat{1}]})$$

$$+ \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{1}]}}^{\text{stable}} \sum_{\substack{a+|k_I|\equiv 0 \\ b+|k_J|\equiv 0 \\ m+|\ell_I|\leq 3g_1-2+|I| \\ \ell+|\ell_J|\leq 3g_2-2+|J|}} r^{\frac{a+b+\left|k_{[\hat{1}]}\right|}{r}} \left\langle \tau_m \, \tau_{\ell_I} \, \Lambda \right\rangle^{(r),(a,k_I)} \left\langle \tau_\ell \, \tau_{\ell_J} \, \Lambda \right\rangle^{(r),(a,k_J)}$$

$$\times R_{m,\ell}^{r,a,b}(\eta_1) \otimes d\xi_{\ell_{[\hat{1}]}}^{r,k_{[\hat{1}]}}(\eta_{[\hat{1}]}),$$

where $d\xi_{\ell_I}^{r,k_I}(\eta_I) = \bigotimes_{i\in I} d\xi_{\ell_i}^{r,k_i}(\eta_i)$,

$$(7.36) \qquad R_{m,\ell}^{r,a,b}(\eta_1) := \left\{ \frac{Y^a(\eta_1)\, Y^b(\eta_1)\, \varphi_{m+1}^{r,a}(v_1)\, \varphi_{\ell+1}^{r,b}(v_1)\, v_1 \, dv_1}{2\varphi_{-1}^{r,0}(v_1)} \right\}_{\eta_1},$$

and

$$(7.37) \qquad R_m^{r,a}(\eta_1,\eta_j) := -d_{\eta_j} \left\{ \frac{\xi_{m+1}^{r,a}(\eta_1)\, d\eta_1}{2\,(\eta_1-\eta_j)\, \varphi_{-1}^{r,0}(v_1)} + \frac{\xi_{m+1}^{r,a}(\widetilde{\eta}_1)\, d\widetilde{\eta}_1}{2\,(\widetilde{\eta}_1-\eta_j)\, \varphi_{-1}^{r,0}(v_1)} \right\}_{\eta_1}.$$

Remark 7.12. The operation of $\{\bullet\}_{\eta_1}$ in (7.37) and (7.36) are well defined because of the fact that $\widetilde{\eta}_1$ is holomorphic in $\eta_1 \in U$, (7.12), Proposition 7.4, and (7.23).

To deduce (1.17) from (7.35), we need the following formulas.

Proposition 7.13.

$$(7.38) \qquad R_{m,\ell}^{r,a,b}(\eta_1) = r \sum_{j=1}^{r} \operatorname{Res}_{\eta=\alpha_j} K_j(\eta,\eta_1) \, d\xi_m^{r,a}(\eta) \otimes d\xi_\ell^{r,b}(\widetilde{\eta})$$

$$(7.39) \qquad R_m^{r,a}(\eta_1,\eta_i) = \sum_{j=1}^{r} \operatorname{Res}_{\eta=\alpha_j} \left[K_j(\eta,\eta_1) \left(B(\eta,\eta_i) \, d\xi_m^{r,a}(\widetilde{\eta}) + B(\widetilde{\eta},\eta_i) \, d\xi_m^{r,a}(\eta) \right) \right],$$

where $B(\eta,\eta_i)$ is defined in (7.8).

Proof. Let $s_j(\gamma_j)$ be the involution image of a small circle γ_j around α_j. We calculate the residue by contour integration.

$$\text{R.H.S. of } (7.38) = r \sum_{j=1}^{r} \operatorname{Res}_{\eta=\alpha_j} K_j(\eta,\eta_1) \, d\xi_m^{r,a}(\eta) \otimes d\xi_\ell^{r,b}(\widetilde{\eta})$$

$$= \frac{r^2 \, d\eta_1}{2} \sum_{j} \frac{1}{2\pi i} \left[\oint_{\gamma_j} \frac{\xi_{m+1}^{r,a}(y)}{(y-\eta_1)(\widetilde{y}^r - y^r)} \frac{d\xi_\ell^{r,b}(\widetilde{y})}{d\widetilde{y}} \, d\widetilde{y} + \oint_{s_j(\gamma_j)} \frac{\xi_{m+1}^{r,a}(\widetilde{y})}{(y-\eta_1)(\widetilde{y}^r - y^r)} \frac{d\xi_\ell^{r,b}(y)}{dy} \, dy \right]$$

$$= \frac{r^2 \, d\eta_1}{2} \sum_{\alpha_j} \operatorname{Res}_{y=\alpha_j} \frac{(1-y^r) \left[\xi_{m+1}^{r,a}(y) \, \xi_{\ell+1}^{r,b}(\widetilde{y}) + \xi_{m+1}^{r,a}(\widetilde{y}) \, \xi_{\ell+1}^{r,b}(y) \right]}{(y-\eta_1) \, y \, (\widetilde{y}^r - y^r)}.$$

In the second line we have used the involution to the second contour integral, and in the third line we have appealed to (7.28). Noticing that

$$\frac{(1-y^r) \left[\xi_{m+1}^{r,a}(y) \, \xi_{\ell+1}^{r,b}(\widetilde{y}) + \xi_{m+1}^{r,a}(\widetilde{y}) \, \xi_{\ell+1}^{r,b}(y) \right]}{y \, (\widetilde{y}^r - y^r)} \in \mathcal{M}_U,$$

we use Lemma 7.11, (7.23), and (7.24) to yield

$$\text{R.H.S.} = -\frac{r^2}{2} \left\{ \frac{(1-\eta_1^r) \left[\xi_{m+1}^{r,a}(\eta_1) \, \xi_{\ell+1}^{r,b}(\widetilde{\eta}_1) + \xi_{m+1}^{r,a}(\widetilde{\eta}_1) \, \xi_{\ell+1}^{r,b}(\eta_1) \right]}{\eta_1 \, (\widetilde{\eta}_1^r - \eta_1^r)} \right\}_{\eta_1} d\eta_1$$

$$= \left\{ \frac{Y^a(\eta_1) \, Y^b(\eta_1) \left[\varphi_{m+1}^{r,a}(v_1) \, \varphi_{\ell+1}^{r,b}(v_1) - h_{m+1}^{r,a}(w_1) \, h_{\ell+1}^{r,b}(w_1) \right] v_1 \, dv_1}{2\varphi_{-1}^{r,0}(v_1)} \right\}_{\eta_1}$$

$$= R_{m,\ell}^{r,a,b}(\eta_1).$$

Similarly, for $\eta_i \notin U$, we have

$$\text{R.H.S of } (7.39) = \sum_{j=1}^{r} \operatorname{Res}_{\eta=\alpha_j} \left[K_j(\eta,\eta_1) \left(B(\eta,\eta_i) \, d\xi_m^{r,a}(\widetilde{\eta}) + B(\widetilde{\eta},\eta_i) \, d\xi_m^{r,a}(\eta) \right) \right]$$

$$= \frac{r}{2} \, d\eta_1 \otimes d\eta_i \left[\sum_{\alpha_j} \frac{1}{2\pi i} \oint_{\gamma_j} \left(\frac{1}{y-\eta_1} - \frac{1}{\widetilde{y}-\eta_1} \right) \left(\frac{\xi_{m+1}^{r,a}(\widetilde{y})}{y-\eta_i} + \frac{\xi_{m+1}^{r,a}(y) \, \widetilde{y}'}{\widetilde{y}-\eta_i} \right) \frac{dy}{\widetilde{y}^r - y^r} \right]$$

$$= r \, d\eta_1 \otimes d\eta_i \left[\sum_{\alpha_j} \operatorname{Res}_{y=\alpha_j} \frac{1}{y-\eta_1} \left(\frac{\xi_{m+1}^{r,a}(\widetilde{y})}{(y-\eta_i)(\widetilde{y}^r - y^r)} + \frac{\xi_{m+1}^{r,a}(y) \, \widetilde{y}'}{(\widetilde{y}-\eta_i)(\widetilde{y}^r - y^r)} \right) \right]$$

$$= -r\, d\eta_1 \otimes d_{\eta_i} \left\{ \frac{\xi_{m+1}^{r,a}(\widetilde{\eta}_1)}{(\eta_1 - \eta_i)(\widetilde{\eta}_1^{\,r} - \eta_1^r)} + \frac{\xi_{m+1}^{r,a}(\eta_1)\,\widetilde{\eta}_1'}{(\widetilde{\eta}_1 - \eta_i)(\widetilde{\eta}_1^{\,r} - \eta_1^r)} \right\}_{\eta_1}$$

$$= d\eta_1 \otimes d_{\eta_i} \left\{ \frac{\xi_{m+1}^{r,a}(\widetilde{\eta}_1)}{2\,(\eta_1 - \eta_i)\,\varphi_{-1}^{r,0}(v_1)} + \frac{\xi_{m+1}^{r,a}(\eta_1)\,\widetilde{\eta}_1'}{2\,(\widetilde{\eta}_1 - \eta_i)\,\varphi_{-1}^{r,0}(v_1)} \right\}_{\eta_1},$$

thanks to Lemma 7.11. Here the sign $'$ indicates differentiation with respect to the variable without the $\sim$-sign. For example, $\widetilde{y}' = d\widetilde{y}/dy$, etc. The last step is to equate the above result with (7.37), which follows from

Lemma 7.14.

$$\left\{ \left(\frac{1}{\eta_1 - \eta_i} + \frac{\widetilde{\eta}_1'}{\widetilde{\eta}_1 - \eta_i} \right) \left(\frac{\xi_{m+1}^{r,a}(\eta_1) + \xi_{m+1}^{r,a}(\widetilde{\eta}_1)}{2\varphi_{-1}^{r,0}(v_1)} \right) \right\}_{\eta_1} = 0.$$

Proof of Lemma. Note that

$$\frac{\xi_{m+1}^{r,a}(\eta_1) + \xi_{m+1}^{r,a}(\widetilde{\eta}_1)}{2\varphi_{-1}^{r,0}(v_1)} = -\frac{r\, Y^a(\eta_1)\, h_{m+1}^{r,a}(0)}{\Delta_1} + \mathcal{O}(\Delta_1),$$

which has a simple pole at each α_j. Since $\widetilde{\eta}_1'\big|_{\eta_1 = \alpha_i} = -1$, the holomorphic function $\frac{1}{\eta_1 - \eta_i} + \frac{\widetilde{\eta}_1'}{\widetilde{\eta}_1 - \eta_i}$ has a zero at each α_j. Therefore, the principal part operation is applied to a holomorphic function in η_1, hence the result is 0. $\qquad\square$

We have now completed the proof of Theorem 1.7. $\qquad\square$

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