GEOMETRY OF CHARACTER VARIETIES OF SURFACE GROUPS

MOTOHICO MULASE*

Abstract. This article is based on a talk delivered at the RIMS–OCAMI Joint International Conference on *Geometry Related to Integrable Systems* in September, 2007. Its aim is to review a recent progress in the Hitchin integrable systems and character varieties of the fundamental groups of Riemann surfaces. A survey on geometric aspects of these character varieties is also provided as we develop the exposition from a simple case to more elaborate cases.

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1. Introduction

The character varieties we consider in this article are the set of equivalence classes

$$\mathrm{Hom}(\pi_1(\Sigma_g), G)/G$$

of representations of a surface group $\pi_1(\Sigma_g)$ into another group G. Here Σ_g is a closed oriented surface of genus g, which is assumed to be $g \geq 2$ most of the time. The action of G on the space of homomorphisms is through the conjugation action. Since this action has fixed points, the quotient requires a special treatment to make it a reasonable space. Despite the simple appearance of the space, it has an essential connection to many other subjects in mathematics ([1, 2, 6, 9, 10, 11, 14, 15, 17, 19, 24, 25, 33, 34]), and the list is steadily growing ([4, 7, 12, 13, 18, 23]). Our subject thus provides an ideal window to observe the scenery of a good part of recent developments in mathematics and mathematical physics.

Each section of this article is devoted to a specific type of character varieties and a particular group G. We start with a finite group in Section 2. Already in this case one can appreciate the interplay between the character variety and the theory of irreducible representations of a finite group. In Sections 3 and 4 we consider the case $G = U_n$. We review the discovery of the relation to two-dimensional Yang-Mills theory and symplectic geometry due to Atiyah and Bott [1]. It forms the turning point of the modern developments on character varieties. We then turn our attention to the case $G = GL_n(\mathbb{C})$ in Sections 5 and 6. Here the key ideas we review are due to Hitchin [14, 15]. In these seminal

*Research supported by NSF grant DMS-0406077 and UC Davis.

papers Hitchin has suggested the subject's possible relations to four-dimensional Yang-Mills theory and the Langlands duality. These connections are materialized recently by Hausel and Thaddeus [13], Donagi and Pantev [4], Kapustin and Witten [18], and many others. Section 7 motivates some of these developments from our study [16] on the Hitchin integrable systems.

2. Character varieties of finite groups and representation theory

The simplest example of character varieties occurs when G is a finite group. The "variety" is a finite set, and the only interesting invariant is its cardinality. Here the *reasonable* quotient $\mathrm{Hom}(\pi_1(\Sigma_g), G)/G$ is *not* the orbit space. A good theory exists only for the *virtual* quotient, which takes into account the information of isotropy subgroups, exactly as we do when we consider orbifolds.

Theorem 2.1 (Counting formula)**.** *The classical counting formula gives*

$$\frac{|\mathrm{Hom}(\pi_1(\Sigma_g), G)|}{|G|} = \sum_{\lambda \in \hat{G}} \left(\frac{\dim \lambda}{|G|} \right)^{\chi(\Sigma_g)}, \tag{2.1}$$

where $\hat{G}$ is the set of irreducible representations of G, $\dim \lambda$ is the dimension of the irreducible representation $\lambda \in \hat{G}$, and $\chi(\Sigma_g) = 2 - 2g$ is the Euler characteristic of the surface.

When $g = 0$, the above formula reduces to a well-known formula in representation theory:

$$|G| = \sum_{\lambda \in \hat{G}} (\dim \lambda)^2. \tag{2.2}$$

Remark 1. The formula for $g = 1$ is known to Frobenius [8]. Burnside asks a related question as an exercise of his textbook [3]. In the late 20th century, the formula was rediscovered by Witten [33] using quantum Yang-Mills theory in two dimensions, and by Freed and Quinn [6] using quantum Chern-Simons gauge theory with the finite group G as its gauge group.

Remark 2. Since 't Hooft [31] we know that a matrix integral admits a ribbon graph expansion, using the Feynman diagram technique [5]. In [23] we ask what types of integrals admit a ribbon graph expansion. Our answer is that an integral over a von Neumann algebra admits such an expansion. We find in [22, 23] that when we apply a formula of [23] to the complex group algebra $\mathbb{C}[G]$, the counting formula (2.1) for all values of g automatically follows. The key fact is the algebra decomposition

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in \hat{G}} \mathrm{End}(\lambda). \tag{2.3}$$

The integral over the group algebra then decomposes into the product of matrix integrals over each simple factor $\mathrm{End}(\lambda)$, which we know how to calculate by 't Hooft's method. Although (2.1) looks like a generalization of (2.2), these formulas actually contain the same amount of information because they are direct consequences of the decomposition (2.3).

Remark 3. We also note that there are corresponding formulas for closed *non-orientable* surfaces [22, 23]. Intriguingly, the formula for non-orientable surfaces are studied in its full generality, though without any mention on its geometric significance, in a classical paper by Frobenius and Schur [9]. The Frobenius-Schur theory automatically appears in the generalized matrix integral over the *real* group algebra $\mathbb{R}[G]$ (see [22]).

Of course (2.1) has an elementary proof, without appealing to quantum field theories or matrix integrals. We record it here only assuming a minimal background of representation theory that can be found, for example, in Serre's textbook [28].

The fundamental group of a compact oriented surface of genus g is generated by $2g$ generators with one relator:

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle,$$

where $[a,b] = aba^{-1}b^{-1}$. Since

$$\mathrm{Hom}(\pi_1(\Sigma_g), G) = \{(s_1, t_1, \dots, s_g, t_g) \in G^{2g} \mid [s_1, t_1] \cdots [s_g, t_g] = 1\}, \tag{2.4}$$

the counting problem reduces to evaluating an *integral*

$$\left|\mathrm{Hom}(\pi_1(\Sigma_g), G)\right| = \int_{G^{2g}} \delta([s_1, t_1] \cdots [s_g, t_g]) ds_1 dt_1 \cdots ds_g dt_g. \tag{2.5}$$

Here the left hand side is the volume of the character variety that is defined by an invariant measure ds on the group G. For the case of a finite group, the volume is simply the cardinality, and the integral is the sum over G^{2g}. The δ-function on G is given by the normalized character of the regular representation

$$\delta(x) = \frac{1}{|G|}\chi_{\mathrm{reg}}(x) = \sum_{\lambda \in \hat{G}} \frac{\dim \lambda}{|G|} \cdot \chi_\lambda(x). \tag{2.6}$$

To compute the integral (2.5), let us first identify the complex group algebra

$$\mathbb{C}[G] = \left\{ x = \sum_{\gamma \in G} x(\gamma) \cdot \gamma \;\middle|\; x(\gamma) \in \mathbb{C} \right\}$$

of a finite group G with the vector space $F(G)$ of functions on G. In this way we can reduce the complexity of the commutator produce in (2.4) into simpler pieces. The *convolution product* of two functions $x(\gamma)$ and $y(\gamma)$ is defined by

$$(x * y)(w) \stackrel{\mathrm{def}}{=} \sum_{\gamma \in G} x(w\gamma^{-1}) y(\gamma) \ ,$$

which makes $(F(G), *)$ an algebra isomorphic to the group algebra. In this identification, the set of class functions $CF(G)$ corresponds to the center $Z\mathbb{C}[G]$ of $\mathbb{C}[G]$. According to the decomposition of this algebra into simple factors (2.3), we have an algebra isomorphism

$$Z\mathbb{C}[G] = \bigoplus_{\lambda \in \hat{G}} \mathbb{C} \ ,$$

where each factor $\mathbb{C}$ is the center of $\mathrm{End}\lambda$. The projection to each factor is given by

$$pr_\lambda : Z\mathbb{C}[G] \ni x = \sum_{\gamma \in G} x(\gamma) \cdot \gamma \longmapsto pr_\lambda(x) \stackrel{\mathrm{def}}{=} \frac{1}{\dim \lambda} \sum_{\gamma \in G} x(\gamma) \chi_\lambda(\gamma) \in \mathbb{C} \ ,$$

where χ_λ is the character of $\lambda \in \hat{G}$. Following Serre [28], let

$$p_\lambda \stackrel{\mathrm{def}}{=} \frac{\dim \lambda}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^{-1}) \cdot \gamma \in Z\mathbb{C}[G], \qquad \lambda \in \hat{G}, \tag{2.7}$$

be a linear bases for $Z\mathbb{C}[G]$. It follows from Schur's orthogonality of the irreducible characters that $pr_\lambda(p_\mu) = \delta_{\lambda\mu}$. Consequently, we have $p_\lambda p_\mu = \delta_{\lambda\mu} p_\lambda$, or equivalently,

$$\begin{aligned}
&\frac{\dim\lambda}{|G|}\sum_{s\in G}\chi_\lambda(s^{-1})\cdot s\cdot\frac{\dim\mu}{|G|}\sum_{t\in G}\chi_\mu(t^{-1})\cdot t\\
&=\frac{\dim\lambda\cdot\dim\mu}{|G|^2}\sum_{w\in G}\left(\sum_{t\in G}\chi_\lambda((wt^{-1})^{-1})\chi_\mu(t^{-1})\right)\cdot w\\
&=\delta_{\lambda\mu}\frac{\dim\lambda}{|G|}\sum_{w\in G}\chi_\lambda(w^{-1})\cdot w\ .
\end{aligned}$$

We thus obtain

$$\chi_\lambda * \chi_\mu = \frac{|G|}{\dim\mu}\delta_{\lambda\mu}\chi_\lambda\ . \tag{2.8}$$

We now turn to the counting formula. Let

$$f_g(w) \overset{\text{def}}{=} \left|\{(s_1,t_1,s_2,t_2,\dots,s_g,t_g)\in G^{2g} \mid [s_1,t_1]\cdots[s_g,t_g]=w\}\right|. \tag{2.9}$$

This is a class function and satisfies $f_g(w) = f_g(w^{-1})$. From the definition, it is obvious that $f_{g_1+g_2} = f_{g_1} * f_{g_2}$. Therefore,

$$f_g = \overbrace{f_1 * \cdots * f_1}^{g\text{-times}}\ . \tag{2.10}$$

Finding f_1 is Exercise 7.68 of Stanley's textbook [29], and the answer is in Frobenius [8]. From Schur's lemma,

$$\sum_{s\in G}\rho_\lambda(s\cdot t\cdot s^{-1}) \tag{2.11}$$

is central as an element of $\mathrm{End}(\lambda)$, where ρ_λ is the irreducible representation corresponding to $\lambda\in\hat{G}$. This is because (2.11) commutes with $\rho_\lambda(w)$ for every $w\in G$. Hence we have

$$\sum_{s\in G}\rho_\lambda(s\cdot t\cdot s^{-1}) = \sum_{s\in G}\frac{\chi_\lambda(s\cdot t\cdot s^{-1})}{\dim\lambda} = \frac{|G|}{\dim\lambda}\chi_\lambda(t),$$

noticing that the character χ_λ is the trace of ρ_λ. Therefore,

$$\begin{aligned}
\dim\lambda\sum_{s\in G}\rho_\lambda(s\cdot t\cdot s^{-1}\cdot t^{-1}w^{-1}) &= \dim\lambda\sum_{s\in G}\rho_\lambda(s\cdot t\cdot s^{-1})\cdot\rho_\lambda(t^{-1}w^{-1})\\
&= |G|\cdot\chi_\lambda(t)\cdot\rho_\lambda(t^{-1}w^{-1})\ .
\end{aligned}$$

Taking trace and summing in $t\in G$ of the above equality, we obtain

$$\frac{\dim\lambda}{|G|}\sum_{s,t\in G}\chi_\lambda(sts^{-1}t^{-1}w^{-1}) = \sum_{t\in G}\chi_\lambda(t)\chi_\lambda(t^{-1}w^{-1}) = (\chi_\lambda*\chi_\lambda)(w^{-1}) = \frac{|G|}{\dim\lambda}\cdot\chi_\lambda(w^{-1}).$$

Switching to the δ-function of (2.6), we find

$$f_1(w) = \int_{G^2}\delta([s,t]w^{-1})dsdt = \sum_{\lambda\in\hat{G}}\frac{|G|}{\dim\lambda}\cdot\chi_\lambda(w^{-1}) = \sum_{\lambda\in\hat{G}}\frac{|G|}{\dim\lambda}\cdot\chi_\lambda(w). \tag{2.12}$$

Note that we can interchange w and w^{-1}, since f_g is integer valued and is invariant under complex conjugation. From (2.8), (2.10) and (2.12), we obtain

Theorem 2.2 (Counting formula for twisted case)**.** *For every $g \geq 1$ and $w \in G$ let*

$$f_g(w)=\big|\{(s_1,t_1,s_2,t_2,\dots,s_g,t_g) \in G^{2g} \mid [s_1,t_1]\cdots[s_g,t_g]=w\}\big|.$$

Then we have a character expansion formula

$$f_g(w)=f_g(w^{-1})=\sum_{\lambda\in\hat{G}}\left(\frac{|G|}{\dim\lambda}\right)^{2g-1}\cdot\chi_\lambda(w)\ . \tag{2.13}$$

The counting formula (2.1) is a special case for $w=1$.

3. Character varieties of U_n as moduli spaces of stable vector bundles

The next natural case of character varieties is for a compact Lie group G, in particular, $G=U_n$. The issue of taking the quotient $\mathrm{Hom}(\pi_1(\Sigma_g),U_n)/U_n$ is much more serious than the finite group case, due to the fact that the trivial representation of $\pi_1(\Sigma_g)$ into U_n is a fixed point of the conjugation action. Consequently, the quotient space does not have a good manifold structure at the trivial representation. One way to avoid this and other quotient difficulties is to restrict our consideration to *irreducible* unitary representations

$$\mathrm{Hom}^{\mathrm{irred}}(\pi_1(\Sigma_g),U_n)/U_n. \tag{3.1}$$

From now on we assume $g \geq 2$. This time the quotient is well-defined as a real analytic space with some minor singularities. According to Narasimhan and Seshadri [25], (3.1) is diffeomorphic to the moduli space, denoted here by $\mathcal{U}_C(n,0)$, of stable holomorphic vector bundles of rank n and degree 0 on a smooth algebraic curve C of genus g. A holomorphic vector bundle E on C is said to be *semistable* if

$$\frac{\deg F}{\mathrm{rank}\, F}\leq\frac{\deg E}{\mathrm{rank}\, E} \tag{3.2}$$

for every holomorphic proper vector subbundle $F\subset E$, and *stable* if the strict inequality holds. If the rank and the degree are relatively prime, then the equality cannot hold in (3.2), hence every semistable vector bundle is automatically stable. The topological structure of a vector bundle E on Σ_g is determined by its rank and the degree. From the expression (3.1) it is clear that the differentiable structure of $\mathcal{U}_{\mathcal{C}}(n,0)$ does not depend on which complex structure we give on Σ_g.

As explained in the newest addition to Mumford's textbook [24] by Kirwan, moduli theory of stable objects can also be understood in terms of the *symplectic quotient* of the space of differentiable connections on C with values in U_n by the group of gauge transformations. Let E be a topologically trivial differentiable U_n-vector bundle on Σ_g, and $\mathcal{A}(\Sigma_g,U_n)$ the space of differentiable connections in E. We denote by $ad(E)$ the associated adjoint u_n-bundle on Σ_g. Since the tangent space to the space of U_n-connections is the space of sections $\Gamma(\Sigma_g,ad(E)\otimes\Lambda^1(\Sigma_g))$, we can define a gauge invariant symplectic form

$$\omega(\alpha,\beta)=\frac{1}{8\pi^2}\int_C \mathrm{tr}(\alpha\wedge\beta),\qquad \alpha,\beta\in\Gamma(\Sigma_g,ad(E)\otimes\Lambda^1(\Sigma_g)) \tag{3.3}$$

on the space of U_n-connections on Σ_g. The Lie algebra of the group $\mathcal{G}(\Sigma_g,U_n)$ of gauge transformations is the space of global sections of $ad(E)$, hence its dual is $\Gamma(\Sigma_g,ad(E)\otimes\Lambda^2(\Sigma_g))$. The moment map of the $\mathcal{G}(\Sigma_g,U_n)$-action on the space of connections is then given by the curvature map

$$\mu_\Sigma:\mathcal{A}(\Sigma_g,U_n)\ni A\longmapsto F_A=dA+A\wedge A\in\Gamma(\Sigma_g,ad(E)\otimes\Lambda^2(\Sigma_g)). \tag{3.4}$$

If we choose $0 \in \Gamma(\Sigma_g, ad(E) \otimes \Lambda^2(\Sigma_g))$ as the reference value of the moment map, then the symplectic quotient

$$\mathcal{A}(\Sigma_g, U_n)/\!\!/\mathcal{G}(\Sigma_g, U_n) = \mu_\Sigma^{-1}(0)/\mathcal{G}(\Sigma_g, U_n) = \mathrm{Hom}(\pi_1(\Sigma_g), U_n)/U_n$$

gives the moduli space of flat U_n-connections on Σ_g. This correspondence is also known as the Riemann-Hilbert correspondence.

If the structure of a compact Riemann surface C is chosen on Σ_g, then a connection in a differentiable vector bundle E on C defines a holomorphic structure in E. This process goes as follows. First we note that there are no type $(0,2)$-forms on C. Therefore, the $(0,1)$-part of the connection is always integrable. We can then define a differentiable section of E to be *holomorphic* if it is annihilated by the $(0,1)$-part of the covanriant derivative. If the connection A is unitary, then it is uniquely determined by it's $(0,1)$-part. The information of A is thus encoded in the complex structure it defines on E. In particular, the moduli space of flat unitary connections modulo gauge equivalence becomes the moduli space of holomorphic vector bundles of degree 0. The stability condition of a holomorphic vector bundle is equivalent to requiring that the corresponding flat connection is irreducible. This in turn corresponds to irreducibility of the unitary representation of $\pi_1(C)$. Since the curvature F_A receives a topological constraint, the moment map (3.4) cannot take an arbitrary value of $\Gamma(\Sigma_g, ad(E) \otimes \Lambda^2(\Sigma_g))$. In particular, 0 is a critical value of the moment map μ_Σ, and hence the symplectic quotient is singular.

Although we have this issue of singularities, the above discussion shows that the U_n-character variety outside its singularities has a natural symplectic structure coming from (3.3) and the process of symplectic quotient, and a complex structure as the moduli space of holomorphic vector bundles if a complex structure is chosen on Σ_g. The symplectic and complex structures are compatible, so outside the singularities the character variety is a complex Kähler manifold. Consequently, its dimension should be even. Actually, we can compute the dimension directly from (2.4). Noticing that $\det[s,t] = 1$ and that the center of U_n acts trivially via conjugation, we have

$$\dim_{\mathbb{R}} \mathrm{Hom}(\pi_1(\Sigma_g), U_n)/U_n = n^2(2g-2) + 2 = 2(n^2(g-1)+1). \tag{3.5}$$

All the considerations become much simpler when the group is $G = U_1$. The condition of (2.4) is vacuous and the character variety is simply a $2g$-dimensional real torus

$$\mathrm{Hom}(\pi_1(\Sigma_g), U_1) = \mathrm{Hom}(H_1(\Sigma_g, \mathbb{Z}), U_1) = (U_1)^{2g}.$$

If a complex structure C is chosen on Σ_g, then the complex line bundle arising from a representation of $\pi_1(\Sigma_g)$ acquires a holomorphic structure, and the character variety becomes the Jacobian:

$$\mathrm{Hom}(\pi_1(C), U_1) \cong \mathrm{Jac}(C) = \mathrm{Pic}^0(C).$$

4. Twisted character varieties of U_n

To study moduli spaces of holomorphic vector bundles on a Riemann surface that are not topologically trivial, we need to consider a variant of character varieties. Let E now be a topological vector bundle of rank n and degree $d \neq 0$ on $C = \Sigma_g$. This time it admits no flat connections, because the degree of E is determined by its connection through the Chern-Weil formula:

$$\deg E = c_1(E) = -\frac{1}{2\pi i}\int_C \mathrm{tr}(F_A).$$

The symplectic quotient of the space of connections in E requires a point in the dual Lie algebra $F_A \in \Gamma(\Sigma_g, ad(E) \otimes \Lambda^2(\Sigma_g))$ that is fixed under the coadjoint action of $\mathcal{G}(\Sigma_g, U_n)$.

Obviously, F_A is coadjoint invariant if it takes central values. A unitary connection A in E is said to be *projectively flat* if its curvature F_A is central. Narasimhan-Seshadri [25] again tells us that the moduli space $\mathcal{U}_C(n,d)$ of stable holomorphic vector bundles on C of rank n and degree d is diffeomorphic to the space of gauge equivalent classes of irreducible projectively flat connections.

Among the projectively flat connections, there is a particularly natural class. Since the curvature F_A of a connection A is a 2-form, we cannot talk about F_A being a constant. But if we apply the Hodge $*$-operator, then the covariant constant condition

$$d_A * F_A = 0 \tag{4.1}$$

makes sense. This is exactly the two-dimensional *Yang-Mills equation* studied by Atiyah and Bott in [1]. A projectly flat solution A of the Yang-Mills equation has its curvature given by

$$F_A = -\frac{2\pi i d}{n} I_n \cdot \mathrm{vol}_C, \tag{4.2}$$

where vol_C is the normalized volume form of C with total volume 1. The *holonomy group* of a connection at a point $p \in C$ is generated by parallel transports along every closed loop that starts at p. The Lie algebra of the holonomy group is the Lie subalgebra of u_n in which the curvature form F_A takes values. For a projectively flat connection, the holonomy group is the center U_1 of U_n. Certainly, the Lie algebra generated by the value (4.2) is $\mathbb{R}$, and the corresponding Lie group is U_1.

The Riemann-Hilbert correspondence gives an identification between a flat connection and a representation of $\pi_1(\Sigma_g)$ into U_n. What is a counterpart of the Riemann-Hilbert correspondence for the case of a projectively flat connection?

When the curvature is non-zero, a parallel transport of a connection does not induce a representation $\pi_1(\Sigma_g) \to U_n$ because it depends on the choice of a loop. The answer to the above question presented in [1] is that *a projective Yang-Mills connection corresponds to a representation of a central extension of* $\pi_1(\Sigma_g)$ *into* U_n. In the following we examine this correspondence for irreducible connections.

We note that $\pi_1(\Sigma_g)$ has a universal central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \hat{\pi}_1(\Sigma_g) \longrightarrow \pi_1(\Sigma_g) \longrightarrow 1, \tag{4.3}$$

where the extended group is defined by

$$\hat{\pi}_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g, c \mid [c, a_i] = [c, b_i] = 1, [a_1, b_1] \cdots [a_g, b_g] = c \rangle,$$

and $\mathbb{Z} \ni k \longmapsto c^k \in \hat{\pi}_1(\Sigma_g)$ determines its center. The central extension we need is a *Lie group* $\hat{\pi}_1(\Sigma_g)_{\mathbb{R}}$ that contains a copy of $\mathbb{R}$ through $\mathbb{R} \ni r \longmapsto c^r \in \hat{\pi}_1(\Sigma_g)_{\mathbb{R}}$, and satisfies that

$$1 \longrightarrow \mathbb{R} \longrightarrow \hat{\pi}_1(\Sigma_g)_{\mathbb{R}} \longrightarrow \pi_1(\Sigma_g) \longrightarrow 1. \tag{4.4}$$

Theorem 4.1 (Atiyah-Bott [1])**.** *The* ***twisted*** *character variety*

$$\mathrm{Hom}^{\mathrm{irred}}(\hat{\pi}_1(\Sigma_g)_{\mathbb{R}}, U_n)/U_n \tag{4.5}$$

of irreducible representations is identified with the space of irreducible unitary Yang-Mills connections in E modulo gauge transformations.

Note that $\mathrm{Hom}(\hat{\pi}_1(\Sigma_g)_{\mathbb{R}}, U_n) = \{(s_1, t_1, \dots, s_g, t_g, \gamma) \in (U_n)^{2g+1} \mid [\gamma, s_i] = [\gamma, t_i] = 1,\ [s_1, t_1] \cdots [s_g, t_g] = \gamma\}$. Since the commutator product is equated to $\gamma \in U_n$ which is not necessarily the identity, the name "twisted" is used in the literature. If a representation

$\hat{\pi}_1(\Sigma_g)_{\mathbb{R}} \to U_n$ is irreducible, then γ is a central element of U_n. Since $\det[s,t] = 1$, we conclude that

$$\gamma = \exp\left(\frac{2\pi i d}{n}\right) \cdot I_n \tag{4.6}$$

for some integer d. Therefore, $\mathrm{Hom}^{\mathrm{irred}}(\hat{\pi}_1(\Sigma_g)_{\mathbb{R}}, U_n)$ consists of n disjoint pieces corresponding to the n possible values for (4.6).

The construction of a Yang-Mills connection from an irreducible representation

$$\rho \in \mathrm{Hom}^{\mathrm{irred}}(\hat{\pi}_1(\Sigma_g)_{\mathbb{R}}, U_n)$$

goes as follows. First we choose a connection a in a complex line bundle L on Σ_g of degree 1. The Yang-Mills equation for a is simply the linear harmonic equation $d * da = 0$ because U_1 is Abelian. So let us choose a harmonic connection a with curvature

$$F_a = -2\pi i \cdot \mathrm{vol}_{\Sigma}. \tag{4.7}$$

Let $h : \hat{\Sigma}_g \to \Sigma_g$ be the universal covering of Σ_g. Then the pull-back line bundle h^*L on $\hat{\Sigma}_g$, viewed as a fiber bundle on Σ_g, has the structure group $U_1 \times \pi_1(\Sigma_g)$. Note that the exact sequence (4.4) induces a surjective homomorphism

$$f : \hat{\pi}_1(\Sigma_g)_{\mathbb{R}} \longrightarrow U_1 \times \pi_1(\Sigma_g)$$

by sending the central generator c to a non-identity element of U_1. We can thus construct a principal $\hat{\pi}_1(\Sigma_g)_{\mathbb{R}}$-bundle P on Σ_g from L, h, and f, in which the lift of a now lives as a Yang-Mills connection with the constant curvature (4.7). Consider the principal U_n-bundle on Σ_g defined by $P \times_\rho U_n$, and its associated rank n vector bundle E through the standard n-dimensional representation of U_n on $\mathbb{C}^n$. Let A be the natural connection in E arising from a. Then by functoriality of the Yang-Mills equation, A is automatically a Yang-Mills connection in E. The holonomy of A is the group generated by $\gamma = \rho(c)$ in U_n, which is central since ρ is irreducible. The value of the curvature F_A of A is quantized according to the topological type of E, which is also determined by $\rho(c) \in U_n$.

To show that every irreducible unitary Yang-Mills connection gives rise to a representation

$$\rho : \hat{\pi}_1(\Sigma_g)_{\mathbb{R}} \to U_n,$$

first we note that the same statement is true for $G = U_1$ and $G = SU_n$. Then we reduce the problem of construction to the hybrid of these two cases. For SU_n, the vector bundle involved is trivial, and an irreducible Yang-Mills connection is necessarily flat. Thus it gives rise to a representation of $\pi_1(\Sigma_g)$. For U_1, the group is Abelian and the question reduces to the standard homology theory. By pulling back a unitary connection through the covering homomorphism

$$U_1 \times SU_n \longrightarrow U_n,$$

we can reduce the general case to the two special cases [1].

An important fact is that if γ of (4.6) is a primitive root of unity, i.e., $G.C.D.(n,d) = 1$, then $\mathcal{U}_C(n,d)$ is a non-singular projective algebraic variety. The smoothness is a consequence of the fact that such a γ is a regular value of the commutator product map

$$\mu : (U_n)^{2g} \ni (s_1, t_1, \ldots, s_g, t_g) \longmapsto [s_1, t_1] \cdots [s_g, t_g] \in SU_n, \tag{4.8}$$

and that the isotropy subgroup of the conjugation action of U_n on $\mu^{-1}(\gamma)$ is always the central U_1. These statements are easily verified through direct calculations (see for example

[12]). Let us choose a point $p = (s_1, t_1, \ldots, s_g, t_g) \in \mu^{-1}(\gamma)$ in the inverse image of a primitive root of unity γ. The differential $d\mu_p$ of μ at p is a linear map between Lie algebras

$$d\mu_p : (u_n)^{\oplus 2g} \longrightarrow su_n.$$

Note that for $s \in U_n$ and $x \in u_n$, we have $ds(x) = x$. Let us first consider the case $g = 1$. We wish to show that

$$\begin{aligned} & d\mu_p(x, y) \\ =& ds(x) \cdot ts^{-1}t^{-1} + s \cdot dt(y) \cdot s^{-1}t^{-1} - sts^{-1} \cdot ds(x) \cdot s^{-1}t^{-1} - sts^{-1}t^{-1} \cdot dt(y) \cdot t^{-1} \\ =& xts^{-1}t^{-1} + sys^{-1}t^{-1} - sts^{-1}xs^{-1}t^{-1} - sts^{-1}t^{-1}yt^{-1} \\ =& \gamma(xs^{-1} - txs^{-1}t^{-1}) + \gamma(syt^{-1}s^{-1} - yt^{-1}) \end{aligned}$$

spans the entire Lie algebra su_n as $(x, y) \in (u_n)^2$ varies. In the above computation products and additions are calculated as $n \times n$ complex matrices, and we have used the commutation relation $sts^{-1}t^{-1} = \gamma$. Recall that $\operatorname{tr}(vw)$ defines a non-degenerate bilinear form on su_n. Suppose now that $\operatorname{tr}(w \cdot d\mu_p(x, y)) = 0$ for all $x, y \in u_n$. For $y = 0$ it follows that

$$\begin{aligned} & \operatorname{tr}(xs^{-1}w) = \operatorname{tr}(txs^{-1}t^{-1}w) \qquad \text{for all } x \in u_n \\ \Longleftrightarrow \quad & s^{-1}w = s^{-1}t^{-1}wt \\ \Longleftrightarrow \quad & w = t^{-1}wt. \end{aligned}$$

Similarly, for $x = 0$, we obtain $w = s^{-1}ws$. Therefore, w commutes with s and t. We can then restrict the relation $[s, t] = \gamma$ to any eigenspace of w of dimension $m \le n$. The determinant condition $\det[s, t] = 1$ yields $\gamma^m = 1$. Hence $m = n$ because γ is primitive, establishing that w is a scalar diagonal matrix. Since $w \in su_n$, we conclude that $w = 0$.

For $g \ge 2$, we use the relation $[s_1, t_1] \cdots [s_g, t_g] = \gamma$ to establish that any $w \in su_n$ that satisfies $\operatorname{tr}(w \cdot d\mu_p(x_1, y_1, \ldots, x_g, y_g)) = 0$ commutes with s_1 and t_1 when restricted to $x_i = y_i = 0$ for $i > 1$. We can then recursively show that w actually commutes with all s_i and t_i. Restricting the commutator product relation to any eigenspace of w as above and using the fact that γ is primitive, we conclude that w is central, and hence equal to $0 \in su_n$. It follows that $\gamma \in SU_n$ is a regular value of (4.8), and consequently $\mu^{-1}(\gamma)$ is a non-singular manifold.

Note that in the above argument we have also shown that the isotropy subgroup of U_n acting on $\mu^{-1}(\gamma)$ through conjugation is the central U_1 at any point of $\mu^{-1}(\gamma)$. Therefore, the quotient

$$\mu^{-1}(\gamma)/U_n = \mathcal{U}_C(n, d)$$

is non-singular if $G.C.D.(n, d) = 1$.

The task of calculating the Poinaré polynomial of this non-singular compact complex algebraic manifold is carried out by Harder-Narasimhan [11], Atiyah-Bott [1] and Zagier [34]. Harder and Narasimhan use Deligne's solution to the Weil conjecture (see for example [26]) as their tool and study the moduli theory over the finite field $\mathbb{F}_q$ for all possible values of $q = p^e$. Atiyah and Bott use 2-dimensional Yang-Mills theory and equivariant Morse-Bott theory to derive the topological structure of $\mathcal{U}_C(n, d)$. Both [11] and [1] lead to a recursion formula for the Poincaré polynomials. Zagier [34] obtains a closed formula, solving the recursion relation.

5. Twisted character varieties of $GL_n(\mathbb{C})$

Twisted character varieties

$$\mathrm{Hom}(\hat{\pi}_1(\Sigma_g), G)/\!\!/G \tag{5.1}$$

for a complex reductive group G have received much attention in recent years from many different points of view [4, 12, 13, 18]. In this section we consider the case $G = GL_n(\mathbb{C})$. The quotient (5.1) is a *geometric invariant theory quotient* of [24], due to the fact that G is not compact. The categorical quotient contains the geometric quotient

$$\mathrm{Hom}^{\mathrm{irred}}(\hat{\pi}_1(\Sigma_g), GL_n(\mathbb{C}))/GL_n(\mathbb{C}).$$

The argument of Section 4 applies here to show that the central generator $c \in \hat{\pi}_1(\Sigma_g)$ is mapped to a central element $\gamma \in GL_n(\mathbb{C})$, which takes the same value as in (4.6). Thus the character variety consists of n disjoint pieces, and a component corresponding to a primitive n-th roots of unity is a non-singular affine algebraic subvariety of complex dimension $2(n^2(g-1)+1)$ contained in $\mathbb{C}^{2gn^2}$. From now on we refer to this non-singular piece at a primitive n-th root of unity γ by

$$\mathcal{X}(\mathbb{C}) = \{\rho \in \mathrm{Hom}^{\mathrm{irred}}(\hat{\pi}_1(\Sigma_g), GL_n(\mathbb{C})) \mid \rho(c) = \gamma\}/GL_n(\mathbb{C}). \tag{5.2}$$

A surprising result recently obtained by Hausel, Rodriguez-Villegas and Katz in [12] is the calculation of the mixed Hodge polynomial of this character variety. Their key idea is Deligne's Hodge theory. It states that the mixed Hodge polynomial of a complex algebraic variety $X(\mathbb{C})$ can be determined if one knows the cardinality of the mod $q = p^e$ reduction $X(\mathbb{F}_q)$ of X for every prime p (or most of them at least) and its power e. For the case of the character variety for $GL_n(\mathbb{C})$, since its defining equation

$$[s_1, t_1] \cdots [s_g, t_g] = \gamma$$

is a set of polynomial equations defined over $\mathbb{Z}[\gamma]$ among the entries of the matrices, the mod q reduction is given by $\mathcal{X}(\mathbb{F}_q)$ if p is not a factor of n. Now the group $GL_n(\mathbb{F}_q)$ is finite, so the cardinality of the character variety is readily available from (2.13)!

Since U_n is the compact real form of $GL_n(\mathbb{C})$, the compact complex manifold $\mathcal{U}_C(n,d)$ is contained as the real part of $\mathcal{X}(\mathbb{C})$ if $\gamma = \exp(2\pi i d/n)$ and $G.C.D.(n,d) = 1$. What is the relation between the complex structure of $\mathcal{X}(\mathbb{C})$ naturally arising from $GL_n(\mathbb{C})$ and that of $\mathcal{U}_C(n,d)$ coming from a complex structure C on the surface Σ_g? This question is addressed in Section 7.

If we view the non-singluar compact complex projective algebraic variety $\mathcal{U}_C(n,d)$ as a real analytic Riemannian manifold whose metric is determined by the Kähler structure, then its *complexification* is the total space of the cotangent bundle $T^*\mathcal{U}_C(n,d)$. This is because the canonical symplectic form on $T^*\mathcal{U}_C(n,d)$ and the Riemannian metric induced from $\mathcal{U}_C(n,d)$ together determine the unique almost complex structure on the cotangent bundle which is integrable. Since $\mathcal{X}(\mathbb{C})$ is a complexification of $\mathcal{U}_C(n,d)$, it contains this cotangent bundle as a complex submanifold:

$$T^*\mathcal{U}_C(n,d) \subset \mathcal{X}(\mathbb{C}). \tag{5.3}$$

Of course this embedding is *never* a holomorphic map with respect to the complex structure of $\mathcal{U}_C(n,d)$. So far we have noticed that there are at least *two* different complex structures in $T^*\mathcal{U}_C(n,d)$. One is what we have just described as a complex submanifold of $\mathcal{X}(\mathbb{C})$, which we denote by J, and the other comes from the cotangent bundle of the complex manifold $\mathcal{U}_C(n,d)$ denoted by I. These complex structures are indeed different, since an affine manifold $\mathcal{X}(\mathbb{C})$ cannot contain a compact complex manifold $\mathcal{U}_C(n,d)$ in it.

In this section we study the structure of $\mathcal{X}(\mathbb{C})$ from the point of view of 2-dimensional Yang-Mills theory following Hitchin [14]. Let us consider a topological complex vector bundle E of rank n and degree d on a Riemann surface C of genus g, and a complex connection $A_{\mathbb{C}}$ in E with values in $gl_n(\mathbb{C})$. We choose a Hermitian fiber metric in E and reduce the structure group to U_n. The skew-Hermitian part A of $A_{\mathbb{C}}$ is a unitary connection which is well-defined under the unitary gauge transformation $\mathcal{G}(C, U_n)$, though the whole gauge transformation $\mathcal{G}(C, GL_n(\mathbb{C}))$ does not preserve the skew-Hermitian part. Note that the action of $\mathcal{G}(C, U_n)$ on the Hermitian part of $A_{\mathbb{C}}$ is a linear transformation because a unitary gauge transformation of the 0 connection is skew-Hermitian. Therefore the Hermitian part Φ of $A_{\mathbb{C}}$ can be identified as a differential 1-form on C with values in $ad_{\mathbb{C}}(E)$, the $gl_n(\mathbb{C})$-bundle associated to $ad(E)$:

$$\Phi \in \Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^1(\Sigma_g)).$$

Using the complex coordinate on C, let ϕ be the type $(1,0)$-part of Φ:

$$\phi = \Phi^{(1,0)} \in \Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^{(1,0)}(C)).$$

Here again ϕ is well-defined under the unitary gauge transformation, and it uniquely determines Φ because of the Hermitian condition. In this way we obtain a $\mathcal{G}(C, U_n)$-space isomorphism

$$\mathcal{A}(C, GL_n(\mathbb{C})) \cong \mathcal{A}(C, U_n) \times \Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^{(1,0)}(C)), \tag{5.4}$$

which identifies $A_{\mathbb{C}}$ with the pair (A, ϕ) thus obtained. We will come back to the point of the action of $\mathcal{G}(C, GL_n(\mathbb{C}))$ on these spaces a little later.

Hitchin shows that the moment map on $\mathcal{A}(C, U_n) \times \Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^{(1,0)}(C))$ for the gauge group $\mathcal{G}(C, U_n)$-action is given by

$$\mu_H : \mathcal{A}(C, U_n) \times \Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^{(1,0)}(C)) \ni (A, \phi) \longmapsto F_A + [\phi, \phi^*] \in \Gamma(C, ad(E) \otimes \Lambda^{(1,1)}(C)),$$

where F_A is the curvature form of A and $[\phi, \phi^*] = \phi \wedge \phi^* + \phi^* \wedge \phi$ is an $ad(E)$-valued (i.e., a locally *skew*-Hermitian) $(1,1)$-form on C. Although $\mathcal{A}(C, U_n) /\!\!/ \mathcal{G}(C, U_n)$ is finite-dimensional, the symplectic quotient $\mu_H^{-1}(0)/\mathcal{G}(C, U_n)$ is still infinite-dimensional due to the second factor $\Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^{(1,0)}(C))$. Hitchin [14] proposes to add another equation to reduce the dimensionality. The *Hitchin equations* are a system of equations

$$\begin{cases} \overline{\partial}_A \phi = 0 \\ F_A + [\phi, \phi^*] = 0\,, \end{cases} \tag{5.5}$$

where $d_A = \partial_A + \overline{\partial}_A$ is the decomposition of the covariant derivative of the connection A into its type $(1,0)$ and $(0,1)$ components that are determined by the complex structure of C. The origin of (5.5) is the dimensional reduction of the 4-dimensional Yang-Mills theory. Hitchin observes that the self-duality equation on $\mathbb{R}^4$ restricted to 2 dimensions by imposing independence in two variables automatically reduces to (5.5).

Since A is a unitary connection in E, it defines a holomorphic structure in E through the covariant Cauchy-Riemann operator $\overline{\partial}_A$. With respect to this complex structure, the first equation $\overline{\partial}_A \phi = 0$ implies that $\phi \in \Gamma(C, ad_{\mathbb{C}}(E) \otimes \Lambda^{(1,0)}(C))$ is holomorphic. We recall that the holomorphic part of $ad(E)$ is the holomorphic endomorphism sheaf $\mathrm{End}(E)$ on C, and the holomorphic part of $\Lambda^{(1,0)}(C)$ is the sheaf of holomorphic 1-forms on C, or the *canonical sheaf* K_C on C. Therefore, a solution of $\overline{\partial}_A \phi = 0$ is a section

$$\phi \in H^0(C, \mathrm{End}(E) \otimes K_C). \tag{5.6}$$

We cannot define the symplectic quotient $\mathcal{A}(C, GL_n(\mathbb{C}))/\!\!/\mathcal{G}(C, GL_n(\mathbb{C}))$ directly as we did before, because $GL_n(\mathbb{C})$ is not compact and the analysis we need to deal with the infinite-dimensional manifolds does not work. The argument of Atiyah and Bott we have used in Section 4 can be certainly applied to $\rho \in \mathcal{X}(\mathbb{C})$ of (5.2), resulting in a projectively flat $gl_n(\mathbb{C})$ Yang-Mills connection $A_{\mathbb{C}}$ on C. It's $(0,1)$ part defines a holomorphic structure in the topological vector bundle E as before, but since the connection is not unitary, we are utilizing only half of the information that $A_{\mathbb{C}}$ has. Hitchin's idea is that the other half of the information goes to $\phi \in H^0(C, \mathrm{End}(E) \otimes K_C)$ through the factorization (5.4). Now the Serre duality

$$H^0(C, \mathrm{End}(E) \otimes K_C) = H^1(C, \mathrm{End}(E))^*$$

and the Kodaira-Spencer deformation theory

$$H^1(C, \mathrm{End}(E)) = T_E\mathcal{U}_C(n, d)$$

show that the pair (E, ϕ) is indeed an element of $T^*\mathcal{U}_C(n, d)$, which is what we expected in (5.3). This pair consisting of a holomorphic vector bundle E and a *Higgs field* ϕ of (5.6) is known as a *Higgs pair* or a *Higgs bundle*.

There is a slight inaccuracy here because we did not impose any stability condition on E. The right notion of stability is that the slope inequality (3.2) holds for every ϕ-invariant proper vector subbundle F. Then the moduli space of unitary gauge equivalent classes of irreducible solutions of the Hitchin equations (5.5) is diffeomorphic to the moduli space of stable Higgs pairs. Here we are assuming that the rank and the degree of E are relatively prime. Obviously, if E itself is stable, then the Higgs bundle (E, ϕ) is stable for every ϕ in $H^0(C, \mathrm{End}(E) \otimes K_C)$. Therefore, the complex cotangent bundle $T^*\mathcal{U}_C(n, d)$ is contained in the moduli space $\mathcal{H}_C(n, d)$ of stable Higgs bundles as an open dense subset. We also note that the stability of a Higgs pair $(E, 0)$ simply means that E is stable.

Now we come back to the action of the group $\mathcal{G}(C, GL_n(\mathbb{C}))$ of complex gauge transformation on the space of complex valued connections $\mathcal{A}(C, GL_n(\mathbb{C}))$. As we have noted earlier, we cannot directly define the symplectic quotient. After reducing the problem to considering Higgs pairs (E, ϕ), still we have the ambiguity of the action of $H^0(C, \mathrm{Aut}(E))$ on the pairs since E is not necessarily stable. But this situation is better than the symplectic quotient, because of the fact that for every *stable* Higgs pair (E, ϕ), we have [14]

$$H^0(C, \mathrm{End}(E, \phi)) = \mathbb{C}.$$

Here an endomorphism of a Higgs bundle (E, ϕ) is defined to be a holomorphic endomorphism ψ of E that commutes with ϕ:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ E \otimes K_C & \xrightarrow[\psi\otimes 1]{} & E \otimes K_C \end{array}$$

Although we know topological structures such as the Poincaré polynomial of $T^*\mathcal{U}_C(n, d)$ from the work of [1] and [11], their methods do not directly apply to the study of the character variety $\mathcal{X}(\mathbb{C})$. The work of Hausel and his collaborators [12] reveals unexpectedly rich structures in the study of the topology of these complex character varieties, such as an unexpected relation to Macdonald polynomials.

6. Hitchin integrable systems

From the point of view of 2-dimensional Yang-Mills theory, we are led to identifying the complex character variety $\mathcal{X}(\mathbb{C})$ as the moduli space $\mathcal{H}_C(n,d)$ of stable Higgs bundles. In this section we show that there is an algebraically completely integrable system on this Hitchin moduli space.

The total space of the complex cotangent bundle $T^*\mathcal{U}_C(n,d)$ is an open non-singular complex submanifold of $\mathcal{H}_C(n,d)$. Since the cotangent bundle is easier to understand than the Hitchin moduli, let us look at it first. Note that $p^*\Lambda^1(\mathcal{U}_C(n,d)) \subset \Lambda^1(T^*\mathcal{U}_C(n,d))$ has a tautological section

$$\eta \in H^0(T^*\mathcal{U}_C(n,d), p^*\Lambda^1(\mathcal{U}_C(n,d))),$$

where $p : T^*\mathcal{U}_C(n,d) \to \mathcal{U}_C(n,d)$ is the projection, and $\Lambda^r(X)$ denotes in this section the sheaf of holomorphic r-forms on a complex manifold X. The differential $\omega_I = d\eta$ of the tautological section defines the canonical holomorphic symplectic form on $T^*\mathcal{U}_C(n,d)$. The suffix I indicates the referrence to the complex structure of $\mathcal{U}_C(n,d)$. The restriction of ω_I on $\mathcal{U}_C(n,d)$, which is the 0-section of the cotangent bundle, is identically 0. Therefore the 0-section is a Lagrangian submanifold of this holomorphic symplectic manifold.

A surprising result of another influential paper [15] of Hitchin's is that $\mathcal{H}_C(n,d)$ is the total space of a Lagrangian torus fibration. The starting point of his discovery is the following intriguing equality as a consequence of the Riemann-Roch formula:

$$\dim_C \mathcal{U}_C(n,d) = n^2(g-1)+1 = 1+(g-1)\sum_{i=1}^n (2i-1) = \dim_C \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}).$$

Let us denote by

$$V_{GL} = V_{GL_n(\mathbb{C})} = \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}). \tag{6.1}$$

As a vector space V_{GL} has the same dimension as $H^0(C, \mathrm{End}(E)\otimes K_C) = T_E^*\mathcal{U}_C(n,d)$. The Higgs field $\phi \in H^0(C, \mathrm{End}(E)\otimes K_C)$ introduced by Hitchin earlier in [14] is a "twisted" endomorphism

$$\phi : E \longrightarrow E \otimes K_C,$$

which induces a homomorphism of the i-th anti-symmetric tensor product spaces

$$\wedge^i(\phi) : \wedge^i(E) \longrightarrow \wedge^i(E\otimes K_C) = \wedge^i(E)\otimes K_C^{\otimes i},$$

or equivalently $\wedge^i(\phi) \in H^0(C, \mathrm{End}(\wedge^i(E))\otimes K_C^{\otimes i})$. Taking its natural trace, we obtain

$$\mathrm{tr} \wedge^i (\phi) \in H^0(C, K_C^{\otimes i}).$$

This is exactly the i-th characteristic coefficient of the twisted endomorphism ϕ:

$$\det(x-\phi) = x^n + \sum_{i=1}^n (-1)^i \mathrm{tr} \wedge^i (\phi)\cdot x^{n-i}. \tag{6.2}$$

By assigning its coefficients, Hitchin [15] defines a holomorphic map, now known as the *Hitchin fibration* or *Hitchin map*,

$$H : \mathcal{H}_C(n,d) \ni (E,\phi) \longmapsto \det(x-\phi) \in \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}) = V_{GL}. \tag{6.3}$$

The map H to a vector space V_{GL} is a collection of $N = n^2(g-1)+1$ globally defined holomorphic functions on $\mathcal{H}_C(n,d)$. The 0-fiber of the Hitchin fibration is the moduli space $\mathcal{U}_C(n,d)$.

What are other fibers of H? To answer this question, the notion of *spectral curves* is introduced in [15]. Generically other fibers are the Jacobians of these spectral curves. The total space of the canonical sheaf $K_C = \Lambda^1(C)$ on C is the cotangent bundle T^*C. Let

$$\pi : T^*C \longrightarrow C$$

be the projection, and

$$\tau \in H^0(T^*C, \pi^*K_C) \subset H^0(T^*C, \Lambda^1(T^*C))$$

be the tautological section of π^*K_C on T^*C. Here again $\omega = d\tau$ is the holomorphic symplectic form on T^*C. The tautological section τ satisfies that $\sigma^*\tau = \sigma$ for every section $\sigma \in H^0(C, K_C)$ viewed as a holomorphic map $\sigma : C \to T^*C$. The characteristic coefficients (6.2) of ϕ give a section

$$s = \det(\tau - \phi) = \tau^{\otimes n} + \sum_{i=1}^{n} (-1)^i \mathrm{tr} \wedge^i (\phi) \cdot \tau^{\otimes n-1} \in H^0(T^*C, \pi^*K_C^{\otimes n}). \tag{6.4}$$

We define the spectral curve C_s associated with a Higgs pair (E,ϕ) as the divisor of 0-points of the section $s = \det(\tau - \phi)$ of the line bundle $\pi^*K_C^{\otimes n}$:

$$C_s = (s)_0 \subset T^*C. \tag{6.5}$$

The spectral curve is the locus of τ that satisfies the characteristic equation $\det(\tau-\phi) = 0$. Thus every point of C_s is an eigenvalue, or spectrum, of the twisted endomorphism ϕ. This is the origin of the name of C_s. The projection π defines a ramified covering map $\pi : C_s \to C$ of degree n.

Another way to look at the spectral curve C_s is to go through algebra. It has an advantage in identifying the fibers of the Hitchin fibration. Since the section $s = \det(\tau - \phi)$ is determined by the characteristic coefficients of ϕ, by abuse of notation we consider s as an element of V_{GL}:

$$s = (s_1, s_2, \dots, s_n) = (-\mathrm{tr}\, \phi, \mathrm{tr} \wedge^2 (\phi), \dots, (-1)^n \mathrm{tr} \wedge^n (\phi)) \in \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}).$$

It defines an $\mathcal{O}_C$-module $(s_1 + s_2 + \cdots + s_n) \otimes K_C^{\otimes -n}$. Let $\mathcal{I}_s$ denote the ideal generated by this module in the symmetric tensor algebra $\mathrm{Sym}(K_C^{-1})$. Since K_C^{-1} is the sheaf of linear functions on T^*C, the scheme associated to this tensor algebra is $\mathrm{Spec}\big(\mathrm{Sym}(K_C^{-1})\big) = T^*C$. The spectral curve as the divisor of 0-points of s is then defined by

$$C_s = \mathrm{Spec}\left(\frac{\mathrm{Sym}(K_C^{-1})}{\mathcal{I}_s}\right) \subset \mathrm{Spec}\big(\mathrm{Sym}(K_C^{-1})\big) = T^*C. \tag{6.6}$$

The set U consisting of points s for which C_s is irreducible and non-singular is an open dense subset of V_{GL} [2]. The genus of C_s can be found as follows. Note that we have

$$\pi_*\mathcal{O}_{C_s} = \mathrm{Sym}(K_C^{-1})/\mathcal{I}_s \cong \bigoplus_{i=0}^{n-1} K_C^{\otimes -i}$$

as an $\mathcal{O}_C$-module. From the Riemann-Roch formula we see that

$$1 - g(C_s) = \chi(C_s, \mathcal{O}_{C_s}) = \chi(C, \pi_*\mathcal{O}_{C_s}) = (1 - g(C))\sum_{i=0}^{n-1}(2i+1) = n^2(1 - g(C)).$$

Hence $g(C_s) = n^2(g-1)+1$. As a consequence, we notice that the dimensions of the Jacobian variety $\mathrm{Jac}(C_s)$ and the moduli space $\mathcal{U}_C(n,d)$ are the same. The theory of spectral curves [2, 15] makes this equality into a precise geometric relation between these two spaces.

The Higgs field $\phi \in H^0(C, \mathrm{End}(E) \otimes K_C)$ gives a homomorphism

$$\varphi : K_C^{-1} \longrightarrow \mathrm{End}(E),$$

which induces an algebra homomorphism, still denoted by the same letter,

$$\varphi : \mathrm{Sym}(K_C^{-1}) \longrightarrow \mathrm{End}(E).$$

Thus φ defines a $\mathrm{Sym}(K_C^{-1})$-module structure in E. Since $s \in V_{GL}$ is the characteristic coefficients of φ, by the Cayley-Hamilton theorem, the homomorphism φ factors through

$$\mathrm{Sym}(K_C^{-1}) \longrightarrow \mathrm{Sym}(K_C^{-1})/\mathcal{I}_s \longrightarrow \mathrm{End}(E).$$

Hence E is actually a module over $\mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$ of rank 1. The rank is 1 because the ranks of E and $\mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$ are the same as $\mathcal{O}_C$-modules. In this way a Higgs pair (E, ϕ) gives rise to a line bundle $\mathcal{L}_E$ on the spectral curve C_s, if it is non-singluar. Since $\mathcal{L}_E$ being an $\mathcal{O}_{C_s}$-module is equivalent to E being a $\mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$-module, we recover E from $\mathcal{L}_E$ simply by $E = \pi_*\mathcal{L}_E$, which has rank n because π is a covering of degree n. From the equality $\chi(C, E) = \chi(C_s, \mathcal{L}_E)$ and Riemann-Roch, we find that $\deg \mathcal{L}_E = d + n(n-1)(g-1)$. To summarize, the above construction defines an inclusion map

$$H^{-1}(s) \subset \mathrm{Pic}^{d+n(n-1)(g-1)}(C_s) \cong \mathrm{Jac}(C_s),$$

if C_s is irreducible and non-singular.

Conversely, suppose we have a line bundle $\mathcal{L}$ of degree $d+n(n-1)(g-1)$ on an irreducible non-singular spectral curve C_s. Then $\pi_*\mathcal{L}$ is a module over $\pi_*\mathcal{O}_{C_s} = \mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$, which defines a homomorphism $\psi : K_C^{-1} \to \mathrm{End}(\pi_*\mathcal{L})$. It is easy to see that the Higgs pair $(\pi_*\mathcal{L}, \psi)$ is stable. Suppose we had a ψ-invariant subbundle $F \subset \pi_*\mathcal{L}$ of rank $k < n$. Since $(F, \psi|_F)$ is a Higgs pair, it gives rise to a spectral curve $C_{s'}$. From the construction, we have an injective morphism $C_{s'} \to C_s$. But since C_s is irreducible, it contains no smaller component. Therefore, $\pi_*\mathcal{L}$ has no ψ-invariant proper subbundle. Thus we have established that

$$H^{-1}(s) \cong \mathrm{Jac}(C_s), \qquad s \in U \subset V_{GL}. \tag{6.7}$$

We note that the vector bundle $\pi_*\mathcal{L}$ is not necessarily stable. It is proved in [2] that the locus of $\mathcal{L}$ in $\mathrm{Pic}^{d+n(n-1)(g-1)}(C_s)$ that gives unstable $\pi_*\mathcal{L}$ has codimension two or more.

Recall that the tautological section $\eta \in H^0(T^*\mathcal{U}_C(n,d), p^*\Lambda^1(\mathcal{U}_C(n,d)))$ is a holomorphic 1-form on $T^*\mathcal{U}_C(n,d) \subset \mathcal{H}_C(n,d)$. Its restriction to the fiber $H^{-1}(s)$ of $s \in U$ for which C_s is non-singular extends to a holomorphic 1-form on the whole fiber $H^{-1}(s) \cong \mathrm{Jac}(C_s)$ since η is undefined only on a codimension 2 subset. Consequently η extends as a holomorphic 1-form on $H^{-1}(U)$. Thus η is well defined on $T^*\mathcal{U}_C(n,d) \cup H^{-1}(U)$. The complement of this open subset in $\mathcal{H}_C(n,d)$ consists of such Higgs pairs (E, ϕ) that E is unstable *and* C_s is singular. Since the stability of E and the non-singular condition for C_s are both open conditions, this complement has codimension at least two. Consequently, both the tautological section η and the holomorphic symplectic form $\omega_I = d\eta$ extend holomorphically to the whole Higgs moduli space $\mathcal{H}_C(n,d)$.

We note that there are no holomorphic 1-forms other than constants on a Jacobian variety. It implies that

$$\omega_I|_{H^{-1}(s)} = d(\eta|_{H^{-1}(s)}) = 0$$

for $s \in U$. The *Poisson structure* on $H^0(\mathcal{H}_C(n,d), \mathcal{O}_{\mathcal{H}_C(n,d)})$ is defined by

$$\{f,g\} = \omega_I(X_f, X_g), \qquad f,g \in H^0(\mathcal{H}_C(n,d), \mathcal{O}_{\mathcal{H}_C(n,d)}),$$

where X_f denotes the Hamiltonian vector field defined by the relation $df = \omega_I(X_f, \cdot)$. Since ω_I vanishes on a generic fiber of H, the holomorphic functions on $\mathcal{H}_C(n,d)$ coming from coordinates of the Hitchin fibration are *Poisson commutative* with respect to the holomorphic symplectic structure ω_I. An *algebraically completely integrable Hamiltonian system* on a holomorphic symplectic manifold (M, ω) of dimension $2m$ is an open holomorphic map $H : M \to \mathbb{C}^m$ such that the coordinate functions are Poisson commutative and a generic fiber is an Abelian variety [32]. Thus $(\mathcal{H}_C(n,d), \omega_I, H)$ is an algebraically completely integrable Hamiltonian system, called the *Hitchin integrable system*.

Theorem 6.1. *The Hitchin fibration*

$$H : \mathcal{H}_C(n,d) \longrightarrow V_{GL}$$

is a Lagrangian Jacobian fibration defined on an algebraically completely integrable system $(\mathcal{H}_C(n,d), \omega_I, H)$. A generic fiber $H^{-1}(s)$ is a Lagrangian with respect to the holomorphic symplectic structure ω_I and is isomorphic to the Jacobian variety of a spectral curve C_s.

7. Symplectic quotient of the Hitchin system and mirror symmetry

Is the Hitchin fibration (6.3) an effective family of deformations of Jacobians? This is the question we address in [16]. The investigation of this question leads to the relation between the Hitchin systems and mirror symmetry discovered by Hausel and Thaddeus [13].

The Jacobian variety $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ acts on $\mathcal{H}_C(n,d)$ by $(E, \phi) \longmapsto (E \otimes L, \phi)$, where $L \in \mathrm{Jac}(C)$ is a line bundle on C of degree 0. The Higgs field is preserved because

$$E^* \otimes E \longmapsto (E \otimes L)^* \otimes (E \otimes L) = E^* \otimes E$$

is unchanged. Thus this action does not contribute to deformations of the spectral curves. It is natural to symplectically quotient it out. On the open subset $T^*\mathcal{U}_C(n,d)$, the $\mathrm{Jac}(C)$ action is symplectomorphic because it is induced by the action on the base space $\mathcal{U}_C(n,d)$. On the other open subset $H^{-1}(U)$ the action is also symplectomorphic because it preserves each fiber which is a Lagrangian. Thus the action of $\mathrm{Jac}(C)$ on $\mathcal{H}_C(n,d)$ is globally symplectomorphic. We claim that the first component of the Hitchin map

$$H_1 : \mathcal{H}_C(n,d) \ni (E, \phi) \longmapsto \mathrm{tr}(\phi) \in H^0(C, K_C)$$

is the moment map of this Jacobian action. Note that $H^1(C, \mathcal{O}_C)$ is the Lie algebra of the Abelian group $\mathrm{Jac}(C)$, hence $H^0(C, K_C)$ is the dual Lie algebra. The claim is obvious because ω_I vanishes on each fiber of the Hitchin fibration on which the $\mathrm{Jac}(C)$ action is restricted, and because dH_1 is the 0-map on any infinitesimal deformation of E. Therefore, we can define the symplectic quotient

$$\mathcal{PH}_C(n,d) \overset{\mathrm{def}}{=} \mathcal{H}_C(n,d) /\!\!/ \mathrm{Jac}(C) = H_1^{-1}(0)/\mathrm{Jac}(C). \tag{7.1}$$

It's dimension is $2(n^2-1)(g-1)$. The letter P stands for "projective."

The moment map H_1 being the trace of ϕ, it is natural to define

$$V_{SL} = V_{SL_n(\mathbb{C})} = \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i}) \subset V_{GL}. \tag{7.2}$$

This is a vector space of dimension $(n^2-1)(g-1)$. Since the $\mathrm{Jac}(C)$-action on $\mathcal{H}_C(n,d)$ preserves fibers of the Hitchin fibration, the map H induces a natural map

$$H_{PGL} : \mathcal{PH}_C(n,d) \longrightarrow V_{SL}. \tag{7.3}$$

It's 0-fiber is $H_{PGL}^{-1}(0) = \mathcal{U}_C(n,d)/\mathrm{Jac}(C)$. To study the symplectic quotient (7.1), let us first analyze this 0-fiber. Following [24] we denote by $\mathcal{SU}_C(n,d)$ the moduli space of stable vector bundles with a fixed determinant line bundle. This is a fiber of the determinant map

$$\mathcal{U}_C(n,d) \ni E \longmapsto \det E \in \mathrm{Pic}^d(C), \tag{7.4}$$

and is independent of the choice of the value of the determinant. This fibration is a non-trivial fiber bundle. The equivariant $\mathrm{Jac}(C)$-action on (7.4) is given by

$$\begin{array}{ccc} \mathcal{U}_C(n,d) & \xrightarrow{\otimes L} & \mathcal{U}_C(n,d) \\ {\scriptstyle\det}\big\downarrow & & \big\downarrow{\scriptstyle\det} \\ \mathrm{Pic}^d(C) & \xrightarrow[\otimes L^{\otimes n}]{} & \mathrm{Pic}^d(C) \end{array} \qquad L \in \mathrm{Jac}(C). \tag{7.5}$$

The isotropy subgroup of the $\mathrm{Jac}(C)$-action on $\mathrm{Pic}^d(C)$ is the group of n-torsion points

$$J_n(C) \overset{\mathrm{def}}{=} \{L \in \mathrm{Jac}(C) \mid L^{\otimes n} = \mathcal{O}_C\} \cong H^1(C, \mathbb{Z}/n\mathbb{Z}).$$

Choose a reference line bundle $L_0 \in \mathrm{Pic}^d(C)$ and consider a degree n covering

$$\nu : \mathrm{Pic}^d(C) \ni L \otimes L_0 \longmapsto L^{\otimes n} \otimes L_0 \in \mathrm{Pic}^d(C), \qquad L \in \mathrm{Jac}(C).$$

Then the pull-back bundle $\nu^*\mathcal{U}_C(n,d)$ on $\mathrm{Pic}^d(C)$ becomes trivial:

$$\nu^*\mathcal{U}_C(n,d) = \mathrm{Pic}^d(C) \times \mathcal{SU}_C(n,d).$$

The quotient of this product by the diagonal action of $J_n(C)$ is the original moduli space:

$$\big(\mathrm{Pic}^d(C) \times \mathcal{SU}_C(n,d)\big)\big/J_n(C) \cong \mathcal{U}_C(n,d). \tag{7.6}$$

It is now clear that

$$\mathcal{U}_C(n,d)/\mathrm{Jac}(C) \cong \mathcal{SU}_C(n,d)/J_n(C).$$

The other fibers of (7.3) are best described in terms of *Prym varieties*. Let $s \in V_{SL} \cap U$ be a point such that C_s is irreducible and non-singular. The covering map $\pi : C_s \to C$ induces an injective homomorphism $\pi^* : \mathrm{Jac}(C) \ni L \longmapsto \pi^*L \in \mathrm{Jac}(C_s)$. This is injective because if $\pi^*L \cong \mathcal{O}_{C_s}$, then by the projection formula we have

$$\pi_*(\pi^*L) \cong \pi_*\mathcal{O}_{C_s} \otimes L \cong \bigoplus_{i=0}^{n-1} L \otimes K_C^{\otimes -i},$$

which has a nowhere vanishing section. Hence $L \cong \mathcal{O}_C$. Take a point $(E,\phi) \in H^{-1}(s)$ and let $\mathcal{L}_E$ be the corresponding line bundle on C_s. Since $\pi_*(\mathcal{L}_E \otimes \pi^*L) \cong E \otimes L$, the action of $\mathrm{Jac}(C)$ on $H^{-1}(s) \cong \mathrm{Jac}(C_s)$ is the canonical subgroup action. Thus we conclude that the fiber $H_{PGL}^{-1}(s)$ is isomorphic to the *dual Prym variety* of the covering $C_s \to C$

$$\mathrm{Prym}^*(C_s/C) \overset{\mathrm{def}}{=} \mathrm{Jac}(C_s)/\mathrm{Jac}(C). \tag{7.7}$$

The *Prym variety* $\mathrm{Prym}(C_s/C)$ of the covering is defined to be the kernel of the norm map

$$\mathrm{Nm} : \mathrm{Jac}(C_s) \ni \mathcal{L} \longmapsto \det(\pi_*\mathcal{L}) \otimes (\det \pi_*\mathcal{O}_{C_s})^* \in \mathrm{Jac}(C). \tag{7.8}$$

Both Prym and dual Prym varieties are Abelian varieties of dimension $g(C_s) - g(C)$. Similarly to the equivariant action (7.5), we have

$$\begin{tikzcd} \mathrm{Jac}(C_s) \arrow[r, "\otimes L"] \arrow[d, "\mathrm{Nm}"'] & \mathrm{Jac}(C_s) \arrow[d, "\mathrm{Nm}"] \\ \mathrm{Jac}(C) \arrow[r, "\otimes L^{\otimes n}"'] & \mathrm{Jac}(C) \end{tikzcd} \qquad L \in \mathrm{Jac}(C). \tag{7.9}$$

By the same argument as we used in (7.6), we obtain

$$\big(\mathrm{Prym}(C_s/C) \times \mathrm{Jac}(C)\big)/J_n(C) \cong \mathrm{Jac}(C_s). \tag{7.10}$$

From (7.7) and (7.10), it follows that $\mathrm{Prym}^*(C_s/C) = \mathrm{Prym}(C_s/C)/J_n(C)$. We have thus established

Theorem 7.1. *The fibration $H_{PGL} : \mathcal{PH}_C(n,d) \to V_{SL}$ is a generically Lagrangian dual Prym fibration.*

How can we construct a Lagrangian *Prym* fibration? The dual Prym variety naturally appears in the above discussion when we quotient out the Jacobian action on the moduli space of vector bundles. Another way to limit the Jacobian action is to restrict the structure group of the vector bundles from $GL_n(\mathbb{C})$ to $SL_n(\mathbb{C})$. So let us consider a character variety

$$\mathrm{Hom}(\hat{\pi}_1(C)_{\mathbb{R}}, SL_n(\mathbb{C}))/\!\!/SL_n(\mathbb{C}).$$

Although the central generator $c \in \hat{\pi}_1(C)$ can take the same value as in (4.6), to have a representation of $\hat{\pi}_1(C)_{\mathbb{R}}$, c has to be mapped to the identity. Thus we go back to the untwisted character variety $\mathrm{Hom}(\pi_1(C), SL_n(\mathbb{C}))/\!\!/SL_n(\mathbb{C})$. The argument of Section 5 leads us to the moduli space of stable Higgs pairs (E, ϕ), where this time $\det(E) = \mathcal{O}_C$ and the Higgs field $\phi : E \to E \otimes K_C$ is traceless since $\mathrm{End}(E)$ is an $sl_n(\mathbb{C})$-bundle. Let us denote this moduli space by $\mathcal{SH}_C(n, 0)$. Here the letter S stands for "special." The natural counterpart of the Hitchin fibration on $\mathcal{SH}_C(n, 0)$ is the map

$$H_{SL} : \mathcal{SH}_C(n,0) \ni (E,\phi) \longmapsto \det(x - \phi) \in V_{SL}. \tag{7.11}$$

It's 0-fiber is $H_{SL}^{-1}(0) = \mathcal{SU}_C(n, 0)$. For a generic $s \in V_{SL}$ for which C_s is irreducible and non-singular, obviously we have $H_{SL}^{-1}(s) \cong \mathrm{Prym}(C_s/C)$.

Theorem 7.2 ([13, 4])**.** *The two Lagrangian Abelian fibrations*

$$\begin{tikzcd} \mathcal{SH}_C(n,0) \arrow[d, "H_{SL}"'] & \mathcal{PH}_C(n,d) \arrow[d, "H_{PGL}"] \\ V_{SL} \arrow[r, equal] & V_{SL} \end{tikzcd} \tag{7.12}$$

are ***mirror dual*** *in the sense of Strominger-Yau-Zaslow* [30].

The mirror duality here means that the bounded derived category $D^b(Coh(\mathcal{SH}_C(n, 0)))$ of coherent analytic sheaves on $\mathcal{SH}_C(n, 0)$ is equivalent to the Fukaya category $Fuk(\mathcal{PH}_C(n, d))$ consisting of Lagrangian subvarieties of $\mathcal{PH}_C(n, d)$ and flat U_1-bundles on them [10]. We can view it as a family of deformations of Furier-Mukai duality [21, 27] between $\mathrm{Prym}(C_s/C)$ and $\mathrm{Prym}^*(C_s/C)$ parametrised on the same base space V_{SL}.

As noted at the end of Section 3, $\mathrm{Jac}(C)$ of an algebraic curve C is the moduli space of flat U_1 connections modulo gauge transformation. This correspondence does not require that C is a curve, because the flatness condition automatically implies the integrability of the

$(0,1)$-part of the connection. Since the Abel-Jacobi map $C \to \text{Jac}(C)$ induces a homology isomorphism

$$H_1(C,\mathbb{Z}) \xrightarrow{\sim} H_1(\text{Jac}(C),\mathbb{Z}),$$

we have an isomorphism

$$\text{Pic}^0(\text{Jac}(C)) \xrightarrow{\sim} \text{Jac}(C),$$

because any representation of the fundamental group in U_1 factors through the Abelian group homomorphism from the homology group. Here Pic^0 indicates the moduli of holomorphic line bundles that are topologically trivial. Thus $\text{Jac}(C)$ is *self-dual*. Now consider a flat U_1 connection A on $\text{Prym}^*(C_s/C)$. It is a holomorphic line bundle on $\text{Jac}(C_s)$ that is invariant under the $\text{Jac}(C)$-action. The restriction of A to $C \subset \text{Jac}(C) \subset \text{Jac}(C_s)$ then defines a holomorphic line bundle on C, which is trivial by the assumption. We notice that this correspondence $\text{Jac}(C_s) \to \text{Jac}(C)$ is exactly the norm map of (7.8). In other words, we obtain the duality

$$\text{Pic}^0(\text{Prym}^*(C_s/C)) \cong \text{Prym}(C_s/C). \tag{7.13}$$

A skyscraper sheaf on $\mathcal{SH}_C(n,0)$ supported at a point (E,ϕ) determines a spectral curve C_s and a point on the Prym variety $\text{Prym}(C_s/C)$, where $s = H_{SL}(E,\phi)$. It then identifies a fiber $H_{PGL}^{-1}(s) \cong \text{Prym}^*(C_s/C)$, which is a Lagrangian subvariety of $\mathcal{PH}_C(n,d)$, and a flat U_1-connection on it because of (7.13). This is the idea of geometric realization of mirror symmetry due to Strominger, Yau and Zaslow [30].

Although complex structures are different, we can identify

$$\begin{cases} \mathcal{SH}_C(n,0) \cong \text{Hom}(\pi_1(C), SL_n(\mathbb{C}))/\!\!/SL_n(\mathbb{C}) \\ \mathcal{PH}_C(n,0) \cong \text{Hom}(\pi_1(C), PGL_n(\mathbb{C}))/\!\!/PGL_n(\mathbb{C}). \end{cases} \tag{7.14}$$

Then the mirror symmetry (7.12) gives a manifestation of *geometric Langlands correspondence* [4, 13, 18], which is a family of Fourier-Mukai duality transformations over the same base space [7]. Thus *the Hitchin integrable systems on character varieties relate the SYZ mirror symmetry and the geometric Langlands correspondence*.

We have noted earlier that $\mathcal{H}_C(n,d)$ has two different complex structures I and J. The complex structure I comes from the moduli space of stable Higgs bundles, and J from a connected component $\mathcal{X}(\mathbb{C})$ of the twisted character variety $\text{Hom}(\hat{\pi}_1(C)_\mathbb{R}, GL_n(\mathbb{C}))/\!\!/GL_n(\mathbb{C})$. The complex manifold $\mathcal{U}_C(n,d)$, assuming $G.C.D.(n,d) = 1$, is projective algebraic, hence has a unique Kähler metric. The Kähler form in a real coordinate is a real symplectic form, which extends to a holomorphic symplectic form ω_J on the complexification $\mathcal{X}(\mathbb{C})$ of $\mathcal{U}_C(n,d)$. Thus ω_J^N defines a holomorphic top form on $\mathcal{X}(\mathbb{C})$, where $N = \dim_\mathbb{C} \mathcal{U}_C(n,d)$. We can then think of $(\mathcal{X}(\mathbb{C}), J, \omega_J^N, \omega_I)$ as a $2N$-dimensional Calabi-Yau manifold. The Hitchin fibration is an example of a *special Lagrangian fibration*, meaning that the restriction of ω_J^N on each fiber $H^{-1}(s)$ gives a Riemannian volume form on $\text{Jac}(C_s)$. Since

$$p : H^{-1}(s) \cong \text{Jac}(C_s) \longrightarrow \mathcal{U}_C(n,d)$$

is a finite covering of degree $2^{3(g-1)} \cdot 3^{5(g-1)} \cdots n^{(2n-1)(g-1)}$ [2], a generic fiber $H^{-1}(s)$ has the same Riemannian volume that is equal to $2^{3(g-1)} \cdot 3^{5(g-1)} \cdots n^{(2n-1)(g-1)}$-times the Kähler volume of $\mathcal{U}_C(n,d)$. Actually, the space $\mathcal{H}_C(n,d) = \mathcal{X}(\mathbb{C})$ is a *hyper Kähler manifold* with complex structures I, J, and $K = IJ$.

Kapustin and Witten [18] noticed that the mirror symmetry (7.12) is a consequence of the dimensional reduction of 4-dimensional super Yang-Mills theory. In their formulation, the Langlands duality corresponds to the physical electro-magnetic duality, and the Fourier-Mukai transform on each fiber of the Hitchin fibrations is the T-duality.

Finally, let us comment on the relation between the Hitchin systems, Prym varieties, and Sato Grassmannians established in [16, 19, 20]. A theorem of [20] basically states that to any morphism $\pi : C_s \to C$ between algebraic curves, a solution of a KP-type integrable system (the n-component KP equations and more general Heisenberg KP equations) is constructed with the following two properties: a) the orbit of the dynamical system on the Grassmannian is the Prym variety $\mathrm{Prym}(C_s/C)$; and b) the evolution equations are linearlized on the Prym variety. To make a connection between the Hitchin integrable systems and the theory of [20], we need to quotient out the trivial deformations of spectral curves $\{C_s\}_{s\in V_{GL}}$ given by a scalar action

$$V_{GL} = \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}) \ni (s_1, s_2, \dots, s_n) \longmapsto (\lambda s_1, \lambda^2 s_2, \dots, \lambda^n s_n) \in V_{GL} \tag{7.15}$$

for $\lambda \in \mathbb{C}^*$. This action corresponds to the scalar multiplication of a Higgs field $\phi \mapsto \lambda \cdot \phi$, which is *not* a symplectomorphism on the Hitchin moduli space because it changes the symplectic form to $\lambda \cdot \omega_I$. Let us define the projective Hitchin moduli space

$$\mathbb{P}(\mathcal{H}_C(n,d)) = \big(\mathcal{H}_C(n,d) \setminus H^{-1}(0)\big)/\mathbb{C}^*.$$

This is no longer a holomorphic symplectic manifold, yet the Hitchin fibration naturally descends to a generically Jacobian fibration

$$H_{GL}^{\mathbb{P}} : \mathbb{P}(\mathcal{H}_C(n,d)) \longrightarrow \mathbb{P}_w(V_{GL})$$

over the weighted projective space of V_{GL} defined by (7.15). Now we have

Theorem 7.3 ([16])**.** *There is a rational map ι from $\mathbb{P}_w(V_{GL})$ into the Grassmannian of* [20] *such that*

(1) *ι is generically an embedding;*
(2) *the orbit of the n-component KP equations starting at $\iota(\mathbb{P}_w(V_{GL}))$ in the Grassmannian is birational to $\mathbb{P}(\mathcal{H}_C(n,d))$; and*
(3) *the dynamical system on $\mathbb{P}(\mathcal{H}_C(n,d))$ defined by the Hitchin integrable system is the pull-back of the n-component KP equations via ι.*

A similar theorem holds for the Prym fibration (7.11), where we use the traceless n-component KP equations to produce Prym varieties as orbits.

There is a common belief coming out of the recent developments on character varieties. It is that to fully appreciate the categorical equivalences of the dualities such as mirror symmetry and geometric Langlands correspondence, the moduli theory based on stable objects is not the right language. We are naturally led to considering *moduli stack* of vector bundles and other categorical objects. Infinite-dimensional geometry of connections [1, 14] played an essential role in understanding the geometry of moduli spaces of stable vector bundles. Infinite-dimensional Sato Grassmannians are more suitable geometric objects for algebraic stacks. Although our current understanding of the relation between the Hitchin systems and Sato Grassmannians is limited, more should be coming as our understanding of the duality deepens from this point of view.

Acknowledgement. I would like to thank the organizers of the RIMS-OCAMI joint conference for their invitation and *exceptional* hospitality during my stay in Kyoto and Nara.

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Department of Mathematics, University of California, Davis, CA 95616–8633
E-mail address: mulase@math.ucdavis.edu