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Duality of Orthogonal and Symplectic Matrix Integrals and Quaternionic Feynman Graphs

Motohico Mulase*, **Andrew Waldron****

Department of Mathematics, One Shields Avenue, University of California, Davis, CA 95616–8633, USA. E-mail: mulase@math.ucdavis.edu; wally@math.ucdavis.edu

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Abstract: We present an asymptotic expansion for quaternionic self-adjoint matrix integrals. The Feynman diagrams appearing in the expansion are ordinary ribbon graphs and their non-orientable counterparts. We show that the $2N \times 2N$ Gaussian Orthogonal Ensemble (GOE) and $N \times N$ Gaussian Symplectic Ensemble (GSE) have exactly the same expansion term by term, except that the contributions from graphs on a non-orientable surface with odd Euler characteristic carry the opposite sign. As an application, we give a new topological proof of the known duality for correlations of characteristic polynomials, demonstrating that this duality is equivalent to Poincaré duality of graphs drawn on a compact surface. Another consequence of our graphical expansion formula is a simple and simultaneous (re)derivation of the *Central Limit Theorem* for GOE, GUE (Gaussian Unitary Ensemble) and GSE: The three cases have exactly the same graphical limiting formula except for an overall constant that represents the type of the ensemble.

Contents

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1. Introduction

The purpose of this paper is to establish an asymptotic expansion for quaternionic self-adjoint matrix integrals in terms of Feynman diagrams and to give a new topological proof of the various characteristic polynomial dualities discovered by [2, 6, 7, 17].

Recent developments in the theory of random matrices exhibit particularly rich structures. Although originally introduced by Wigner as a model for heavy nuclei, random matrices appear almost ubiquitously in modern mathematics. Mathematical applications pertain, for example, to number theory, combinatorics, probability theory, and geometry of moduli spaces of Riemann surfaces (see for example, [1, 3, 10, 11, 15, 16, 21, 25, 27] and articles in [4] and references cited therein). In physics, 't Hooft's discovery [24] that quantum chromodynamics (QCD) simplifies in the limit where the number of colors N (*i.e.* gauge group $SU(N)$) is large relied on a graphical expansion in terms of "fat" or "ribbon" graphs. Hermitian matrix integrals appear in this context as generating functions for oriented ribbon graphs [3, 5].

Graphical expansions of gauge theories with other gauge groups were studied in the early 1980s (for example, see [18, 8, 9]). In particular, it was recognized that $SO(2N)$-gauge theory and $Sp(N)$-gauge theory are identical in their graphical expansions, except that the parameter N in the $SO(2N)$-theory has to be replaced with $-N$ [18]. This *duality* is also noted in more recent works (see for example, [20, 28]). A characteristic feature that distinguishes these gauge theories from the $SU(N)$-gauge theory is the appearance of graphs drawn on non-orientable surfaces. Real symmetric matrix integrals have been used as generating functions for these non-orientable ribbon graphs [10, 14, 23, 26]. Although the integration in the GOE and GSE matrix integrals is over real and quaternionic self-adjoint matrices, rather than $so(2N)$ and $sp(N)$ Lie algebra valued fields of the gauge theory case, on the basis of the $Sp(N)$-gauge theory and "$SO(-2N)$-gauge theory" equivalence, an $N \times N$ quaternionic self-adjoint matrix integral should also give a generating function for non-orientable ribbon graphs identical with the $2N \times 2N$ real symmetric matrix integral, with the parameter N replaced by $-N$. We show that this is indeed the case and our method also implies a simple graphical proof of the gauge theory result of [18]. As discussed in the Conclusion, since our proof is based on the construction of a new topological invariant of punctured surfaces, it generalizes to a large class of models.

In this article, we develop a graphical expansion technique for an $N \times N$ self-adjoint quaternionic matrix integral, and directly verify its duality with a real symmetric matrix integral of size $2N$. As an immediate consequence of the graphical expansion formulas, we give a new topological proof of the known duality for k-fold correlations of characteristic polynomials of $N \times N$ matrices for *Gaussian Orthogonal, Gaussian Unitary,* and *Gaussian Symplectic Ensembles* [2, 6, 7, 17]. For the GUE model expressed in terms of ribbon graphs, this N-k duality [6, 7] is precisely the Poincaré duality of graphs drawn on a compact oriented surface. Similarly, the relation between GOE and GSE correlations stems from the combination of Poincaré duality, this time including non-orientable surfaces, and our graphical expansion formula. It is interesting to note that the machinery of fermionic integrations employed in [6, 7] is equivalent to a very simple switch from a graph on a surface to its dual graph.

An $N \times N$ matrix X with entries in the ring of quaternions

$$\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$$

is *self-adjoint* if $X^\top = \overline{X}$, where $\overline{X}$ denotes the quaternionic conjugation of X defined by

$$\overline{x + iy + jz + kw} = x - iy - jz - kw \in \mathbb{H}.$$

Our result for the self-adjoint quaternionic matrix integral is

$$\log \left(\frac{\int [dX] \exp\left(- N \operatorname{tr} X^2 + \sum_j \frac{2N t_j}{j} \operatorname{tr} X^j \right)}{\int [dX] \exp(-N \operatorname{tr} X^2)} \right)$$
$$= \sum_{\Gamma \in \mathfrak{G}} \frac{(-2N)^{\chi(S_\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_j t_j^{v_\Gamma^{(j)}}. \tag{1.1}$$

Exact conventions are given later, at present it suffices to indicate that the sum is over all graphs Γ drawn on compact orientable and non-orientable surfaces S_Γ, and $\chi(S_\Gamma)$ is the Euler characteristic of the surface S_Γ uniquely defined by the graph Γ. Our proof of this result is based on viewing the GOE integral over real symmetric matrices as fundamental. The crucial observation is that the contribution of any given graph Γ in the GOE, GUE and GSE is a topological invariant of the surface S_Γ with f_Γ *marked points* on it, where f_Γ denotes the number of faces of the cell-decomposition of S_Γ defined by the graph Γ. The connectivity of the space of triangulations of two dimensional surfaces then allows any graphical contribution to be calculated from a simple representative graph for any given topology.

Writing the results for all three ensembles in a uniform notation (see (5.2)) makes the expected duality

$$\begin{aligned} \mathrm{GOE} &\longleftrightarrow \widetilde{\mathrm{GSE}} \\ \mathrm{GUE} &\longleftrightarrow \mathrm{GUE} \\ \mathrm{GSE} &\longleftrightarrow \widetilde{\mathrm{GOE}} \end{aligned} \tag{1.2}$$

manifest. The middle line for the GUE is a (trivial) self-duality. The tilde on the right hand side indicates that equality holds upon doubling/halving the matrix size and an overall sign change for contributions of graphs where the Euler characteristic $\chi(S_\Gamma)$ is odd.

Although not discussed in the main text, it is indeed possible to generalize the usual Schwinger trick to quaternionic source terms, and represent a Gaussian symplectic integral as non-commutative quaternionic differentiations. The result is a sum over both orientable and non-orientable ribbon graphs, and is easily verified to agree with ours for simple graphs. This method is described in Appendix A. We also note that a partial result for quaternionic expansions has been obtained in [13].

If we reduce our integral (1.1) to a symplectic Penner model by setting $t_1 = t_2 = 0$ and

$$t_j = -z^{\frac{j}{2}-1}, \quad j \geq 3,$$

then we can explicitly compute the asymptotic expansion in z of

$$\lim_{m\to\infty} \log \left(\frac{\int_{\mathbb{R}^N} |\Delta(k)|^{2\alpha} \prod_{i=1}^N \exp\left(-\sum_{j=2}^{2m} \frac{k_i^j}{j} \left(\frac{z}{\alpha N}\right)^{j/2-1} \right) dk_i}{\int_{\mathbb{R}^N} |\Delta(k)|^{2\alpha} \prod_{i=1}^N \exp\left(-\frac{k_i^2}{2} \right) dk_i} \right) \tag{1.3}$$

utilizing the Selberg integration formula and the asymptotic analysis technique of [19], where α is either a positive integer or its reciprocal. We demonstrate that the duality for GOE and GSE of (1.1) extends to the same type of duality between an arbitrary positive integer α and $1/\alpha$ for (1.3) with the sign change for all terms with odd powers of N in the asymptotic expansion.

The orthogonal Penner model gives the orbifold Euler characteristic of the moduli spaces of smooth real algebraic curves with an arbitrary number of marked points [10]. Also, the original Penner model [21] provides the orbifold Euler characteristic of the moduli spaces of pointed algebraic curves over $\mathbb{C}$ [11]. The GOE-GSE duality shows that the symplectic Penner model is identical to the orthogonal Penner model, except for doubling the matrix size and an overall sign change for contributions from surfaces of odd Euler characteristic.

From the graphical expansion formulas for matrix integrals, one can uniformly derive the *Central Limit Theorem* for Gaussian random matrix ensembles. This result follows as a direct consequence of 't Hooft's original large N limit in which planar ribbon graphs dominate: we derive a precise limiting formula for GOE, GUE and GSE matrix ensembles in terms of planar two-vertex ribbon graphs. The formula is the same for all three ensembles except for an overall constant, which is parallel to the equivalence of $SO(N)$, $SU(N)$ and $Sp(N)$-gauge theories it large N.

The material is organized as follows: In Sect. 2 we introduce the matrix integrals studied in this paper. Our conventions for the topological data of surfaces are given in Sect. 3 as well as our theorem and its proof for the graphical expansion of matrix integrals. Examples, including a comparison with the first few terms of the Penner model, are given in Sect. 4. The GOE-GSE duality appears in Sect. 5. The central formula of this paper is Eq. (5.2) which gives the graphical expansion for the GOE, GUE and GSE simultaneously in a manifestly duality invariant form. Its application to characteristic polynomial duality is in Sect. 6. The extended version of the duality for Penner type models is found in Sect. 7 while detailed derivations of the formulæ there are presented in Appendix B. Section 8 concerns the *Central Limit Theorem* for Gaussian random matrix ensembles. In the Conclusions (Sect. 9) we discuss possible further generalizations. In particular, the construction of a graphical topological invariant of surfaces with marked points necessary for the proof of our main result is rather general and may be applied to higher algebraic structures.

2. Matrix Integrals

The object of our study is the integral over self-adjoint matrices[1]

$$Z^{(\beta)}(t, N) = \frac{\int [dX]_{(\beta)} \exp\left(-\frac{1}{4} \operatorname{tr} X^2 + \sum_{j=1}^{\infty} \frac{t_j}{2j} \operatorname{tr} X^j \right)}{\int [dX]_{(\beta)} \exp\left(-\frac{1}{4} \operatorname{tr} X^2 \right)} \tag{2.1}$$

[1] We employ various normalizations throughout the paper, so it is convenient to divide through by the free matrix integral.

as a function of the "coupling constants" $t = (t_1, t_2, t_3, \dots)$ and the size N of the matrix variable X. Here

$$X = S + \sum_{i=1}^{\beta-1} e_i A_i \tag{2.2}$$

is built from real, $N \times N$, symmetric and antisymmetric matrices S and A_i, respectively. The parameter β takes values 1, 2 or 4 depending on whether we study real, complex or quaternionic self-adjoint matrices and in turn Gaussian orthogonal, unitary or symplectic ensembles (GOE, GUE, GSE). The imaginary units e_i are then drawn from one of three sets,

$$e_i \in \begin{cases} \emptyset\,, & \beta = 1\,, \\ \{i : i^2 = -1\}\,, & \beta = 2\,, \\ \{i, j, k : i^2 = j^2 = k^2 = ijk = -1\}\,, & \beta = 4\,. \end{cases} \tag{2.3}$$

The self-adjoint condition

$$X^\dagger \equiv \overline{X}^\top = X\,, \qquad \overline{e_i} = -e_i\,, \tag{2.4}$$

is implied by antisymmetry of the matrices A_i. Finally, the measure $[dX]_{(\beta)}$ is the translation invariant Lebesgue measure of the vector space of real dimension $\frac{1}{2}N(\beta(N-1)+2)$ spanned by independent matrix elements of S and A_i. This measure is invariant, respectively, under orthogonal, unitary and symplectic transformations

$$X \longmapsto U^\dagger X U, \tag{2.5}$$

where $U^\dagger U = 1$ and $U = U_0 + \sum_{i=1}^{\beta-1} e_i U_i$ for real $N \times N$ matrices U_0 and U_i.

The matrix integral (2.1) is a holomorphic function in $(t_1, t_2, \dots, t_{2m})$ if we fix $m > 0$ and restrict the coupling constants to satisfy

$$\mathrm{Re}(t_{2m}) < 0 \qquad \text{and} \qquad t_{2m+1} = t_{2m+2} = t_{2m+3} = \cdots = 0.$$

Under this restriction, $Z^\beta(t, N)$ has a unique Taylor expansion in $(t_1, t_2, \dots, t_{2m-1})$ and an *asymptotic* expansion in t_{2m} as $t_{2m} \to 0$ while keeping $\mathrm{Re}(t_{2m}) < 0$. Let us introduce a weighted degree of the coupling constants by $\deg(t_n) = n$. Then the asymptotic expansion of the truncated integral $Z^\beta(t, N)$ has a well-defined limit as $m \to \infty$ in the ring

$$(\mathbb{Q}[N])\,[[t_1, t_2, t_3, \dots]]$$

of formal power series in infinitely many variables with coefficients in the polynomial ring of N with rational coefficients [19]. The subject of our study in what follows is this asymptotic expansion of $Z^\beta(t, N)$ as a function in infinitely many variables.

3. Graphical Expansion

The graphs appearing in our asymptotic expansion of the matrix integrals (2.1) are those drawn on orientable as well as *non-orientable* surfaces. To avoid confusion with an already well-established convention that ribbon graphs are drawn on orientable surfaces, we propose the terminology *Möbius graphs*. Let us recall that a ribbon graph Γ is a graph with a cyclic order chosen at each vertex for half-edges adjacent to it. Equivalently, it is a graph drawn on a compact oriented surface S giving a cell-decomposition of it. The complement $S \setminus \Gamma$ of the graph Γ on S is the disjoint union of f_Γ open disks (or faces) of the surface. Since a ribbon graph Γ defines a unique oriented surface on which it is drawn as the 1-skeleton of a cell-decomposition, we denote the surface by S_Γ.

Similarly, a Möbius graph is drawn on a compact surface, orientable or non-orientable, giving a cell-decomposition of the surface. It can be viewed as a ribbon graph with twisted edges. A Möbius graph Γ also uniquely defines the surface S_Γ in which it is embedded.

Let $\mathfrak{G}$ be the set of connected Möbius graphs. A graph $\Gamma \in \mathfrak{G}$ consists of a finite number of vertices and edges. Let $v_\Gamma^{(j)}$ denote the number of j-valent vertices of Γ. Then the number of vertices and edges are given by

$$v_\Gamma = \sum_j v_\Gamma^{(j)} \qquad \text{and} \qquad e_\Gamma = \frac{1}{2} \sum_j j v_\Gamma^{(j)} . \tag{3.1}$$

The unique compact surface S_Γ has f_Γ faces and its Euler characteristic is

$$\chi(S_\Gamma) = v_\Gamma - e_\Gamma + f_\Gamma . \tag{3.2}$$

We will also need the number of faces with a given number of edges, so denote the number of j-gons in the cell-decomposition of S_Γ by $f_\Gamma^{(j)}$ whereby the Poincaré dual formulæ to Eqs. (3.1) are

$$f_\Gamma = \sum_j f_\Gamma^{(j)} \qquad \text{and} \qquad e_\Gamma = \frac{1}{2} \sum_j j f_\Gamma^{(j)} . \tag{3.3}$$

A Möbius graph Γ also determines the orientability of S_Γ and we define

$$\natural_\Gamma = \begin{cases} 1 & S_\Gamma \text{ orientable}, \\ -1 & S_\Gamma \text{ non-orientable}. \end{cases} \tag{3.4}$$

By *genus* of S_Γ we mean

$$g(S_\Gamma) = 1 - 2^{-\frac{1+\natural_\Gamma}{2}} \chi(S_\Gamma) . \tag{3.5}$$

Thus $\chi(S_\Gamma) = 2 - 2g(S_\Gamma)$ for an orientable surface and $\chi(S_\Gamma) = 1 - g(S_\Gamma)$ if it is non-orientable. We also define the parity of $\chi(S_\Gamma)$ by

$$\sharp_\Gamma = (-1)^{\chi(S_\Gamma)} , \tag{3.6}$$

while many results can be written compactly in terms of

$$\Sigma_\Gamma = \frac{1}{2}\Big(1 + \sharp_\Gamma\Big) - \natural_\Gamma = \begin{cases} 0 & \Gamma \text{ orientable}, \\ 1 & \Gamma \text{ non-orientable}, \ \chi(S_\Gamma) \text{ odd}, \\ 2 & \Gamma \text{ non-orientable}, \ \chi(S_\Gamma) \text{ even}. \end{cases} \tag{3.7}$$

Our main result is:

Theorem 3.1. *The logarithm of the asymptotic expansion of the matrix integral* $Z^{(\beta)}(t, N)$ *is expressed as a sum over connected Möbius graphs:*

$$\log\left(Z^{(\beta)}(t,N)\right) = \sum_{\Gamma\in\mathfrak{G}} \frac{(-4+6\beta-\beta^2)^{1-\frac{1}{2}\Sigma_\Gamma-\frac{1}{2}\chi(S_\Gamma)}(2-\beta)^{\Sigma_\Gamma}\beta^{f_\Gamma-1}N^{f_\Gamma}}{|\mathrm{Aut}(\Gamma)|}\prod_j t_j^{v_\Gamma^{(j)}}$$
$$\in (\mathbb{Q}[N])\,[[t_1,t_2,t_3,\ldots]]. \tag{3.8}$$

Remark.

(1) We define $(2-\beta)^{\Sigma_\Gamma} = 1$ when $\beta = 2$ and $\Sigma_\Gamma = 0$.
(2) For every connected Möbius graph Γ, the monomial $\prod_j t_j^{v_\Gamma^{(j)}}$ is a finite product of total degree $2e_\Gamma$.
(3) The only reason to consider $\log Z^\beta(t,N)$ is because it yields a compact formula (3.8): $Z^\beta(t,N)$ itself has an expansion in terms of graphs although a given summand may have a mixture of orientable and non-orientable connected components.

The automorphism group $\mathrm{Aut}(\Gamma)$ of a Möbius graph Γ is a group of automorphisms of the cellular complex S_Γ consisting of v_Γ vertices, e_Γ edges and f_Γ faces. When S_Γ is orientable, the group $\mathrm{Aut}(\Gamma)$ may contain orientation-reversing automorphisms as well. We note that a cyclic rotation of half-edges around a vertex corresponds to the invariance of the trace under a cyclic permutation

$$\mathrm{tr}\,(M_1M_2M_3\cdots M_n) = \mathrm{tr}\,(M_nM_1M_2\cdots M_{n-1}), \tag{3.9}$$

and an orientation-reversing flip of vertex with adjacent half-edges corresponds to the invariance of the trace of symmetric matrices

$$\mathrm{tr}\,(S_1S_2S_3\cdots S_n) = \mathrm{tr}\,(S_nS_{n-1}S_{n-2}\cdots S_1) \tag{3.10}$$

reversing the order of multiplication.

The proof of the theorem involves two main ingredients. The first is to view GUE and GSE matrix integrals as the coupling of a singlet or triplet of skew-symmetric matrix integrals to the fundamental GOE integral. When $\beta = 1$, Eq. (3.8) is the Möbius graphical expansion of a symmetric matrix integral [10, 26, 23, 14]:

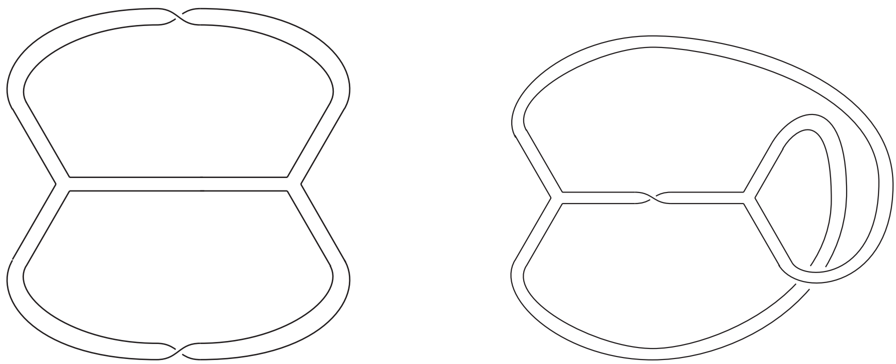


Fig. 3.1. Two equivalent Möbius graphs consisting of two vertices, three edges, and one face. The graphs are interchanged by a *vertex flip*

$$\log\left(Z^{(\beta=1)}(t,N)\right) = \sum_{\Gamma\in\mathfrak{G}} \frac{N^{f_\Gamma}}{|\mathrm{Aut}(\Gamma)|} \prod_j t_j^{v_\Gamma^{(j)}} . \tag{3.11}$$

This formula follows immediately from the fact that the Wick contraction of any pair of symmetric matrices $S = (S_{ab})$ obeys

$$\langle S_{ab} S_{cd}\rangle = \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} , \tag{3.12}$$

which is denoted graphically as an edge of a Möbius graph (see Fig. 3.2). These edges connect vertices of the type

$$\frac{t_j}{2j} \operatorname{tr} S^j = \frac{t_j}{2j} \sum_{a_1,\dots,a_j=1}^{N} S_{a_1a_2} S_{a_2a_3} \cdots S_{a_ja_1} , \tag{3.13}$$

as depicted in Fig. 3.3. The factor $(2j)^{-1}$ in (3.13) is precisely the one required to cancel the over-counting implied by the identities (3.9) and (3.10). Therefore the overall weight of any given graph is as quoted in (3.11).

In contrast, for a pair of antisymmetric matrices $A = (A_{ab})$ the Wick contraction yields

$$\langle A_{ab} A_{cd}\rangle = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} . \tag{3.14}$$

This is depicted in Fig. 3.4.

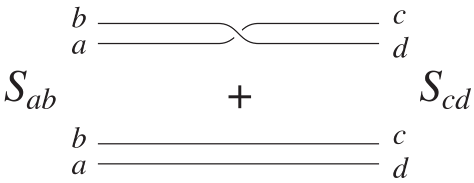


Fig. 3.2. Propagator for symmetric matrices

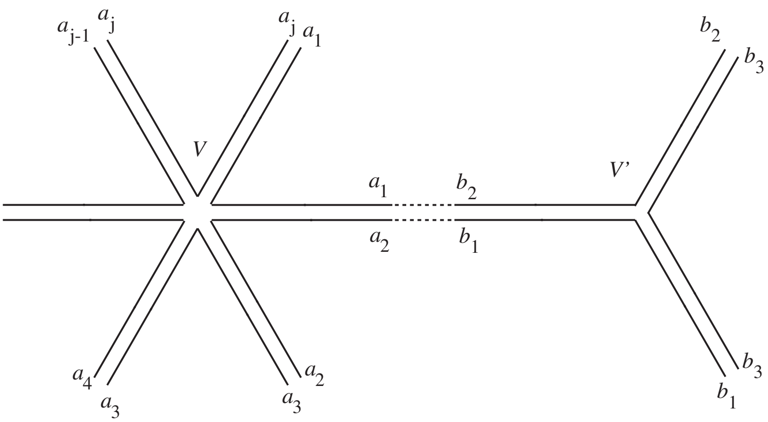


Fig. 3.3. Two vertices connected at the half edge labeled by a_1a_2 to the half edge labeled by b_2b_1. For the GOE a second graph with a twisted edge connecting a_1a_2 to b_1b_2 is also present since the ribbon edges are not directed

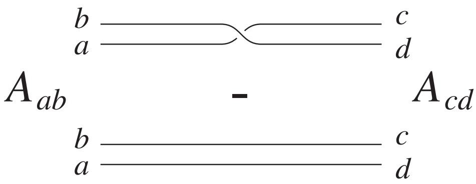


Fig. 3.4. Propagator for antisymmetric matrices

The minus sign for untwisted edges will play a crucial rôle in what follows. Notice that since the exponent of the Gaussian part of (2.1) is

$$-\frac{1}{4}\sum_{a=1}^{N} S_{aa}^2 - \frac{1}{2}\sum_{1\le a<b\le N}\left(S_{ab}^2 + \sum_{i=1}^{\beta-1} A_{iab}^2\right), \tag{3.15}$$

there are no non-vanishing Wick contractions between symmetric and antisymmetric matrices. Therefore the model (2.1) is a sum over graphs with edges of either of the two types (3.12) or (3.14). Furthermore $\langle A_i A_j\rangle$ is only non-vanishing when $i = j$ so edges "emitted" at vertices are only connected when they carry the same imaginary units e_i.

If we call $e_0 = 1$ and $A_0 = S$, a j-valent vertex now looks like

$$\frac{1}{2j}\,\mathrm{tr}\,X^j = \frac{1}{2j}\,\mathrm{tr}\sum_{\alpha=0}^{\beta-1} e_\alpha A_\alpha = \sum_{\{(\alpha_1,\dots,\alpha_j)\}/\sim}\frac{\rho(\alpha_1,\dots,\alpha_j)}{2j}\,\mathrm{tr}\prod_{k=1}^{j} A_{\alpha_k} e_{\alpha_k}\,. \tag{3.16}$$

The last sum runs over all j-tuples of integers $0, 1, \dots, \beta-1$ modulo the equivalence relation

$$(\alpha_1,\dots,\alpha_j) \sim (\beta_1,\dots,\beta_j) \quad \text{iff} \quad \mathrm{tr}\,(A_{\alpha_1}\cdots A_{\alpha_j}) = \mathrm{tr}\,(A_{\beta_1}\cdots A_{\beta_j})\,. \tag{3.17}$$

The multiplicities $\rho(\alpha_1,\dots,\alpha_j)$ are non-zero *only* when the product

$$e_{\alpha_1}\cdots e_{\alpha_j} = \pm 1\,. \tag{3.18}$$

In other words, the vertex (3.16) is real. Observe that the numbers

$$\frac{2j}{\rho(\alpha_1,\dots,\alpha_j)} \tag{3.19}$$

count the number of automorphisms of a vertex with a given configuration of units e_α sprinkled at every edge, with respect to rotations and vertex flips. For example, for $\beta = 4$ and $j = 3$,

$$\begin{aligned}\frac{1}{6}\,\mathrm{tr}\,X^3 = \frac{1}{6}\,&\mathrm{tr}\,(1.S)^3\\ &+\frac{1}{2}\,\mathrm{tr}\left(1.S\; i.A_1\; i.A_1 + 1.S\; j.A_2\; j.A_2 + 1.S\; k.A_3\; k.A_3\right)\\ &+\frac{1}{1}\,\mathrm{tr}\left(i.A_1\; j.A_2\; k.A_3\right).\end{aligned} \tag{3.20}$$

On the first line, the trace of three identical symmetric matrices can be cycled and reversed, so $2j = 6$ is the correct automorphism factor. On the second line the terms with a single symmetric and a pair of antisymmetric matrices can only be flipped yielding two automorphisms. The term on the third line has no automorphisms. We will keep track of the matrix type by "sprinkling" units $\{1, e_i\}$ over the set of Möbius graphs (see Fig. 3.5).

Orchestrating the above observations yields the following:

Lemma 3.2. *The matrix integral* $\log\left(Z^{(\beta)}(t, N)\right)$ *may be computed as a sum over Möbius graphs* Γ *with weight*

$$\frac{\mu_\Gamma N^{f_\Gamma}}{|\mathrm{Aut}(\Gamma)|} \tag{3.21}$$

multiplied by a single power of t_j *for each* j*-valent vertex in* Γ*. The factor* μ_Γ *is calculated by*

(1) Writing down all possible configurations of units $e_\alpha \in \{1, e_1, \dots e_{\beta-1}\}$ *at each vertex such that their product is* ± 1*.*
(2) Counting these signed configurations with an additional minus sign for every untwisted edge with imaginary units e_i *at each end.*

The proof of this lemma follows from our previous remarks and by examining the Wick contraction for antisymmetric matrices (3.14). A sample computation of μ_Γ is given in Fig. 3.5.

Our proof is completed by computing the numbers μ_Γ, which requires a second major ingredient:

Lemma 3.3. *The quantity* μ_Γ *is a topological invariant of the* n*-punctured surface* $S_\Gamma \setminus \{p_1, p_2, \dots, p_n\}$*, where the removed points are all distinct and* $n = f_\Gamma$ *is the number of faces of the cell decomposition defined by* Γ*.*

The proof of this crucial lemma is almost a triviality. Graphs are identified if they are equivalent under rotations and flips of their vertices (Fig. 3.1), so it suffices to show that μ_Γ is invariant under contraction of an untwisted edge. This operation is depicted in Fig. 3.6.

This relation is obvious since the products of units at either vertex must be real and the Wick contraction corresponding to an untwisted edge with units e_α at either end comes with an overall factor $+1$. We remark that an analogous relation involving a vertex flip for the contraction of a twisted edge follows.

We now compute the invariant μ_Γ for a graph of arbitrary topology. Since μ_Γ is topological, we may employ a standard graph for each distinct topology labeled by the orientability $\natural_\Gamma$, number of faces f_Γ and genus $g(S_\Gamma)$. The key theorem from topology we need here is the connectivity of the space of all triangulations of a compact surface with a fixed number of vertices under the action of a *diagonal flip* [12] (see Fig. 3.7). A diagonal flip of a triangulation is exactly the *fusion move* of the dual Möbius graph.

For orientable surfaces, the connectivity of concern implies the path connectivity of the moduli space $\mathfrak{M}_{g,n}$ of smooth complete algebraic curves defined over $\mathbb{C}$ with n marked points. The connectivity of triangulations of non-orientable surfaces has been also established.

Now note that because of the invariance of μ_Γ under edge contraction, we can expand all vertices of valence greater than 3 and contract all vertices of valence 1 and 2 to create

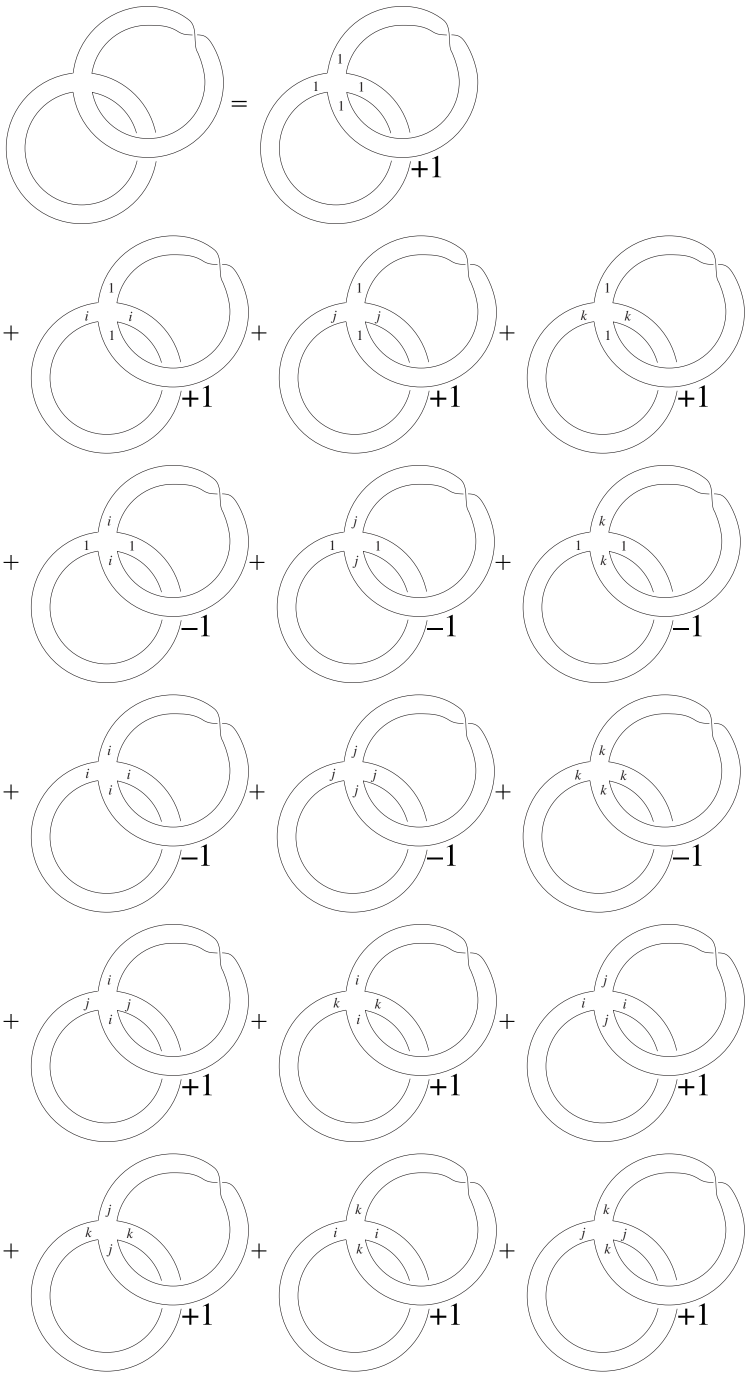


Fig. 3.5. Computation of μ_Γ for a non-orientable Möbius graph Γ. Here $\chi(S_\Gamma) = 0$. The surface S_Γ is a Klein bottle. The contributions ± 1 add to yield $\mu_\Gamma = 4 = (2-\beta)^2$ (for $\beta = 4$)

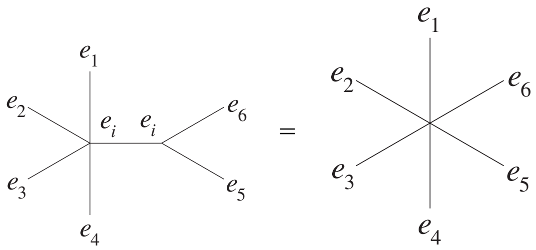


Fig. 3.6. Edge contraction

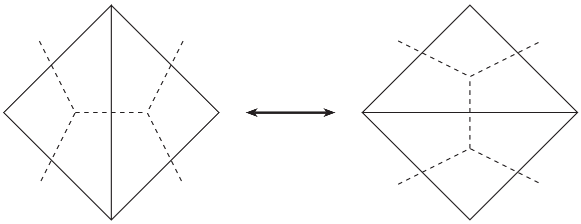


Fig. 3.7. The diagonal flip operation for a triangulation

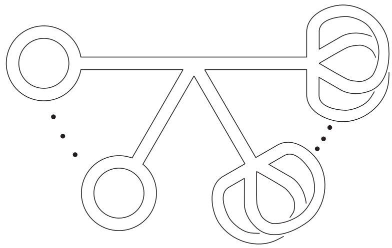
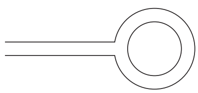


Fig. 3.8. A standard Möbius graph that represents an orientable surface of genus g with n marked points. It consists of $n-1$ tadpoles on the left and g bi-petal flowers on the right

a trivalent graph. Since the dual of a trivalent graph is a triangulation of the surface and the number of vertices of the triangulation is the number $n = f_\Gamma$ of faces of the Möbius graph Γ, the connectivity of the space of triangulations implies that our invariant μ_Γ is a constant for each topology of a given n-punctured surface. In the actual computation, it is convenient to use the following representatives for each of the three topological classes:

(1) Orientable ($\natural = 1$); a standard graph is given in Fig. 3.8.
(2) Non-orientable ($\natural = -1$), odd genus; with standard graph given in Fig. 3.9.
(3) Non-orientable ($\natural = -1$), even genus; a standard graph is in Fig. 3.10.

Finally, it is easy to calculate μ_Γ for each one particle irreducible component (subgraphs which remain connected after cutting a single edge):

 $= \beta\,,$ (3.22)

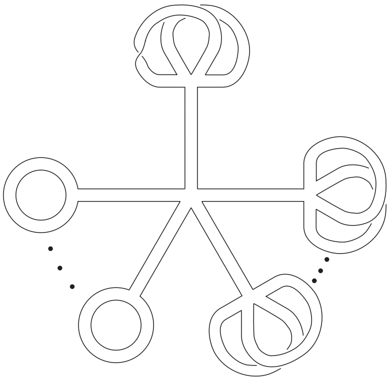


Fig. 3.9. A standard Möbius graph drawn on a non-orientable surface of genus $g = 2k+1$ with n marked points. In addition to $n-1$ tadpoles on the left it has k orientable bi-petal flowers on the right and a non-orientable one added at the top

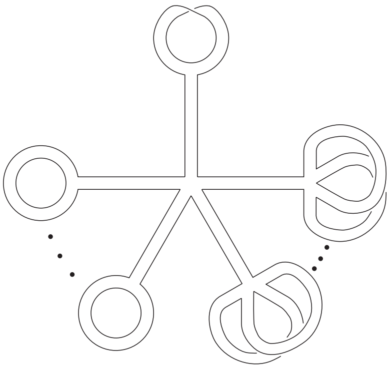
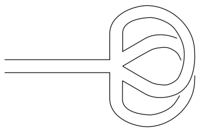
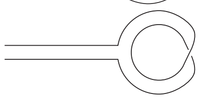


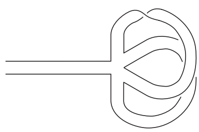
Fig. 3.10. A standard Möbius graph representing a non-orientable surface of genus $g = 2k$ with n marked points. It has $n-1$ orientable tadpoles on the left, k bi-petal orientable flowers on the right and a non-orientable tadpole added at the top



$$= 1 + 3(\beta - 1) - (\beta - 1)(\beta - 2)\,, \tag{3.23}$$



$$= 2 - \beta\,, \tag{3.24}$$



$$= (2 - \beta)^2\,. \tag{3.25}$$

Each of the above calculations is rather similar: A single line emitted from a tadpole subgraph can only ever carry the unit $e_0 = 1$. As an example, (3.23) arises from (i) placing all 1's on the remaining four lines, (ii) 1's on one loop and imaginary units e_i on the other which can be done in $2(\beta - 1)$ ways, (iii) the same imaginary unit on both loops giving $\beta - 1$ possibilities, (iv) different imaginary units which can only be achieved for the quaternions and in $6 = (\beta - 1)(\beta - 2)$ different ways incurring a minus sign since $-1 = ijij$ (say).

Therefore we find

$$\mu_\Gamma = (-4 + 6\beta - \beta^2)^{1-\frac{1}{2}\Sigma_\Gamma - \frac{1}{2}\chi(S_\Gamma)}(2-\beta)^{\Sigma_\Gamma}\beta^{f_\Gamma - 1}\,. \tag{3.26}$$

This result combined with the first lemma above completes the proof of our main theorem.

Finally, we prove the gauge theory result of [18]. In the $SO(2N)$ case the integration is over antisymmetric matrix valued gauge fields. The expansion is again in terms of Möbius graphs with a relative minus sign for each twisted edge. As shown in [18], in the $Sp(N)$ theory, a twisted edge gets a minus sign as well as a directed factor $J = -J^\top$ (the $2N \times 2N$ symplectic metric) on each line. Therefore the symplectic and orthogonal graphs differ only by a possible relative minus sign which can again be computed for the standard graphs in Figs. 3.8–3.10 since this relative sign is also invariant under edge contraction. The result is, of course, a factor $(-)^{\chi(S_\Gamma)}$.

4. Examples

A simple consistency check is the case $\beta = 2$. Clearly, non-orientable graphs give a vanishing contribution as required for hermitian matrix integrals. To obtain a more conventional normalization, we rescale $X \longrightarrow \beta^{1/2} X$ in (2.1) and absorb all but one power of β in the couplings t, so that

$$\log\left(\frac{\int [dX]_{(\beta)} \exp\left(-\frac{\beta}{4}\operatorname{tr} X^2 + \sum_{j=1}^\infty \frac{\beta t_j}{2j}\operatorname{tr} X^j\right)}{\int [dX]_{(\beta)} \exp\left(-\frac{\beta}{4}\operatorname{tr} X^2\right)}\right)$$

$$= \sum_{\Gamma \in \mathfrak{G}} \frac{(-4+6\beta-\beta^2)^{1-\frac{1}{2}\Sigma_\Gamma - \frac{1}{2}\chi(S_\Gamma)}(2-\beta)^{\Sigma_\Gamma}\beta^{\chi(S_\Gamma)-1}N^{f_\Gamma}}{|\mathrm{Aut}(\Gamma)|}\prod_j t_j^{v_\Gamma^{(j)}}. \tag{4.1}$$

Hence when $\beta = 2$,

$$\log\left(\frac{\int [dX]_{(2)} \exp\left(-\frac{1}{2}\operatorname{tr} X^2 + \sum_{j=1}^\infty \frac{t_j}{j}\operatorname{tr} X^j\right)}{\int [dX]_{(2)} \exp\left(-\frac{1}{2}\operatorname{tr} X^2\right)}\right)$$

$$= \sum_{\Gamma \in \mathfrak{G}} \frac{2N^{f_\Gamma}}{|\mathrm{Aut}(\Gamma)|}\prod_j t_j^{v_\Gamma^{(j)}}. \tag{4.2}$$

We note that

$$\sum_{\Gamma \in \mathfrak{G}} \frac{2N^{f_\Gamma}}{|\mathrm{Aut}(\Gamma)|}\prod_j t_j^{v_\Gamma^{(j)}} = \sum_{\Gamma \in \mathfrak{R}} \frac{N^{f_\Gamma}}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|}\prod_j t_j^{v_\Gamma^{(j)}}, \tag{4.3}$$

where $\mathfrak{R}$ denotes the set of all connected ribbon graphs and $\mathrm{Aut}_{\mathfrak{R}}(\Gamma)$ the automorphism group of a ribbon graph disallowing orientation-reversing automorphisms. To see (4.3), let Γ be an oriented ribbon graph. Then either (a) Γ and its flip $\check{\Gamma}$ are isomorphic as a ribbon graph; or (b) they are different ribbon graphs. In case (a), we have

$|\mathrm{Aut}(\Gamma)| = 2|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|$. If (b) is the case, then $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}_{\mathfrak{R}}(\Gamma)$, but Γ and $\check{\Gamma}$ appear as different graphs on the right hand side while they are the same as a Möbius graph in $\mathfrak{G}$.

Together, Eqs. (4.3) and (4.2) constitute the well-known result for the hermitian matrix integral [3].

A less trivial test for the $\beta = 4$, GSE model is a comparison with the Penner model. To begin with we re-express (3.8) in yet another normalization, together with a specialization $t_1 = t_2 = 0$:

$$\log\left(\frac{\int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\, X^2 + \sum_{j=3}^{\infty} \frac{t_j}{j}\,\mathrm{tr}\, X^j\right)}{\int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\, X^2\right)}\right) = \sum_{\Gamma\in\mathfrak{G}} \frac{(-1)^{\chi(S_\Gamma)}}{|\mathrm{Aut}(\Gamma)|} (2N)^{f_\Gamma} \prod_{j\geq 3} t_j^{v_\Gamma^{(j)}}. \tag{4.4}$$

Here we have employed the useful identity

$$(-1)^{\Sigma_\Gamma} = (-1)^{\chi(S_\Gamma)}. \tag{4.5}$$

The Penner substitution

$$t_j \longrightarrow -z^{j/2-1}, \quad j \geq 3 \tag{4.6}$$

yields the graphical expansion

$$\lim_{m\to\infty} \log\left(\frac{\int [dX]_{(4)} \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j}\,\mathrm{tr}\, X^j\right)}{\int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\, X^2\right)}\right) = \sum_{\Gamma\in\mathfrak{G}} \frac{(-1)^{e_\Gamma}}{|\mathrm{Aut}(\Gamma)|} (2N)^{f_\Gamma} (-1)^{\chi(S_\Gamma)} (-z)^{e_\Gamma - v_\Gamma}. \tag{4.7}$$

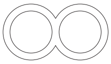
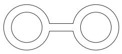
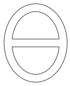
Equation (4.7) is valid as an asymptotic expansion of the integral for $z \to 0$ while keeping $z > 0$, as an element of a formal power series ring $(\mathbb{Q}[N])[[z]]$.

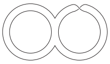
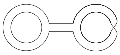
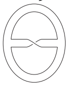
The integral on the left hand side can be evaluated explicitly (see (7.2) and (7.3) of Sect. 7). The leading terms as an expansion in z are

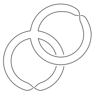
$$\lim_{m\to\infty} \log\left(\frac{\int [dX]_{(4)} \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j}\,\mathrm{tr}\, X^j\right)}{\int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\, X^2\right)}\right) = \left(-\frac{1}{12}N - \frac{1}{2}N^2 + \frac{2}{3}N^3\right) z + \mathcal{O}(z^2). \tag{4.8}$$

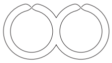
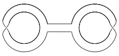
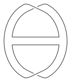
On the other hand, our graphical expansion yields

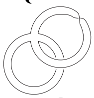
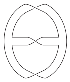
$$\sum_{\Gamma\in\mathfrak{G}} \frac{(-)^{e_\Gamma}(-2N)^{b_\Gamma} z^{e_\Gamma-v_\Gamma}}{|\mathrm{Aut}(\Gamma)|} \tag{4.9}$$

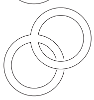
$= (-2N)^3 z \Big\{ \frac{(-1)^2}{4}$  $+ \frac{(-1)^3}{4}$  $+ \frac{(-1)^3}{12}$  $\Big\}$

$+ (-2N)^2 z \Big\{ \frac{(-1)^2}{2}$  $+ \frac{(-1)^3}{2}$  $+ \frac{(-1)^3}{4}$ 

$+ \frac{(-1)^2}{8}$  $\Big\}$

$+ (-2N)\, z \Big\{ \frac{(-1)^2}{4}$  $+ \frac{(-1)^3}{4}$  $+ \frac{(-1)^3}{4}$ 

$+ \frac{(-1)^2}{4}$  $+ \frac{(-1)^3}{12}$ 

$+ \frac{(-1)^2}{8}$  $\Big\} + \mathcal{O}(z^2)$

$$= \Big(-\frac{1}{12}N - \frac{1}{2}N^2 + \frac{2}{3}N^3 \Big)\, z + \mathcal{O}(z^2)\,.$$

Needless to say agreement is perfect. In fact, as we shall show in Sect. 7, agreement to all orders amounts to known results for the orbifold Euler characteristic of the moduli space of real algebraic curves.

5. Duality for Matrix Integrals

An additional change of variables $X \longrightarrow N^{1/2}X$ in (4.1), absorption of all but a single N in the couplings t, as well as the substitution

$$\beta = 2\alpha\,, \tag{5.1}$$

yields

$$\log\left(\frac{\int [dX]_{(2\alpha)} \exp\Big(-\frac{N\alpha}{2}\,\mathrm{tr}\,X^2 + \sum_{j=1}^{\infty} \frac{N\alpha t_j}{j}\,\mathrm{tr}\,X^j \Big)}{\int [dX]_{(2\alpha)} \exp\Big(-\frac{N\alpha}{2}\,\mathrm{tr}\,X^2 \Big)} \right)$$

$$= \sum_{\Gamma\in\mathfrak{G}} 2\,(\alpha^{1/2}N)^{\chi(S_\Gamma)} \times \frac{(3-\alpha^{-1}-\alpha)^{1-\frac{1}{2}\Sigma_\Gamma-\frac{1}{2}\chi(S_\Gamma)}(\alpha^{-1/2}-\alpha^{1/2})^{\Sigma_\Gamma}}{|\mathrm{Aut}(\Gamma)|} \prod_j t_j^{v_\Gamma^{(j)}}. \tag{5.2}$$

This formula is invariant under

$$\alpha \longrightarrow \alpha^{-1} \text{ and } N \longrightarrow -\alpha N\,. \tag{5.3}$$

Remark. (1) The duality holds graph by graph[2].
(2) The $\alpha = 1$ GUE model is self-dual since $\chi(S_\Gamma)$ is even for orientable graphs.
(3) The graphical expansion of the $\alpha = 2$, $N \times N$ GSE model is identical to that of the $\alpha = 1/2$ GOE model if the size of the matrices are doubled and the contribution of every Möbius graph embedded in a non-orientable surface of odd Euler characteristic is multiplied by -1.

One might wonder whether matrix integrals exist whose graphical expansion coincides exactly with the image of (5.2) under the duality (5.3). Although we have no definite answer to this question at this point, in the next section we show that the combination of Poincaré duality and the one discovered here underly the dualities for correlations of characteristic polynomials [2, 6, 7, 17].

6. Characteristic Polynomial Duality

The average of products of characteristic polynomials obey dualities between GOE and GSE models [7]:

$$\frac{\int [dS]_{(1)}^{N\times N} \exp\left(-\frac{N}{2}\operatorname{tr} S^2\right)\prod_{\ell=1}^{k}\det_{N\times N}(\lambda_\ell - S)}{\int [dS]_{(1)}^{N\times N}\exp\left(-\frac{N}{2}\operatorname{tr} S^2\right)} = \frac{\int [dX]_{(4)}^{k\times k}\exp(-N\operatorname{tr} X^2)\,\mathbb{H}\mathrm{det}_{k\times k}^{N}(\Lambda - \sqrt{-1}\,X)}{\int [dX]_{(4)}^{k\times k}\exp(-N\operatorname{tr} X^2)}\,, \tag{6.1}$$

as well as a self duality for the GUE case[3] [6]:

$$\frac{\int [dX]_{(2)}^{N\times N} \exp\left(-\frac{N}{2}\operatorname{tr} X^2\right)\prod_{\ell=1}^{k}\det_{N\times N}(\lambda_\ell - X)}{\int [dX]_{(2)}^{N\times N}\exp\left(-\frac{N}{2}\operatorname{tr} X^2\right)} = \frac{\int [dY]_{(2)}^{k\times k}\exp\left(-\frac{N}{2}\operatorname{tr} Y^2\right)\det_{k\times k}^{N}(\Lambda - \sqrt{-1}\,Y)}{\int [dY]_{(2)}^{k\times k}\exp\left(-\frac{N}{2}\operatorname{tr} Y^2\right)}\,. \tag{6.2}$$

The duality relates expectations of k-fold products of distinct characteristic polynomials of $N \times N$ matrices to averages over the N^{th} power of determinants of certain $k \times k$ matrices. Here, $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$ is a diagonal $k \times k$ matrix of real entries. The quaternionic determinant $\mathbb{H}\det$ in (6.1) is defined by

$$\mathbb{H}\mathrm{det}_{k\times k} M = \det_{2k\times 2k}^{1/2} C(M)\,, \tag{6.3}$$

[2] Physicists would call this a T-duality – valid order by order in perturbation theory.

[3] One might question the reality of the integrals on the right-hand sides of (6.2) and (6.1) since an explicit $\sqrt{-1}$ appears in the determinants. It is clear, however, from both the graphical expansions below and the original derivation in [6, 7] that both integrals are real.

where the $2k \times 2k$ matrix $C(M)$ is obtained from the $k \times k$ quaternion valued matrix M by replacing the quaternionic units by their representation in terms of Pauli matrices[4] $1 \to I_{2\times 2}$, $e_i \to i\sigma_i$ $(i = 1, 2, 3)$.

To begin with, we demonstrate that the N-k duality for GUE models follows from the usual ribbon graph expansion along with Poincaré duality of graphs on a compact oriented surface. The first step is to represent the determinants as vertices of the graphical expansion. Let us assume that the parameter λ_ℓ satisfies $\lambda_\ell > \lambda > 0$ for every ℓ and some positive λ, and let Ω_λ denote the set of all $N \times N$ hermitian matrices whose eigenvalues are contained in the bounded interval $[-\lambda, \lambda]$. Then for every $X \in \Omega_\lambda$, we have a convergent power series expansion in λ_ℓ^{-1}:

$$\det\left(I - \frac{X}{\lambda_\ell}\right) = \exp\left(\mathrm{tr}\ \log\left(I - \frac{X}{\lambda_\ell}\right)\right) = \exp\left(-\sum_{j=1}^{\infty} \frac{1}{j}\lambda_\ell^{-j}\mathrm{tr}\ X^j\right).$$

Therefore,

$$\begin{aligned}
&\int [dX]_{(2)}^{N\times N} e^{-\frac{N}{2}\mathrm{tr}\ X^2} \prod_{\ell=1}^{k} \det{}_{N\times N}\left(I - \frac{X}{\lambda_\ell}\right) \\
&\quad = \int_{\Omega_\lambda} [dX]_{(2)}^{N\times N} e^{-\frac{N}{2}\mathrm{tr}\ X^2} \exp\left(-\sum_{j=1}^{\infty}\frac{1}{j}\mathrm{tr}_{k\times k}\Lambda^{-j}\mathrm{tr}_{N\times N}X^j\right) \\
&\qquad + \int_{\Omega_\lambda^c} [dX]_{(2)}^{N\times N} e^{-\frac{N}{2}\mathrm{tr}\ X^2} \prod_{\ell=1}^{k} \det{}_{N\times N}\left(I - \frac{X}{\lambda_\ell}\right),
\end{aligned} \tag{6.4}$$

where Ω_λ^c is the complement of Ω_λ in the space of all $N \times N$ hermitian matrices. Since Ω_λ is a compact space, the first integral on the right-hand side of (6.4) is a convergent power series in $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_k^{-1}$. Set $t_j = -\mathrm{tr}\ \Lambda^{-j}$. Then $\mathrm{Re}(t_j) < 0$, and as $\lambda \to +\infty$, $t_j \to 0$. Thus the ribbon graph expansion provides each coefficient of the power series expansion of this integral in t_j as $\lambda \to +\infty$. The second integral on the right-hand side of (6.4) is a polynomial in λ_ℓ^{-1} whose coefficients converge to 0 as λ goes to infinity since Ω_λ^c approaches the empty set and the integrand is bounded. Therefore, we obtain an asymptotic expansion formula

$$\begin{aligned}
&\log\left(\frac{\int [dX]_{(2)}^{N\times N}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ X^2\right)\prod_{\ell=1}^{k}\det{}_{N\times N}\left(I - \frac{X}{\lambda_\ell}\right)}{\int [dX]_{(2)}^{N\times N}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ X^2\right)}\right) \\
&\quad = \sum_{\Gamma\in\mathfrak{R}} \frac{1}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|}(-1)^{v_\Gamma} N^{f_\Gamma - e_\Gamma}\prod_j (\mathrm{tr}\ \Lambda^{-j})^{v_\Gamma^{(j)}} \\
&\quad \in (\mathbb{Q}[N])[[\lambda_1^{-1}, \ldots, \lambda_k^{-1}]].
\end{aligned} \tag{6.5}$$

[4] The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The computation of

$$\log\left(\frac{\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)\ \det{}^N(I-\sqrt{-1}\ Y\Lambda^{-1})}{\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)}\right) \tag{6.6}$$

can be performed similarly: First we decompose the space of all $k\times k$ hermitian matrices into two pieces, one consisting of matrices with eigenvalues in $[-\lambda,\lambda]$, and the other its complement. If $\lambda_\ell > \lambda$ for every ℓ, then $\det^N(I-\sqrt{-1}\ Y\Lambda^{-1})$ can be expanded as before. Asymptotically as an element of $(\mathbb{Q}[N])[[\lambda_1^{-1},\dots,\lambda_k^{-1}]]$, we have

$$\begin{aligned}&\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)\ \det(I-\sqrt{-1}\ Y\Lambda^{-1})^N\\ &\quad=\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)\ \exp\left(-N\sum_j\frac{(\sqrt{-1})^j}{j}\mathrm{tr}\ (Y\Lambda^{-1})^j\right).\end{aligned} \tag{6.7}$$

The appearance of the term $\mathrm{tr}\ (Y\Lambda^{-1})^m$ instead of $\mathrm{tr}\ Y^m$ occurring in (6.7) replaces products of traces over identity matrices

$$N^{f_\Gamma}=\prod_j(\mathrm{tr}\ I^j)^{f_\Gamma^{(j)}} \tag{6.8}$$

incurred in (3.8) as one travels around each face of the graph Γ, by

$$\prod_j(\mathrm{tr}\ \Lambda^{-j})^{f_\Gamma^{(j)}}\ . \tag{6.9}$$

(Recall that $f_\Gamma^{(j)}$ denotes the number of j-gons in the cell-decomposition of S_Γ defined by the graph Γ.) Therefore,

$$\begin{aligned}&\log\left(\frac{\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)\ \det{}^N(I-\sqrt{-1}\ Y\Lambda^{-1})}{\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)}\right)\\ &\quad=\log\left(\frac{\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)\ \exp\left(-N\sum_j\frac{(\sqrt{-1})^j}{j}\mathrm{tr}\ (Y\Lambda^{-1})^j\right)}{\int[dY]^{k\times k}_{(2)}\ \exp\left(-\frac{N}{2}\ \mathrm{tr}\ Y^2\right)}\right)\\ &\quad=\sum_{\Gamma\in\mathfrak{R}}\frac{1}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|}(-1)^{v_\Gamma}N^{v_\Gamma-e_\Gamma}(\sqrt{-1})^{2e_\Gamma}\prod_j(\mathrm{tr}\ \Lambda^{-j})^{f_\Gamma^{(j)}}\\ &\quad=\sum_{\Gamma\in\mathfrak{R}}\frac{1}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|}(-1)^{f_\Gamma}N^{v_\Gamma-e_\Gamma}\prod_j(\mathrm{tr}\ \Lambda^{-j})^{f_\Gamma^{(j)}},\end{aligned} \tag{6.10}$$

where we used

$$(-1)^{v_\Gamma-e_\Gamma}=(-1)^{\chi(S_\Gamma)-f_\Gamma}=(-1)^{f_\Gamma}.$$

Let us denote by Γ^* the dual graph of a ribbon graph Γ drawn on a compact oriented surface S_Γ. We note that $\mathrm{Aut}_{\mathfrak{R}}(\Gamma) \cong \mathrm{Aut}_{\mathfrak{R}}(\Gamma^*)$ and

$$\begin{cases} v_\Gamma^{(j)} = f_{\Gamma^*}^{(j)}\,, \\ e_\Gamma = e_{\Gamma^*}\,, \\ f_\Gamma^{(j)} = v_{\Gamma^*}^{(j)}\,. \end{cases} \tag{6.11}$$

Since one and two valent vertices are included in the set of ribbon graphs $\mathfrak{R}$, the map

$$* : \mathfrak{R} \longrightarrow \mathfrak{R}$$

is a bijection. [Contrast this situation to the Penner model in Sect. 4, where the couplings $t_1 = t_2 = 0$ and Poincaré duality does not apply.] Therefore,

$$\begin{aligned} &\sum_{\Gamma\in\mathfrak{R}} \frac{1}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|}(-1)^{v_\Gamma} N^{f_\Gamma - e_\Gamma} \prod_j (\mathrm{tr}\ \Lambda^{-j})^{v_\Gamma^{(j)}} \\ &= \sum_{\Gamma\in\mathfrak{R}} \frac{1}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma^*)|}(-1)^{v_{\Gamma^*}} N^{f_{\Gamma^*} - e_{\Gamma^*}} \prod_j (\mathrm{tr}\ \Lambda^{-j})^{v_{\Gamma^*}^{(j)}} \\ &= \sum_{\Gamma\in\mathfrak{R}} \frac{1}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|}(-1)^{f_\Gamma} N^{v_\Gamma - e_\Gamma} \prod_j (\mathrm{tr}\ \Lambda^{-j})^{f_\Gamma^{(j)}}. \end{aligned} \tag{6.12}$$

This implies that the matrix integrals (6.5) and (6.6) have the same asymptotic expansion.

The N-k duality in Eq. (6.2) is a polynomial identity of degree Nk in $(\mathbb{Q}[N])[\lambda_1, \lambda_2, \dots, \lambda_k]$, where we define $\deg(\lambda_\ell) = 1$. We must now consider also disconnected graphs, since there is no logarithm. The coefficient of the degree $Nk - d$ term of (6.2) is therefore determined by a partition $d = 2(e_1 + e_2 + \cdots + e_m)$ corresponding to the product of m connected graphs consisting of e_i edges. The contributions of connected graphs are computed in (6.5) and (6.6). We note that the duality (6.12) holds for every surface even when the number of edges is fixed. Therefore, the asymptotic equality we have derived implies the polynomial identity (6.2). In other words, the N-k duality of [6] is a simple consequence of the Poincaré duality of graphs on a compact oriented surface.

Our derivation of the characteristic polynomial duality between the GOE and GSE models goes quite similarly. Here again we see that the duality is a consequence of our graphical expansion formula (3.8) and Poincaré duality:

Using the same trick for characteristic polynomials as in the GUE case, from the expansion formula (3.8) we obtain an asymptotic expansion formula for the GOE side of the duality

$$\begin{aligned} &\log\left(\frac{\int [dS]_{(1)}^{N\times N}\ \exp\left(-\frac{2N}{4}\ \mathrm{tr}\ S^2\right)\prod_{\ell=1}^k \det\left(I - \frac{S}{\lambda_\ell}\right)}{\int [dS]_{(1)}^{N\times N}\ \exp\left(-\frac{N}{4}\ \mathrm{tr}\ S^2\right)}\right) \\ &= \log\left(\frac{\int [dS]_{(1)}^{N\times N}\ \exp\left(-\frac{2N}{4}\ \mathrm{tr}\ S^2\right)\ \exp\left(-\sum_{j=1}^\infty \frac{2\,\mathrm{tr}\ \Lambda^{-j}}{2j}\mathrm{tr}\ S^j\right)}{\int [dS]_{(1)}^{N\times N}\ \exp\left(-\frac{2N}{4}\ \mathrm{tr}\ S^2\right)}\right) \\ &= \sum_{\Gamma\in\mathfrak{G}} \frac{1}{|\mathrm{Aut}(\Gamma)|}(-1)^{v_\Gamma} 2^{v_\Gamma - e_\Gamma} N^{f_\Gamma - e_\Gamma} \prod_j (\mathrm{tr}\ \Lambda^{-j})^{v_\Gamma^{(j)}}. \end{aligned} \tag{6.13}$$

(Note the non-standard normalization of the Gaussian exponent yields the factor 2^{-e_Γ}.) Its GSE counterpart requires some care: The "characteristic polynomial" of a $k \times k$ quaternionic matrix X is defined by

$$\mathbb{H}\det(\Lambda - \sqrt{-1}\,X) = \det{}^{1/2}(\Lambda I_{2k\times 2k} - \sqrt{-1}\,C(X))$$

and

$$\mathrm{tr}\,X^j = \frac{1}{2}\mathrm{tr}\,C(X)^j.$$

Thus if all eigenvalues of X are in $[-\lambda, \lambda]$ and $\lambda_\ell > \lambda > 0$, then

$$\mathbb{H}\mathrm{det}(I - \sqrt{-1}\,X\Lambda^{-1}) = \exp\left(-\frac{1}{2}\sum_{j=1}^{\infty}\frac{2\,(\sqrt{-1})^j}{j}\,\mathrm{tr}\,(X\Lambda^{-1})^j\right).$$

Therefore, we have an asymptotic expansion

$$\begin{aligned}
&\log\left(\frac{\int [dX]_{(4)}^{k\times k}\,\exp(-N\,\mathrm{tr}\,X^2)\,\mathbb{H}\mathrm{det}_{k\times k}^N(I - \sqrt{-1}\,X\Lambda^{-1})}{\int [dX]_{(4)}^{k\times k}\,\exp(-N\,\mathrm{tr}\,X^2)}\right)\\
&= \log\left(\frac{\int [dX]_{(4)}^{k\times k}\exp\left(-\frac{Nk}{k}\,\mathrm{tr}\,X^2\right)\exp\left(-N\sum_{j=1}^{\infty}\frac{2(\sqrt{-1})^j}{2j}\mathrm{tr}\,(X\Lambda^{-1})^{-j}\right)}{\int [dX]_{(4)}^{k\times k}\exp\left(-\frac{Nk}{k}\,\mathrm{tr}\,X^2\right)}\right)\\
&= \sum_{\Gamma\in\mathfrak{G}}\frac{1}{|\mathrm{Aut}(\Gamma)|}(-1)^{\Sigma_\Gamma + e_\Gamma + v_\Gamma}2^{f_\Gamma - e_\Gamma}N^{v_\Gamma - e_\Gamma}\prod_j(\mathrm{tr}\,\Lambda^{-j})^{f_\Gamma^{(j)}}\\
&= \sum_{\Gamma\in\mathfrak{G}}\frac{1}{|\mathrm{Aut}(\Gamma)|}(-1)^{f_\Gamma}2^{f_\Gamma - e_\Gamma}N^{v_\Gamma - e_\Gamma}\prod_j(\mathrm{tr}\,\Lambda^{-j})^{f_\Gamma^{(j)}},
\end{aligned}\tag{6.14}$$

where we have used the fact that

$$(-1)^{\Sigma_\Gamma + e_\Gamma + v_\Gamma} = (-1)^{f_\Gamma}$$

that follows from (4.5). (In addition the non-standard Gaussian exponent normalization now accounts for the absence of explicit factors k in the graphical expansion.) We now see that (6.13) and (6.14) are equal again through the dual construction of a Möbius graph.

The polynomial identity (6.1) follows from the equality of the asymptotic expansions. This time each term of (6.1) may have contributions from both orientable and non-orientable graphs, but since the dual graph construction works for each surface, the equality holds.

7. The Penner Model

The Penner model for the hermitian matrix integral provides an effective tool to compute the orbifold Euler characteristic of the moduli space of smooth algebraic curves defined over $\mathbb{C}$ with an arbitrary number of marked points [21, 11]. It was discovered in [10] that the Penner model of the real symmetric (or GOE) matrix integral yields the orbifold Euler characteristic of the moduli spaces of real algebraic curves. Due to the GOE-GSE duality (5.3), we see that the symplectic Penner model is identical to the GOE case, except for the matrix size and an overall sign for contributions of non-orientable surfaces. As we shall see, of the three main classes of Möbius graphs – oriented, non-orientable odd $\chi(S_\Gamma)$ and non-orientable even $\chi(S_\Gamma)$ – only the first two survive the Penner substitution for the couplings, or in other words, the orbifold Euler characteristic vanishes when $\chi(S_\Gamma)$ is even. Therefore the third symplectic Penner type model is not an independent topological quantity.

We also show that the generalized Penner model expressed in terms of Vandermonde determinants to powers in $2(\mathbb{N} \cup 1/\mathbb{N})$ exhibits an extended duality that agrees with (5.3) when the power of the Vandermonde is restricted to 1, 2 or 4. Many explicit formulæ and derivations are reserved for Appendix B.

The symplectic Penner model introduced in Sect. 4 reads

$$\begin{aligned}
&\lim_{m\to\infty} \log\left(\frac{\int [dX]_{(4)} \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j} \operatorname{tr} X^j\right)}{\int [dX]_{(4)} \exp\left(-\frac{1}{2} \operatorname{tr} X^2\right)}\right)\\
&\quad = \sum_{\Gamma\in\mathfrak{G}} \frac{(-1)^{e_\Gamma}}{|\mathrm{Aut}(\Gamma)|} (2N)^{f_\Gamma} (-1)^{\chi(S_\Gamma)} (-z)^{e_\Gamma - v_\Gamma}\,, \qquad (7.1)
\end{aligned}$$

to be viewed as an element of the formal power series ring $(\mathbb{Q}[N])[[z]]$. This integral is indeed explicitly computable. Symplectic invariance of the measure and integrand allows us to diagonalize the matrix variable $X \longrightarrow \operatorname{diag}(k_1, k_2, \ldots, k_N)$ so that:

$$\begin{aligned}
&\lim_{m\to\infty} \log\left(\frac{\int [dX]_{(4)} \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j} \operatorname{tr} X^j\right)}{\int [dX]_{(4)} \exp\left(-\frac{1}{2} \operatorname{tr} X^2\right)}\right)\\
&\quad = \lim_{m\to\infty} \log\left(\frac{\int_{\mathbb{R}^N} \Delta^4(k) \prod_{i=1}^N \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j} k_i^j\right) dk_i}{\int_{\mathbb{R}^N} \Delta^4(k) \prod_{i=1}^N \exp\left(-\frac{k_i^2}{2}\right) dk_i}\right), \qquad (7.2)
\end{aligned}$$

where

$$\Delta(k) = \prod_{i<j} (k_i - k_j)$$

is the Vandermonde determinant. Using the asymptotic expansion technique established in [19], the Selberg integral formula and the Stirling formula, an explicit asymptotic expansion, even valid for every $\alpha \in \mathbb{N}$, can be computed for the integral

$$K(z, N, \alpha) = \lim_{m\to\infty} \log\left(\frac{\int_{\mathbb{R}^N} \Delta^{2\alpha}(k) \prod_{i=1}^N \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j} k_i^j\right) dk_i}{\int_{\mathbb{R}^N} \Delta^{2\alpha}(k) \prod_{i=1}^N \exp\left(-\frac{k_i^2}{2}\right) dk_i}\right). \qquad (7.3)$$

(In what follows $K(z, N, \alpha)$ should be regarded as the asymptotic expansion, not the underlying integral; an explicit expression is given in Appendix B.) Specializing to the

$\alpha = 1$ hermitian case yields a very compact result corresponding to the original formula of Penner [21]:

$$\begin{aligned} K(z, N, 1) &= \sum_{\substack{g \geq 0, n > 0 \\ 2-2g-n<0}} \frac{(2g+n-3)!(2g-1)}{(2g)!n!} b_{2g} N^n (-z)^{2g+n-2} \\ &= \sum_{\Gamma \in \mathfrak{R}} \frac{(-1)^{e_\Gamma}}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|} N^{f_\Gamma} (-z)^{e_\Gamma - v_\Gamma} . \end{aligned} \tag{7.4}$$

Identifying n with f_Γ and g as the genus yields the well known generating function of the Euler characteristic $\chi(\mathfrak{M}_{g,n})$ of the moduli of complex algebraic curves of genus g and n marked points.

For the $\alpha = 2$ symplectic case a similar simplification occurs and (7.3) can be written in terms of the $\alpha = 1$ hermitian result plus additional terms corresponding to non-orientable surfaces of even genus $g = 2q$ with $m + 1 - 2q$ marked points

$$\begin{aligned} K(z, N, 2) &= \frac{1}{2} K(z, 2N, 1) \\ &\quad - \frac{1}{2} \sum_{\substack{q \geq 0, n > 0 \\ 1-2q-n<0}} \frac{(2q+n-2)!(2^{2q-1}-1)}{(2q)!\, n!} b_{2q} (2N)^n (-z)^{2q+n-1}. \end{aligned} \tag{7.5}$$

Comparing (7.5) with (7.1), we obtain

$$\sum_{\Gamma:\, f_\Gamma = n,\ g(S_\Gamma) = 2q} \frac{(-1)^{e_\Gamma}}{|\mathrm{Aut}(\Gamma)|} = \frac{1}{2} \frac{(2q+n-2)!(2^{2q-1}-1)}{(2q)!\, n!} b_{2q}, \tag{7.6}$$

where the summation is over all connected non-orientable Möbius graphs with $n > 0$ faces that are drawn on a non-orientable surface of genus $2q$ satisfying a hyperbolicity condition $1 - 2q - n < 0$. The formula (7.6) is in exact agreement with the formula for the orbifold Euler characteristic of the moduli space of smooth *real* algebraic curves of genus $2q$ with n marked points that can be found in [10 and 20].

To study the GOE-GSE duality for the Penner model, we need the expansion of the analog of formula (7.3) valid for the single power of the Vandermonde determinant relevant to the GOE model. Indeed an integral formula for

$$J(z, N, \gamma) = \lim_{m \to \infty} \log \left(\frac{\int_{\mathbb{R}^N} |\Delta(k)|^{2/\gamma} \prod_{i=1}^N \exp\Big(- \sum_{j=2}^{2m} \frac{z^{j/2-1}}{j} k_i^j \Big) dk_i}{\int_{\mathbb{R}^N} |\Delta(k)|^{2/\gamma} \prod_{i=1}^N \exp\Big(- \frac{k_i^2}{2} \Big) dk_i} \right) , \tag{7.7}$$

valid for every positive integer $\gamma \in \mathbb{N}$ (*i.e.* for all powers $2/\mathbb{N}$ of the Vandermonde) was derived in [10] in order to compute (7.6). (See Appendix B.)

The matrix integrals $K(z, N, \alpha)$ and $J(z, N, \gamma)$ are closely related: Obviously

$$J(z, N, 1) = K(z, N, 1) . \tag{7.8}$$

The identity

$$J(2z,2N,2)=\frac{1}{2}J(z,2N,1)-\Big(K(z,N,2)-\frac{1}{2}K(z,2N,1)\Big)$$
$$=\frac{1}{2}J(z,2N,1)$$
$$+\frac{1}{2}\sum_{\substack{q\geq 0,n>0\\ 1-2q-n<0}}\frac{(2q+n-2)!(2^{2q-1}-1)}{(2q)!\,n!}b_{2q}(2N)^n(-z)^{2q+n-1}\,, \quad (7.9)$$

expresses the $\gamma=2$ GOE case in terms of the $\gamma=1$ oriented hermitian result and a sum over non-orientable contributions with odd Euler characteristic. Notice, as claimed above, in the graphical expansions of the orthogonal Penner model $J(2z,2N,2)$ and the symplectic Penner model $K(z,N,2)$, non-orientable surfaces of odd genera do not contribute. This corresponds to the fact that the orbifold Euler characteristic of the moduli space of smooth real algebraic curves of odd genus $2q+1$ with n marked points is 0 for any value of $q\geq 0$ and $n>0$.

More importantly, observe that the GSE and GOE formulæ (7.5) and (7.9) almost coincide except that the GOE formula is for matrix size $2N$ and the non-orientable odd Euler characteristic terms differ by an overall sign. (The appearance of $2z$ rather than z will be cured by the appropriate normalization given below and in the master formula (5.2).) This is precisely the duality (5.3).

Finally, we show that the duality between GOE and GSE extends to arbitrary positive integers for the two types of Penner integrals introduced in this section. Let $r\in\mathbb{N}\cup 1/\mathbb{N}$ and set

$$I(z,N,r)$$
$$=\lim_{m\to\infty}\log\left(\frac{\int_{\mathbb{R}^N}|\Delta(k)|^{2r}\prod_{i=1}^N\exp\Big(-\sum_{j=2}^{2m}\frac{k_i^j}{j}\big(\frac{z}{rN}\big)^{j/2-1}\Big)dk_i}{\int_{\mathbb{R}^N}|\Delta(k)|^{2r}\prod_{i=1}^N\exp\Big(-\frac{k_i^2}{2}\Big)dk_i}\right)$$
$$\in(\mathbb{Q}[N,N^{-1},r,r^{-1}])[[z]]. \quad (7.10)$$

Then we have

$$I(z,N,r)=\begin{cases}K\left(\frac{z}{\alpha N},N,\alpha\right)\,, & r=\alpha\in\mathbb{N}\,,\\ J\left(\frac{\gamma z}{N},N,\gamma\right)\,, & r=\gamma^{-1}\in 1/\mathbb{N}\,.\end{cases} \quad (7.11)$$

From inspection of the explicit asymptotic expansion formulæ of (B.1) and (B.9) presented in Appendix B, we obtain an extended duality

$$I(z,N,r)=I(z,-rN,r^{-1}) \quad (7.12)$$

for an arbitrary positive integer r. This is in agreement with the duality (5.3) for $r=1,2$.

8. The Central Limit Theorem

To prove a central limit theorem for large matrix size N, we need to show that the leading dependence is Gaussian in the coupling constants t_j. More precisely, define the Gaussian expectation value of $f(X)$ for GOE ($\alpha = 1/2$), GUE ($\alpha = 1$), and GSE ($\alpha = 2$) as

$$\langle f(X) \rangle_{(N,\alpha)} = \frac{\int [dX]_{(2\alpha)} \exp\left(-\frac{N\alpha}{2} \operatorname{tr} X^2\right) f(X)}{\int [dX]_{(2\alpha)} \exp\left(-\frac{N\alpha}{2} \operatorname{tr} X^2\right)}, \tag{8.1}$$

and consider

$$V(t, N, \alpha) = \frac{\left\langle \exp\left(\sum_j \frac{\alpha t_j}{j} \operatorname{tr} X^j\right)\right\rangle_{(N,\alpha)}}{\exp\left(\sum_j \frac{\alpha t_j}{j} \left\langle \operatorname{tr} X^j \right\rangle_{(N,\alpha)}\right)}. \tag{8.2}$$

The expansion formula (5.2) shows that the contribution of a connected Möbius graph $\Gamma \in \mathfrak{G}$ to $\log \left\langle \exp\left(\sum_j \frac{\alpha t_j}{j} \operatorname{tr} X^j\right)\right\rangle_{(N,\alpha)}$ is

$$2\alpha^{\frac{1}{2}\chi(S_\Gamma)} N^{\chi(S_\Gamma) - v_\Gamma} \frac{(3 - \alpha^{-1} - \alpha)^{1 - \frac{1}{2}\Sigma_\Gamma - \frac{1}{2}\chi(S_\Gamma)} (\alpha^{-1/2} - \alpha^{1/2})^{\Sigma_\Gamma}}{|\operatorname{Aut}(\Gamma)|} \prod_j t_j^{v_\Gamma^{(j)}}. \tag{8.3}$$

Since

$$\sum_j \frac{\alpha t_j}{j} \left\langle \operatorname{tr} X^j \right\rangle_{(N,\alpha)}$$

is the sum of all contributions from 1-vertex Möbius graphs, we see that $\log V(t, N, \alpha)$ has no terms coming from 1-vertex graphs. In particular, it has no terms with a positive power of N. Indeed, the power of N in (8.3) is strictly positive only when $\chi(S_\Gamma) = 2$ and $v_\Gamma = 1$, *i.e.* Γ is an orientable planar graph with one vertex. Therefore, $\lim_{N\to\infty} \log V(t, N, \alpha)$ consists of contributions from graphs that have two or more vertices and $\chi(S_\Gamma) - v_\Gamma = 0$. But this is possible only when $\chi(S_\Gamma) = v_\Gamma = 2$. In other words, Γ is an orientable planar graph with exactly two vertices contributing (see (8.3))

$$\frac{2\alpha}{|\operatorname{Aut}(\Gamma)|} t_{j_1} t_{j_2},$$

where j_1 and j_2 are the valences of the two vertices of Γ. Altogether, we have established the following

Theorem 8.1. *The Central Limit Theorem for GOE* ($\alpha = 1/2$)*, GUE* ($\alpha = 1$) *and GSE* ($\alpha = 2$) *ensembles is*

$$\lim_{N\to\infty} \log \left(\frac{\left\langle \exp\left(\sum_j \frac{\alpha t_j}{j} \operatorname{tr} X^j\right)\right\rangle_{(N,\alpha)}}{\exp\left(\sum_j \frac{\alpha t_j}{j} \left\langle \operatorname{tr} X^j \right\rangle_{(N,\alpha)}\right)} \right) = \sum_{\substack{\Gamma \text{ connected, oriented, planar} \\ 2\text{-vertex ribbon graph}}} \frac{\alpha}{|\operatorname{Aut}_{\mathfrak{R}}(\Gamma)|} t_{j_1} t_{j_2}, \tag{8.4}$$

where j_1 and j_2 are the valences of the two vertices of the ribbon graph Γ.

We notice that the formula is the same for all three ensembles except for the overall factor of α. In particular, only oriented planar ribbon graphs contribute in the large N limit. This mechanism was observed long ago by 't Hooft in the hope that large N quantum chromodynamics could be solved exactly [24].

9. Conclusions

Let us tabulate the patina of results gathered here:

- The asymptotic expansion of the three Gaussian random matrix ensembles is expressed as a sum over Möbius graphs.
- These expansions are related by a duality

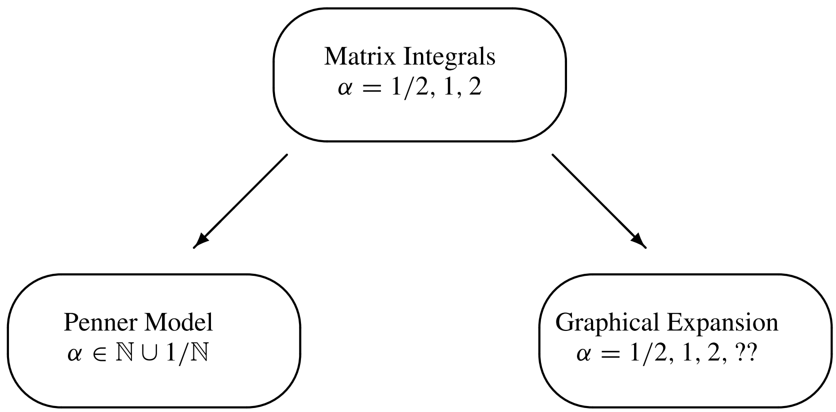
$$\alpha \longrightarrow \alpha^{-1} \text{ and } N \longrightarrow -\alpha N. \tag{9.1}$$

The $\alpha = 1$ GUE model is self-dual and sums over only ribbon graphs. The duality between $\alpha = 1/2$ GOE and $\alpha = 2$ GSE models amounts to an equality of graphical expansions up to a factor

$$(-1)^{\chi(S_\Gamma)}$$

for any graph Γ.
- When specialized to Penner model couplings, the Selberg integral representation yields an asymptotic expansion for all $\alpha \in \mathbb{N} \cup 1/\mathbb{N}$ and the duality (9.1) holds for this extended set of α's.

Therefore, the first and probably most interesting question one might pose is whether our graphical expansion formulæ can also generalized to the extended set of $\alpha \in \mathbb{N} \cup 1/\mathbb{N}$, i.e.



Let us briefly postpone a discussion of this issue while enumerating several other questions for which we have no immediate answers:

(1) Do there exist matrix models where the duality holds exactly, without a factor $(-1)^{\chi(S_\Gamma)}$?
(2) What is the significance of the minus sign in the transformation

$$N \to -\alpha N\,?$$

Is there an interpretation where traces over $N \times N$ matrices are replaced by a supertrace and in turn bosonic matrix integrals by fermionic ones (cf. [9])?

(3) Why the factor α in the transformation

$$N \to -\alpha N\,?$$

Is there a generalization of the GSE dual to the GOE at odd values of N?

After this disquisitive interlude, we return to the postponed question. Let us examine the generality of Lemma 3.3 which claimed that a topological invariant of a punctured surface S_Γ was obtained by counting signed configurations of units $\{1, e_1 \dots , e_{2\alpha-1}\}$ on the associated graph Γ. Its proof relied on the following: (i) The units all square to ±1 and $\{\pm1, \pm e_i\}$ is a group. (ii) At any vertex, their product was ±1 and therefore real. (iii) Units whose square was $(-)1$ were joined by Wick contractions of (anti)symmetric matrices.

Therefore generalized matrix models whose graphical expansion is completely determined via our methods can be written down based on a larger group of "imaginary" units $\{\pm1, \pm f_a; \pm e_i \,:\, f_a^2 = 1, e_i^2 = -1\}$. Simple examples are generated by considering elements drawn from Clifford algebras. *i.e.*, the Pauli matrix representation of the quaternions can be generalized to larger sets of higher dimensional Dirac matrices. Although these Clifford type models seem not to generate theories with the new values of α exhibited in the Penner model, it would be interesting to investigate whether new matrix models of this type can indeed be constructed.

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Appendix

A. Quaternionic Feynman Calculus

Since the quaternions are the last real associative division algebra, it is natural to develop a manifestly quaternionic Feynman calculus. Again, let us consider GOE, GUE and GSE models all at once via the unified notation

$$X = S + \sum_{i=1}^{\beta-1} e_i A_i\,, \tag{A.1}$$

where the $N \times N$ matrices X are real, complex or quaternionic self-adjoint

$$X^\dagger \equiv \overline{X}^\top = X\,, \qquad \bar{e}_i \equiv -e_i\,, \tag{A.2}$$

depending on the value of $\beta = 1, 2$ or 4, respectively.

To begin, we need a shift identity

$$\exp\left(\frac{1}{2}\,\mathrm{tr}\left(B^\top X + X B^\top\right)\right) f(B) = f(X + B)\,, \tag{A.3}$$

where the "background variable"

$$B \equiv \widehat{S} + \sum_i e_i \widehat{A}_i = B^\dagger \tag{A.4}$$

and the $N \times N$ matrix of derivatives ∂ is given by

$$\partial_{ab} = \begin{cases} \frac{1}{2}\left(\frac{\partial}{\partial \widehat{S}^{ab}} - \sum_i e_i \frac{\partial}{\partial \widehat{A}_i^{ab}}\right), & a \neq b\,, \\ \frac{\partial}{\partial \widehat{S}^{aa}}\,, & a = b\,. \end{cases} \tag{A.5}$$

The identity (A.3) holds thanks to the commutation relations

$$\left[\frac{1}{2}\mathrm{tr}\left(\partial^\top X + X\partial^\top\right), B\right] = X\,. \tag{A.6}$$

Note also that

$$\left[\partial, \frac{1}{2}\,\mathrm{tr}\left(B^\top X + XB^\top\right)\right] = X\,, \tag{A.7}$$

although $[\partial, B] = I$ only for the real and complex cases $\beta = 1, 2$.

We may now rewrite matrix integration as differentiation[5]

$$\begin{aligned}
&\int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\,X^2 + \sum_{j=1}^\infty \frac{t_j}{j}\,\mathrm{tr}\,X^j\right) \\
&= \int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\,X^2 + \sum_{j=1}^\infty \frac{t_j}{j}\,\mathrm{tr}\,(X+B)^j\right)\Bigg|_{B=O} \\
&= \int [dX]_{(4)} \exp\left(-\frac{1}{2}\mathrm{tr}\left(X - \partial^\top\right)^2 + \frac{1}{2}\,\mathrm{tr}\,\partial^{\top 2}\right)\exp\left(\sum_{j=1}^\infty \frac{t_j}{j}\,\mathrm{tr}\,B^j\right)\Bigg|_{B=O} \\
&= \left(\int [dX]_{(4)} \exp\left(-\frac{1}{2}\,\mathrm{tr}\,X^2\right)\right)\,\exp\left(\frac{1}{2}\,\mathrm{tr}\,\partial^2\right)\exp\left(\sum_{j=1}^\infty \frac{t_j}{j}\,\mathrm{tr}\,B^j\right)\Bigg|_{B=O}\,.
\end{aligned} \tag{A.8}$$

The first factor on the last line is just an overall normalization while the two exponentials can be expanded in terms of Feynman diagrams: the n^{th} order term in the expansion of each exponential is interpreted as either n edges or n vertices, respectively.

Let us give some details: The operator $\frac{1}{2}$ tr ∂^2 acting on a quantum variable, yields

$$[\frac{1}{2}\,\mathrm{tr}\,\partial^2, B] = \partial^\top\,, \tag{A.9}$$

which is represented graphically as attaching a ribbon edge to a vertex since the operator $\frac{1}{2}$ tr ∂^2 is to be viewed as an edge. Note that by this rule, a vertex emitting a B_{ab} is replaced with one emitting ∂_{ba} which amounts to a twist. Attaching the other end to an adjacent vertex yields

$$\left(\partial_{ab} B_{cd}\right) = \frac{1}{2}\,\beta\,\delta_{ac}\delta_{bd} + \frac{1}{2}\,(2-\beta)\,\delta_{ad}\delta_{bc}\,. \tag{A.10}$$

[5] The result is equivalent to the one obtained using a Schwinger source term. The reformulation presented here is often called the background field formalism; a simple account may be found in the on-line textbook [22].

Note that the brackets on the left hand sides above indicate that we are computing the derivative rather than allowing it to continue acting to the right as an operator. In particular, observe that for the GUE case, $\beta = 2$, so no twisted ribbon graphs can appear.

For the GOE and GUE models we are done, one simply attaches all possible edges to vertices using the above rules and finds the usual known results. The symplectic case is more subtle however, thanks to the quaternionic non-commutativity of ∂ and B. In particular

$$\Big(\partial_{ab} f(B)\, B_{cd}\Big) \neq \Big(\partial_{ab} f(B)\Big)\, B_{cd} + f(B)\, \Big(\partial_{ab} B_{cd}\Big) \tag{A.11}$$

for some function f of the quaternionic matrix B. *i.e.*, the quaternion valued operator ∂ does not satisfy the Leibniz rule. However, a generalized Leibniz rule does apply: First note that for any

$$\mathcal{Q} = \mathcal{S} + \sum_{\alpha} e_\alpha \mathcal{A}_\alpha\,, \tag{A.12}$$

where the real matrices $\mathcal{S}$ and $\mathcal{A}_\alpha$ need not have any definite symmetry properties (so that $\mathcal{Q}$ is not necessarily self adjoint) we have[6]

$$\sum_{\alpha} e_\alpha \mathcal{Q}\, e_\alpha = -\,\mathcal{Q} + (2-\beta)\,\overline{\mathcal{Q}}\,. \tag{A.13}$$

Therefore we have the generalized Leibniz rule

$$\partial_{ab}\Big(\mathcal{Q}\, B_{cd}\Big) - \Big(\partial_{ab}\mathcal{Q}\Big)\, B_{cd} = \delta_{ac}\delta_{bd}\,\mathcal{Q} - \frac{1}{2}(2-\beta)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\overline{\mathcal{Q}}\,. \tag{A.14}$$

Note that for $\beta = 1, 2$ the right hand side is equal to $\mathcal{Q}\Big(\partial_{ab} B_{cd}\Big)$, expressing the commutativity of real and complex numbers. This relation is the central graphical rule for our quaternionic Feynman calculus and is depicted in Fig. A.1. Note that it reverts to the rule (A.10) when $\mathcal{Q} = 1$.

It is now possible to compute any term in the expansion of (A.8) in terms of graphs. However, for quaternionic matrices, when connecting vertices with ribbon edges, intermediate vertices and unconnected edges may be twisted and/or flipped according to (A.1). It is easy, but tedious to verify that the results for simple graphs coincide with our general formula (5.2).

---

[6] When $\beta = 1$, the sum on the left hand side is empty and equal to zero, while the right-hand side vanishes for real $\mathcal{Q}$.

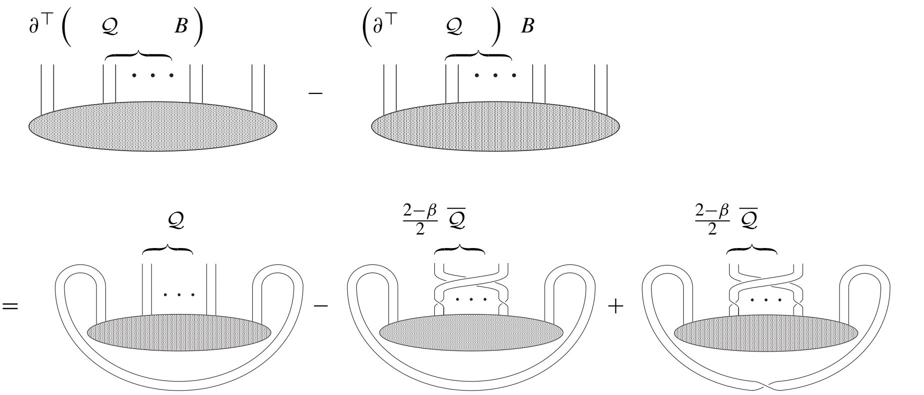

Fig. A.1. Quaternionic Feynman rule. Observe how connecting vertices with edges change vertices yet to be connected

B. Generalized Penner Model

In this Appendix, we derive the asymptotic expansion formula

$$
\begin{aligned}
&K(z, N, \alpha)\\
&\quad = \lim_{m\to\infty} \log\left(\frac{\int_{\mathbb{R}^N} \Delta^{2\alpha}(k)\prod_{i=1}^N \exp\left(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j}\, k_i^j\right) dk_i}{\int_{\mathbb{R}^N} \Delta^{2\alpha}(k)\prod_{i=1}^N \exp\left(-\frac{k_i^2}{2}\right) dk_i}\right)\\
&\quad = \sum_{m=1}^{\infty} \frac{b_{2m}}{2m\ (2m-1)}\ N z^{2m-1} + \sum_{m=1}^{\infty} \frac{1}{4m} (-1)^m \alpha^m N^m z^m\\
&\qquad + \frac{1}{2} \sum_{m=1}^{\infty}\sum_{q=0}^{[\frac{m}{2}]} \frac{(-1)^m (m-1)!\, b_{2q}}{(2q)!(m+1-2q)!} \alpha^m (\alpha^{1-2q}-1) N^{m+1-2q} z^m\\
&\qquad - \sum_{m=1}^{\infty}\sum_{q=0}^{[\frac{m}{2}]}\sum_{s=0}^{[\frac{m+1}{2}]-q} \frac{(-1)^m (m-1)!\, b_{2q} b_{2s}}{(2q)!(2s)!(m+2-2q-2s)!} \alpha^{m+1-2q} N^{m+2-2q-2s} z^m ,
\end{aligned}
\tag{B.1}
$$

which is valid for every positive integer α. Here the b_n's are the Bernoulli numbers defined by

$$
\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \frac{t}{e^t - 1} .
$$

The key techniques are the Selberg integration formula, Stirling's formula for $\Gamma(1/z)$ and the asymptotic analysis of [19]. First we note that as an asymptotic series in z when

$z \to 0$ while keeping $z > 0$, we have

$$\lim_{m\to\infty} \log\left(\int_{\mathbb{R}^N} \Delta^{2\alpha}(k)\prod_{i=1}^{N} \exp\Big(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j}\, k_i^j\Big) dk_i\right)$$
$$= \log\left(\Big(z^{\frac{1}{2}} e^{\frac{1}{z}} z^{\frac{1}{z}}\Big)^n z^{\frac{\alpha N(N-1)}{2}} \int_{[0,\infty)^N} \Delta^{2\alpha}(k)\prod_{i=1}^{N} e^{k_i} k_i^{\frac{1}{z}}\, dk_i\right). \qquad \text{(B.2)}$$

(For the mechanism changing the integration from $\mathbb{R}^N$ to $[0, \infty)^N$, we refer to [19].) This integral can be calculated by the Selberg integration formula:

$$\int_{[0,\infty)^N} \Delta^{2\alpha}(k)\prod_{i=1}^{N} e^{k_i} k_i^{\frac{1}{z}}\, dk_i = \prod_{j=0}^{N-1} \frac{\Gamma(1+\alpha+j\alpha)\Gamma(1+\frac{1}{z}+j\alpha)}{\Gamma(1+\alpha)}.$$

Therefore, we have

$$\lim_{m\to\infty} \log\left(\int_{\mathbb{R}^N} \Delta^{2\alpha}(k)\prod_{i=1}^{N} \exp\Big(-\sum_{j=2}^{2m} \frac{z^{j/2-1}}{j}\, k_i^j\Big) dk_i\right)$$
$$= c + \frac{N}{2}\log z + \frac{N}{z} + \frac{N}{z}\log z + \frac{\alpha N(N-1)}{2}\log z + \log\prod_{j=0}^{N-1}\Gamma\left(1+\frac{1}{z}+j\alpha\right), \qquad \text{(B.3)}$$

where c is the constant term independent of z. Since (B.1) does not have any constant term relative to z, here and below we ignore all constant terms independent of z (but possibly N dependent). The product of Γ-functions can be calculated by the recursion formula, noticing that α is an integer:

$$\prod_{j=0}^{N-1}\Gamma\left(1+\frac{1}{z}+j\alpha\right) = \Gamma(1/z)^N \prod_{i=0}^{N-1}\prod_{j=0}^{i\alpha}\left(\frac{1}{z}+i\alpha-j\right)$$
$$= \Gamma(1/z)^N\left(\frac{1}{z}\right)^N \prod_{i=0}^{N-1}\prod_{j=1}^{\alpha}\left(\frac{1}{z}+i\alpha+j\right)^{N-i-1}$$
$$= \Gamma(1/z)^N\left(\frac{1}{z}\right)^N \prod_{i=1}^{N-1}\prod_{j=0}^{\alpha-1}\left(\frac{1+z(1+(i-1)\alpha+j)}{z}\right)^{N-i}. \qquad \text{(B.4)}$$

We now apply Stirling's formula for $\log \Gamma(1/z)$ to obtain, up to a constant term:

$$\begin{aligned}\log \prod_{j=0}^{N-1} \Gamma\left(1+\frac{1}{z}+j\alpha\right) = &-\frac{N}{z}\log z - \frac{N}{z} + \frac{N}{2}\log z + \sum_{m=1}^{\infty} \frac{b_{2m}}{2m(2m-1)} N z^{2m-1} \\ &-N\log z - \frac{\alpha N(N-1)}{2}\log z \\ &+\sum_{m=1}^{\infty}\sum_{i=1}^{N-1}\sum_{j=0}^{\alpha-1} \frac{1}{m}(-1)^{m-1}(N-i)(1+(i-1)\alpha+j)^m z^m.\end{aligned} \tag{B.5}$$

We note that all negative powers of z and $\log z$ related terms in (B.3) cancel out using (B.5). Finally, we obtain

$$\begin{aligned}K(z,N,\alpha) = &\sum_{m=1}^{\infty} \frac{b_{2m}}{2m(2m-1)} N z^{2m-1} \\ &+\sum_{m=1}^{\infty}\sum_{i=0}^{N-1}\sum_{j=1}^{\alpha} \frac{1}{m}(-1)^{m-1}(N-1-i)(i\alpha+j)^m z^m.\end{aligned} \tag{B.6}$$

This last sum of powers can be calculated using Bernoulli polynomials, from which (B.1) follows.

Using a formula for Bernoulli numbers,

$$(1-2n)b_{2n} = \sum_{q=0}^{n}\binom{2n}{2q} b_{2q} b_{2n-2q} = \sum_{q=0}^{n}\binom{2n}{2q} b_{2q} b_{2n-2q} 2^{2q}, \quad n \neq 1,$$

and noting that $b_2 = 1/6$, we recover the original formula of Penner for $\alpha = 1$ [21]:

$$\begin{aligned}K(z,N,1) &= -\sum_{m=1}^{\infty} \frac{b_{2m}}{2m} N z^{2m-1} + \sum_{m=1}^{\infty}\sum_{q=0}^{[\frac{m}{2}]} \frac{(m-1)!(2q-1)}{(2q)!(m+2-2q)!} b_{2q} N^{m+2-2q}(-z)^m \\ &= \sum_{\substack{g\geq 0, n>0 \\ 2-2g-n<0}} \frac{(2g+n-3)!(2g-1)}{(2g)!n!} b_{2g} N^n (-z)^{2g+n-2} \\ &= \sum_{\Gamma\in\mathfrak{R}} \frac{(-1)^{e_\Gamma}}{|\mathrm{Aut}_{\mathfrak{R}}(\Gamma)|} N^{f_\Gamma}(-z)^{e_\Gamma - v_\Gamma}.\end{aligned} \tag{B.7}$$

For $\alpha = 2$, (B.1) simplifies again:

$$\begin{aligned}K(z,N,2) = &-\sum_{m=1}^{\infty} \frac{b_{2m}}{2m} N z^{2m-1} \\ &+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{q=0}^{[\frac{m}{2}]} \frac{(m-1)!(2q-1)}{(2q)!(m+2-2q)!} b_{2q} 2^{m+2-2q} N^{m+2-2q}(-z)^m\end{aligned}$$

$$- \frac{1}{2} \sum_{m=1}^{\infty} \sum_{q=0}^{[\frac{m}{2}]} \frac{(m-1)!(2^{2q-1}-1)}{(2q)!(m+1-2q)!} b_{2q} 2^{m+1-2q} N^{m+1-2q} (-z)^m. \tag{B.8}$$

Note that the first two lines of (B.8) are identical to the Penner model $\frac{1}{2}K(z, 2N, 1)$.

The following integral formula, again valid for every positive integer $\gamma \in \mathbb{N}$, has been established in [10]:

$$\begin{aligned}
&J(z, N, \gamma) \\
&= \lim_{m\to\infty} \log \left(\frac{\int_{\mathbb{R}^N} |\Delta(k)|^{2/\gamma} \prod_{i=1}^N \exp\left(- \sum_{j=2}^{2m} \frac{z^{j/2-1}}{j} k_i^j \right) dk_i}{\int_{\mathbb{R}^N} |\Delta(k)|^{2/\gamma} \prod_{i=1}^N \exp\left(- \frac{k_i^2}{2} \right) dk_i} \right) \\
&= \sum_{m=1}^{\infty} \frac{b_{2m}}{2m\,(2m-1)} \frac{N}{\gamma} \left(\frac{z}{\gamma} \right)^{2m-1} + \sum_{m=1}^{\infty} \frac{1}{4m} (-1)^m N^m \left(\frac{z}{\gamma} \right)^m \\
&\quad - \frac{1}{2} \sum_{m=1}^{\infty} \sum_{q=0}^{[\frac{m}{2}]} \frac{(-1)^m (m-1)! b_{2q}}{(2q)!(m+1-2q)!} \left(1 - \frac{1}{\gamma^{1-2q}} \right) N^{m+1-2q} \left(\frac{z}{\gamma} \right)^m \\
&\quad - \sum_{m=1}^{\infty} \sum_{q=0}^{[\frac{m}{2}]} \sum_{s=0}^{[\frac{m+1}{2}]-q} \frac{(-1)^m (m-1)! b_{2q} b_{2s}}{(2q)!(2s)!(m+2-2q-2s)!} \cdot \frac{1}{\gamma^{1-2s}} N^{m+2-2q-2s} \left(\frac{z}{\gamma} \right)^m.
\end{aligned} \tag{B.9}$$

Our duality (7.12) for the generalized Penner model follows from comparing (B.1) with (B.9).

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