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Complete Integrability of the Kadomtsev–Petviashvili Equation

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The total hierarchy of the Kadomtsev–Petviashvili (KP) equation is transformed to the system of linear partial differential equations with constant coefficients. The complete integrability of the KP equation is proved by using this linear system. The existence and uniqueness theorem of the Cauchy problem of the KP hierarchy is obtained. © 1984 Academic Press, Inc.

The purpose of this paper is to show the complete integrability of the total hierarchy of the Kadomtsev–Petviashvili equation (KP hierarchy) in the sense of Frobenius. Let us consider a linear total differential equation

$$dU = \Omega U \tag{1}$$

for a Lie group valued unknown function U defined on some manifold, where Ω is a given 1-form with values in the Lie algebra of the Lie group. Equation (1) is said to be *completely integrable in the sense of Frobenius* if and only if Ω satisfies the 0-curvature condition

$$d\Omega - \tfrac{1}{2}[\Omega, \Omega] = 0. \tag{2}$$

In this paper, I transform the KP hierarchy to (1) defined on an infinite dimensional fibre bundle over the infinite-dimensional affine space whose fibres are the formal Lie group of the Lie algebra consisting of all the formal ordinary micro- (or pseudo-) differential operators. Our linear equation (1) is simple enough to be solved just by algebraic procedures. This enables us to solve the initial value problem of the KP hierarchy only by *quadrature* in our algebraic category. We can also determine the space of all solutions to the KP hierarchy.

The KP hierarchy involves many kinds of soliton equations such as the Korteweg–de Vries (KdV) hierarchy and the Boussinesq hierarchy [1]. They have been regarded as typical examples of nonlinear equations, however, I

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would like to state here that even soliton equations, though they describe several nonlinear phenomena, have a beautiful *linear* structure as their essential nature hidden deeply behind them. It is remarkable that such purely nonlinear equations can be transformed into a system of rather simple linear partial differential equations by using infinite-dimensional formal Lie groups.

It is difficult to extend the classical definition of the complete integrability to infinite-dimensional cases. However, I propose in this paper that we can use the statement of the classical theorem of Frobenius as the *definition* of the complete integrability in infinite dimensional cases and that this definition fits our sense because soliton equations such as the KdV equation, the Boussinesq equation and so on, turn out to be completely integrable under this definition.

The definition of the KP hierarchy and the algebraic settings are given in Section 1 following [1, 2, 6]. In Section 2, I define the formal Lie groups of the Lie algebras of ordinary (micro-) differential operators and, with this preparation, present our main theorems.

Notation

$K[[x]]$ = ring of all the formal power series in x with coefficients in K.
$K((x))$ = field of quotients of $K[[x]]$.
$\mathbb{N}$ = set of nonnegative integers.
$\mathbb{Z}$ = set of all integers.

1. Algebraic Settings of the KP Hierarchy

First of all we construct our stage: The trivial Lie algebra bundle $\mathbf{E}_T$ defined over the infinite-dimensional affine space $T = \varinjlim_n K^n$ with inductive limit toplogy, where K is an arbitrary field of characteristic 0. The fibre of $\mathbf{E}_T$ is the Lie algebra

$$E = R((\partial^{-1}))$$

consisting of all the formal ordinary micro-differential operators with coefficients in a commutative differential algebra R defined over K together with a derivation

$$\partial: R \to R.$$

We require that R has the following properties:

(i) R is a K-algebra with the identity 1;

(ii) R is closed under indefinite integration, i.e., for any $f \in R$ there is $g \in R$ such that $\partial g = f$;

(iii) R is closed under exponentiation, i.e. for every $f \in R$ the expression $\sum_{n=0}^{\infty} (1/n!) f^n$ is also an element of R.

The associative K-algebra structure and the Lie algebra structure of E is studied in [2, 6]. An element f in R is called a constant if $\partial f = 0$.

The Lie algebra E has two typical Lie subalgebras; the Lie algebra $D = R[\partial]$ consists of all the ordinary differential operators with coefficients in R, and $E^{(-1)}$ consists of all the micro-differential operators of order at most -1. The left R-module direct sum decomposition

$$E = D \oplus E^{(-1)}, \tag{3}$$

plays an important role in the whole theory. Every operator $P \in E$ is written uniquely as

$$P = P_+ + P_-, \tag{4}$$

where $P_+ \in D$ and $P_- \in E^{(-1)}$.

The "nilpotent" part $E^{(-1)}$ of E has a formal Lie group $G = 1 + E^{(-1)}$ and this group G acts on E by adjoint action which preserves the order of elements in E.

We also consider the trivial bundles $\mathbf{D}_T$, $\mathbf{E}_T^{(-1)}$ and $\mathbf{G}_T$ on T with fibres D, $E^{(-1)}$ and G, respectively. Since T is infinite dimensional, we have to be careful to talk about *sections* of these bundles. So first let us define the set of all the *functions* on T with values in R by

$$\mathscr{R} = R[[t_1, t_2, t_3, \dots]], \tag{5}$$

where we define $\operatorname{ord}(t_n) = n$ for $n = 1, 2, 3, \dots$ according to the inductive limit topology of T. Note that $\mathscr{R}$ satisfies again the condition (i) ~ (iii). The set of all the *formal sections* of $\mathbf{E}_T$ and $\mathbf{D}_T$ are defined by

$$\mathscr{E} = \Gamma(T, \mathbf{E}_T) \underset{dfn}{=} \mathscr{R}((\partial^{-1})) \tag{6}$$

and

$$\mathscr{D} = \Gamma(T, \mathbf{D}_T) \underset{dfn}{=} \mathscr{R}[\partial]. \tag{7}$$

Since $\mathbf{E}_T^{(-1)}$ and $\mathbf{G}_T$ are subbundles of $\mathbf{E}_T$, we can also define the sets $\mathscr{E}^{(-1)} = \Gamma(T, \mathbf{E}_T^{(-1)})$ and $\mathscr{G} = \Gamma(T, \mathbf{G}_T)$ of all the formal sections.

An element $L \in \mathscr{E}$ is called a *Lax operator* if

$$\mathscr{E} = \mathscr{R}((L^{-1})). \tag{8}$$

A Lax operator can be taken as a monic first-order operator without loss of generality. Moreover changing L via the inner automorphism $L \mapsto uLu^{-1}$ by $u \in \mathscr{R}$, if necessary, we can assume that

$$L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots, \tag{9}$$

where $u_{-1}, u_{-2}, \cdots \in \mathscr{R}$, because R is closed under exponentiation. From now on, we always assume that Lax operators are of the form (9).

The simplest Lax operator is ∂ itself and the group $\mathscr{G}$ connects ∂ and any other Lax operators; for every Lax operator L there exists $S \in \mathscr{G}$ such that

$$L = S\,\partial S^{-1} \qquad ([2, 6]). \tag{10}$$

We call this S a *gauge operator* of L. The ambiguity of a gauge operator S of L is exactly the right multiplication $S \mapsto SC$ by $C \in \mathscr{G}_C = \{C \in \mathscr{G} \mid [C, \partial] = 0\}$ because $L = S\,\partial S^{-1} = (SC)\,\partial (SC)^{-1}$.

Now let us introduce connection forms on $\mathbf{E}_T$. The formal 1-form

$$Z = \sum_{n=1}^{\infty} (L^n)_+ \, dt_n \tag{11}$$

is called the Zakharov–Shabat (ZS) connection form of L and

$$Z^c = -\sum_{n=1}^{\infty} (L^n)_- \, dt_n \tag{12}$$

is called the complementary Zakharov–Shabat (cZS) connection of L, where $t_1, t_2, \ldots$ are the coordinates of T.

Let d denote the exterior differentiation on T.

Definition 1 ([1, 8]). The equation

$$dL - [Z, L] = 0 \tag{13}$$

is called the Lax equation of the KP hierarchy and

$$dZ - \tfrac{1}{2}[Z, Z] = 0 \tag{14}$$

is called the Zakharov–Shabat (ZS) equation of the KP hierarchy.

Remarks. (i) The Lax equation (13) asserts that L is a horizontal section of $\mathbf{E}_T$ with respect to the ZS connection Z and (14) asserts that the ZS connection is flat. Note that equation (13) is essentially nonlinear because Z contains L in its definition.

(ii) In the case of $R = K[[x]]$ with derivation $\partial = d/dx$, our Lax equation (13) is nothing but the total hierarchy of the KP equation

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] \tag{15}$$

for unknown functions $u_{-1}, u_{-2}, \dots$ which are the coefficients of L [1]. Substituting $u = u_{-1}$, $y = t_2$ and $t = t_3$, we restore the original KP equation (two-dimensional KdV equation)

$$\tfrac{3}{4}u_{yy} - (u_t - \tfrac{1}{4}u_{xxx} - 3uu_x)_x = 0. \tag{16}$$

(iii) It is known that the above two equations are equivalent under definition (11) ([1, 6]). Moreover if L satisfies the Lax equation then

$$dL - [Z^c, L] = 0 \tag{17}$$

and

$$dZ^c - \tfrac{1}{2}[Z^c, Z^c] = 0 \tag{18}$$

are also satisfied.

LEMMA 1 ([7]). *A Lax operator L satisfies the Lax equation* (13) *if and only if there is a gauge operator $S \in \mathscr{S}$ of L satisfying the* gauge equation

$$dS = Z^c S. \tag{19}$$

Proof. Suppose L satisfies the Lax equation (13). Let $S_0 \in \mathscr{S}$ be one of the gauge operators of L; i.e., $L = S_0\, \partial S_0^{-1}$. Define

$$Z_0^c = S_0^{-1} Z^c S_0 - S_0^{-1} dS_0.$$

This is the gauge transformation of the cZS connection Z^c by the bundle automorphism S_0^{-1}. Since Z^c is a flat connection, Z_0^c is also a flat connection. Now let us show that Z_0^c has only constant coefficients;

$$\begin{aligned}
[Z_0^c, \partial] &= S_0^{-1}[S_0 Z_0^c S_0^{-1}, S_0\, \partial S_0^{-1}]\, S_0 \\
&= S_0^{-1}[Z^c - dS_0 S_0^{-1}, L]\, S_0 \\
&= S_0^{-1}([Z^c, L] - [dS_0 S_0^{-1}, L])\, S_0 \\
&= S_0^{-1}(dL - dL)\, S_0 \\
&= 0.
\end{aligned}$$

Then there is an element $C \in \mathscr{S}_c$ such that $Z_0^c = dC \cdot C^{-1}$. Since

$$C^{-1} Z_0^c C - C^{-1} dC = 0,$$

we conclude that the gauge transformation of Z^c by $S^{-1} = (S_0 C)^{-1}$ is 0;

$$(S_0 C)^{-1} Z^c (S_0 C) - (S_0 C)^{-1}\, d(S_0 C) = S^{-1}(Z^c - dS \cdot S^{-1})\, S = 0.$$

Note that $S\,\partial S^{-1}=S_0 C\,\partial C^{-1}S_0^{-1}=S_0\,\partial S_0^{-1}=L$. Thus we have found the gauge operator S satisfying the gauge equation (19).

The converse is obvious. ∎

Remark. The gauge operator S of L satisfying the gauge equation still has ambiguity which comes from the multiplication

$$S \mapsto SC$$

of $C \in G_C=\{C \in G \mid [C, \partial]=0\}$ from the right.

DEFINITION 2. For every Lax operator L satisfying the Lax equation, a gauge operator S of L is called a *Sato–Wilson* (*SW*) *operator* if S satisfies the gauge equation (19).

Lemma 1 says that the gauge transformation of Z^c by a SW operator is just 0. Now let us compute the gauge transformation of the ZS connection Z by this SW operator.

LEMMA 2. *Let S be the SW operator of Lemma* 1. *Then*

$$S\Omega S^{-1}+dS\cdot S^{-1}=Z, \tag{20}$$

where $\Omega=\sum_{n=1}^{\infty}\partial^n\,dt_n$.

Proof.

$$\begin{aligned}
S\Omega S^{-1} &= \sum_{n=1}^{\infty} L^n\,dt_n \\
&= \sum_{n=1}^{\infty}(L^n)_+\,dt_n+\sum_{n=1}^{\infty}(L^n)_-\,dt_n \\
&= Z-Z^c \\
&= Z-dS\cdot S^{-1}. \quad ∎
\end{aligned}$$

Note that Ω obviously satisfies the 0-curvature condition

$$d\Omega-\tfrac{1}{2}[\Omega,\Omega]=0.$$

This explains why Z should satisfy the ZS equation (14).

2. The Linear Total Differential Equation Which Is Equivalent to the KP Hierarchy

Let us start with the definition of formal Lie groups of the big Lie algebras $\mathscr{E}$ and $\mathscr{D}$.

DEFINITION 3 ([3]). The *formal completion* of $\mathscr{E}$ and $\mathscr{D}$ are defined by

$$\hat{\mathscr{E}} = \left\{ \sum_{\nu \in \mathbb{Z}} p_\nu \partial^\nu \mid p_\nu \in \mathscr{R}, \exists N \in \mathbb{Z}, \operatorname{ord}_t(p_\nu) > \nu - N \ \forall \nu \gg 0 \right\}$$

and

$$\hat{\mathscr{D}} = \left\{ \sum_{\nu=0}^{\infty} p_\nu \partial^\nu \mid p_\nu \in \mathscr{R}, \exists N \in N, \operatorname{ord}_t(p_\nu) > \nu - N \ \forall \nu \gg 0 \right\},$$

where $\operatorname{ord}_t(p_\nu)$ denotes the order of p_ν as a formal power series in $t = (t_1, t_2, t_3, ...)$.

DEFINITION 4 ([3]). The formal Lie groups of $\mathscr{D}$ and $\mathscr{E}$ are defined by

$$\hat{\mathscr{D}}^x = \{P \in \hat{\mathscr{D}} \mid P|_{t=0} = 1, \exists P^{-1} \in \hat{\mathscr{D}}\}$$

and

$$\hat{\mathscr{E}}^x = \{P \in \hat{\mathscr{E}} \mid P|_{t=0} \in G, \exists P^{-1} \in \hat{\mathscr{E}}\},$$

where $P|_{t=0}$ is the restruction of P at $t = 0 \in T$.

We have a corresponding decomposition of $\mathscr{E} = \mathscr{D} \oplus \mathscr{E}^{(-1)}$ in group level. The following lemma is the key lemma of the main theorems.

LEMMA 3. *For every $U \in \hat{\mathscr{E}}^x$, there exists unique $S \in \mathscr{G}$ and $Y \in \hat{\mathscr{D}}^x$ such that*

$$U = S^{-1} Y.$$

Thus we have a group decomposition

$$\hat{\mathscr{E}}^x = \mathscr{G} \cdot \hat{\mathscr{D}}^x,$$

where $\mathscr{G} \cap \hat{\mathscr{D}}^x = \{1\}$.

Proof. Let $U = \sum_{\nu \in \mathbb{Z}} u_\nu \partial^\nu \in \hat{\mathscr{E}}^x$. It is sufficient to prove that there is a unique element $S = 1 + \sum_{\mu=-\infty}^{-1} s_\mu \partial^\mu \in \mathscr{G}$ such that $SU = \hat{\mathscr{D}}^x$. So let us solve the equation $(SU)_- = 0$ for unknown S. Now,

$$\begin{aligned}
0 &= (SU)_- \\
&= \left[\left(\sum_{\mu=-\infty}^{0} s_\mu \partial^\mu \right) \left(\sum_{\nu \in \mathbb{Z}} u_\nu \partial^\nu \right) \right]_- \\
&= \left[\sum_{\mu=-\infty}^{0} \sum_{\nu \in \mathbb{Z}} \sum_{i=1}^{\infty} \binom{\mu}{i} s_\mu (\partial^i u_\nu) \partial^{\mu+\nu-i} \right]_- \\
&= \left[\sum_{\mu=-\infty}^{0} \sum_{k \in \mathbb{Z}} \sum_{i=0}^{\infty} \binom{\mu}{i} s_\mu (\partial^i u_{k-\mu+i}) \partial^k \right]_- \\
&= \sum_{k=-\infty}^{-1} \left[u_k + \sum_{\mu=-\infty}^{-1} \sum_{i=0}^{\infty} \binom{\mu}{i} s_\mu (\partial^i u_{k-\mu+i}) \right] \partial^k.
\end{aligned}$$

Thus we obtain a system of linear algebraic equations:

$$\sum_{\nu=-\infty}^{-1} \sum_{i=0}^{\infty} \binom{\nu}{i} (\partial^i u_{\mu+i-\nu}) s_\nu = -u_\mu \qquad \text{for} \quad \mu = -1, -2, -3, \dots . \tag{21}$$

Let $M = [\sum_{i=0}^{\infty} \binom{\nu}{i} \partial^i u_{\mu+i-\nu}]_{\mu,\nu=-1,-2,-3,\dots}$. Note that for every fixed μ and ν, the infinite sum $\sum_{i=0}^{\infty} \binom{\nu}{i} \partial^i u_{\mu+i-\nu}$ determines an element in $\mathscr{R}$ because of the growth order condition imposed in Definition 3. The matrix M is a well-defined infinite matrix with components in $\mathscr{R}$.

Since $U|_{t=0} \in G$, the (μ, ν)-component $\sum_{i=0}^{\infty} \binom{\nu}{i} \partial^i u_{\mu+i-\nu}|_{t=0}$ of $M|_{t=0}$ is 1 if $\mu = \nu$ and 0 if $\mu > \nu$. Thus we know that

$$M|_{t=0} = \begin{array}{cc} \begin{bmatrix} \ddots & & & * \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{bmatrix} & \begin{array}{c} \vdots \\ -3 \\ -2 \\ -1 \end{array} \\ \begin{array}{cccc} \cdots & -3 & -2 & -1 \end{array} & \end{array} .$$

Hence M is invertible, and we can solve the linear algebraic equation (21) by

$$\begin{bmatrix} \vdots \\ \\ s_{-3} \\ s_{-2} \\ s_{-1} \end{bmatrix} = -M^{-1} \begin{bmatrix} \vdots \\ \\ u_{-3} \\ u_{-2} \\ u_{-1} \end{bmatrix} . \quad \blacksquare$$

Under these preparations we can now state our main theorems.

THEOREM 1. *The Lax equation* (13) *of the KP hierarchy is equivalent to the linear total differential equation*

$$dU = \Omega U \tag{1}$$

for $U \in \hat{\mathscr{E}}^\times$, *and this equation is completely integrable in the sense of Frobenius.*

Proof. Suppose we have a solution $L \in \mathscr{E}$ of the Lax equation (13). Then we also have the ZS connection Z, the cZS connection Z^c both satisfying the 0-curvature condition and an SW operator $S \in \mathscr{S}$ satisfying the gauge equation (19). Since Z is a flat connection on $\mathbf{D}_T$, we can find $Y \in \hat{\mathscr{D}}^\times$ such that

$$Z = dY\, Y^{-1}.$$

Define $U = S^{-1} \cdot Y \in \hat{\mathscr{E}}^\times$. Then it is obvious that U satisfies the linear equation (1) because of Lemma 2.

Conversely, let $U \in \mathscr{E}^{x}$ be the solution of (1). Decompose it into $U = S^{-1} \cdot Y$ as in Lemma 3. Now define

$$L = S\,\partial S^{-1},$$
$$Z = dY \cdot Y^{-1},$$
$$Z^c = dS \cdot S^{-1}.$$

Since $S\Omega S^{-1} = SdU \cdot U^{-1}S^{-1} = Z - Z^c$ and Z^c is a 1-form with values in $\mathscr{E}^{(-1)}$, we see that Z^c is nothing but the cZS connection of L and S is a SW operator. By Lemma 1, L should be a solution to the Lax equation (13). ∎

Remarks. (i) Let $U = \sum_{\nu \in \mathbb{Z}} u_\nu \partial^\nu$. Then

$$dU - \Omega U = \sum_{n=1}^{\infty} \left[\sum_{\nu \in \mathbb{Z}} \frac{\partial u_\nu}{\partial t_n} \partial^\nu - \sum_{\nu \in \mathbb{Z}} \sum_{i=0}^{n} \binom{n}{i} (\partial^i u_\nu) \partial^{\nu+n-i} \right] dt_n$$
$$= \sum_{n=1}^{\infty} \sum_{\nu \in \mathbb{Z}} \left[\frac{\partial u_\nu}{\partial t_n} - \sum_{i=0}^{n} \binom{n}{i} \partial^i u_{\nu-n+i} \right] \partial^\nu dt_n.$$

Thus our Eq. (1) is nothing but a system

$$\frac{\partial u_\nu}{\partial t_n} = \sum_{i=0}^{n} \binom{n}{i} \partial^i u_{\nu-n+i} \qquad \text{for} \quad n = 1, 2, 3, \ldots \text{ and } \nu \in \mathbb{Z} \tag{22}$$

of linear partial differential equations with *constant coefficients*.

(ii) We can solve the initial value problem of this linear equation (1). Suppose we have an initial data $U(0) \in G$ at $t = 0$. Then the solution to (1) is given by the following very simple expression

$$U(t) = \exp\left[\sum_{n=1}^{\infty} t_n \partial^n \right] U(0). \tag{23}$$

THEOREM 2. *The initial value problem of the Lax equation is uniquely solvable in our formal category.*

Proof. Let us start with a Lax operator $L \in E$ at $t = 0$. So $L = \partial + u_{-1} \partial^{-1} + u_{-2} \partial^{-2} + \cdots$ and $u_{-1}, u_{-2}, \ldots \in R$. We can find $S \in G$ such that $L = S\,\partial S^{-1}$. Now take any $S_1, S_2 \in G$ so that both of them give the same L. Then there exists $C \in G_c$ such that $S_1 = S_2 C$.

Let $U_i(t) = \exp[\sum_{n=1}^{\infty} t_n \partial^n] \cdot S_i^{-1}$, $i = 1, 2$. Since C commutes with $\exp[\sum_{n=1}^{\infty} t_n \partial^n]$, the solution $S_i(t)$ to the gauge equation (19) defined by $U_i(t)$ satisfies the relation $S_1(t) = S_2(t) \cdot C$. Then this ambiguity does not reflect on the Lax operator

$$L(t) = S_1(t)\,\partial S_1(t)^{-1}$$
$$= S_2(t)\,\partial S_2(t)^{-1}.$$

This is the desired solution to the initial value problem of the Lax equation. ∎

Note that the whole process is algebraic except the calculation from L to S which is a computation of infinitely many recursive indefinite integrals. Thus we can say that the Lax equation (13) can be solved by *quadrature* alone.

COROLLARY. *The space of all the solutions to the Lax equation* (13) *is the space* G/G_c *of the initial data of Lax operators.*

In the case of $R = K[[x]]$, the group G, which is the space of all the solutions to the gauge equation (19), has a structure of infinite-dimensional affine space with projective limit topology [4]. A sufficient condition of convergence of our formal solution to the KP hierarchy will be given elsewhere.

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