THE LAPLACE TRANSFORM, MIRROR SYMMETRY, AND THE TOPOLOGICAL RECURSION OF EYNARD-ORANTIN

MOTOHICO MULASE

Abstract. This paper is based on the author's talk at the 2012 Workshop on Geometric Methods in Physics held in Białowieża, Poland. The aim of the talk is to introduce the audience to the Eynard-Orantin topological recursion. The formalism is originated in random matrix theory. It has been predicted, and in some cases it has been proven, that the theory provides an effective mechanism to calculate certain quantum invariants and a solution to enumerative geometry problems, such as open Gromov-Witten invariants of toric Calabi-Yau threefolds, single and double Hurwitz numbers, the number of lattice points on the moduli space of smooth algebraic curves, and quantum knot invariants. In this paper we use the Laplace transform of generalized Catalan numbers of an arbitrary genus as an example, and present the Eynard-Orantin recursion. We examine various aspects of the theory, such as its relations to mirror symmetry, Gromov-Witten invariants, integrable hierarchies such as the KP equations, and the Schrödinger equations.

Contents

1. Introduction

The purpose of this paper is to give an introduction to the Eynard-Orantin topological recursion [22], by going through a simple mathematical example. Our example is constructed from the Catalan numbers, their higher-genus analogues, and the mirror symmetry of these numbers.

There have been exciting new developments around the Eynard-Orantin theory in the last few years that involve various quantum topological invariants, such as single and double Hurwitz numbers, open Gromov-Witten invariants, and quantum knot polynomials. A big picture is being proposed, from which, for example, we can understand the relation between the A-polynomial [11] of a knot and its colored Jones polynomials as the same as the **mirror symmetry** in string theory.

From the rigorous mathematical point of view, the predictions on this subject coming from physics are conjectural. In mathematics we need a simple example, for which we can prove all the predicted properties, and from which we can see what is going on in a more general context. The aim of this paper is to present such an example of the Eynard-Orantin theory.

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The formalism of our interest is originated in the large N asymptotic analysis of the correlation functions of resolvents of a random matrix of size $N \times N$ [3, 18]. The motivation of Eynard and Orantin [22] is to find applications of the computational mechanism beyond random matrix theory. Their formula takes the shape of an integral recursion equation on a given Riemann surface Σ called the **spectral curve** of the theory. At that time already Mariño was developing the idea of **remodeled B-model** of topological string theory on a Riemann surface Σ in [37]. He noticed the geometric significance of [22], and formulated a precise theory of remodeling B-model with Bouchard, Klemm, and Pasquetti in [7]. This work immediately attracted the attention of the mathematics community. The currently accepted picture is that the remodeled B-model defines symmetric differential forms on Σ via the Eynard-Orantin recursion, and that these differentials forms are the **Laplace transform** of the quantum topological invariants that appear on the A-model side of the story. In this context *the Laplace transform plays the role of the mirror symmetry.*

This picture tells us that once we identify the spectral curve Σ, we can calculate the quantum topological invariants in terms of complex analysis on Σ. The effectiveness of this mechanism has been mathematically proven for single Hurwitz numbers [21, 45], orbifold (or double) Hurwitz numbers [6], enumeration of the lattice points of $\mathcal{M}_{g,n}$ [10, 47, 48], the Poincaré polynomials of $\mathcal{M}_{g,n}$ [41], the Weil-Petersson volume and its higher analogues of $\overline{\mathcal{M}}_{g,n}$ [23, 35, 38, 39, 42], and the higher-genus Catalan numbers [17]. A spectacular conjecture of [7] states that the Laplace transform of the open Gromov-Witten invariants of an arbitrary toric Calabi-Yau threefold satisfies the Eynard-Orantin topological recursion. A significant progress toward this conjecture has been made in [24].

Furthermore, an unexpected application of the Eynard-Orantin theory has been proposed in knot theory [2, 5, 9, 12, 26, 31]. A key ingredient there is the **quantum curve** that characterizes quantum knot invariants.

The word *quantum* means many different things in modern mathematics. For example, a quantum curve is a holonomic system of linear differential equations whose Lagrangian is an algebraic curve embedded in the cotangent bundle of a base curve. Quantum knot invariants, on the other hand, are invariants of knots defined by representation theory of quantum algebras, and quantum algebras are deformations of usual algebras. In such a diverse usage, the only common feature is the aspect of non-commutative deformations. Therefore, when two completely different quantum objects turn out to be the same, we expect a deep mathematical theory behind the scene. In this vein, within the last two years, mathematicians and physicists have discovered a new, miraculous mathematical procedure, although still conjectural, that directly relates quantum curves and quantum knot invariants.

The notion of quantum curves appeared in Aganagic, Dijkgraaf, Klemm, Mariño, and Vafa [1], and later in Dijkgraaf, Hollands, Sułkowski, and Vafa [13, 14]. When the A-model we start with has a vanishing obstruction class in algebraic K-theory, then it is expected that a quantum curve exists, and it is a differential operator. Let us call it P. A quantum knot invariant is a function. Call it Z. Then the conjectural relation is simply the **Schrödinger equation** $PZ = 0$. For this equation to make sense, in addition to the very existence of P, we need to identify the variables appearing in P and Z. The key observation is that both P and Z are defined on the same Riemann surface, and that it is exactly the spectral curve of the Eynard-Orantin topological recursion, being realized as a Lagrangian immersion. Moreover, the total symbol of the operator P defines the Lagrangian immersion.

What is the significance of this Schrödinger equation $PZ = 0$, then? Recently Gukov and Sułkowski [31], based on [12], provided the crucial insight that when the underlying spectral curve is defined by the A-polynomial of a knot, the algebraic K-theory obstruction vanishes, and the equation $PZ = 0$ becomes the same as the AJ-conjecture of Garoufalidis [28]. This means that the Eynard-Orantin theory conjecturally computes colored Jones polynomial as the partition function Z of the theory, starting from a given A-polynomial.

In what follows, we present a simple example of the story. Although our example is not related to knot theory, it exhibits all key ingredients of the theory, such as the Schrödinger equation, relations to quantum topological invariants, the Eynard-Orantin recursion, the KP equations, and mirror symmetry.

At the Białowieża Workshop in summer 2012, Professor L. D. Faddeev gave a beautiful talk on the quantum dilogarithm, Bloch groups, and algebraic K-theory [25]. Our example of this paper does not illustrate the fundamental connection to these important subjects, because our spectral curve (2.4) has genus 0, and the K-theoretic obstruction to quantization, similar to the idea of K_2-Lagrangian of Kontsevich, vanishes. Further developments are expected in this direction.

2. Mirror dual of the Catalan numbers and their higher genus extensions

The **Catalan numbers** appear in many different places of mathematics and physics, often quite unexpectedly. The *Wikipedia* lists some of the mathematical interpretations. The appearance in string theory [49] is surprising. Here let us use the following definition:

(2.1) $\quad C_m =$ the number of ways to place $2m$ pairs of parentheses in a legal manner.

A *legal* manner means the usual way we stack them together. If we have one pair, then $C_1 = 1$, because ( ) is legal, while)(is not. For $m = 2$, we have (( )) and ( )( ), hence $C_2 = 2$. Similarly, $C_3 = 5$ because there are five legal combinations:

$$((( ))), (( ))( ), (( )( )), ( )(( )), ( )( )( ).$$

This way of exhaustive listing becomes harder and harder as m grows. We need a better mechanism to find the value, and also a general *closed* formula, if at all possible. Indeed, we have the *Catalan recursion equation*

$$C_m = \sum_{a+b=m-1} C_a C_b, \tag{2.2}$$

and a closed formula

$$C_m = \frac{1}{m+1}\binom{2m}{m}. \tag{2.3}$$

Although our definition (2.1) does not make sense for $m = 0$, the closed formula (2.3) tells us that $C_0 = 1$, and the recursion (2.2) works only if we define $C_0 = 1$. We will give a proof of these formulas later.

Being a ubiquitous object, the Catalan numbers have many different generalizations. What we are interested here is not those kind of generalized Catalan numbers. We want to define *higher-genus* Catalan numbers. They are necessary if we ask the following question:

Question 2.1. *What is the mirror symmetric dual object of the Catalan numbers?*

The mirror symmetry was conceived in modern theoretical physics as a duality between two different Calabi-Yau spaces of three complex dimensions. According to this idea, the universe consists of the visible 3-dimensional spatial component, 1-dimensional time component, and an invisible 6-dimensional component. The invisible component of the universe is considered as a complex 3-dimensional Calabi-Yau space, and the quantum nature of the universe, manifested in quantum interactions of elementary particles and black holes, is believed to be hidden in the geometric structure of this invisible manifold. The surprising discovery is that the same physical properties can be obtained from two different settings: a Calabi-Yau space X with its Kähler structure, or another Calabi-Yau space Y with its complex structure. The duality between these two sets of data is the mirror symmetry.

The phrase, "having the same quantum nature of the universe," does not give a mathematical definition. The idea of Kontsevich [34], the **Homological Mirror Symmetry**, is to define the mirror symmetry as the equivalence of *derived* categories. Since categories do not necessarily require underlying spaces, we can talk about mirror symmetries among more general objects. For instance, we can ask the above question.

What I'd like to explain in this paper is that the answer to the question is a simple *function*

$$x = z + \frac{1}{z}. \tag{2.4}$$

It is quite radical: the mirror symmetry holds between the Catalan numbers and a function like (2.4)!

If we naively understand the homological mirror symmetry as the derived equivalence between symplectic geometry (the A-model side) and holomorphic complex geometry (the B-model side), then it is easy to guess that (2.4) should define a B-model. According to Ballard [4], the mirror symmetric partner to this function is the projective line $\mathbb{P}^1$, together with its standard Kähler structure. The higher-genus Catalan numbers we are going to define below are associated with the Kähler geometry of $\mathbb{P}^1$. Their mirror symmetric partners are the symmetric differential forms that the Eynard-Orantin theory defines on the Riemann surface of the function $x = z + \frac{1}{z}$.

It is more convenient to give a different definition of the Catalan numbers that makes the higher-genus extension more straightforward. Consider a graph Γ drawn on a sphere S^2 that has only one vertex. Since every edge coming out from this vertex has to come back, the vertex has an even degree, say $2m$. This means $2m$ **half-edges** are incident to the unique vertex. Let us place an outgoing arrow to one of the half-edges near at the vertex (see Figure 2.1). Since Γ is drawn on S^2, the large loop of the left of Figure 2.1 can be placed as in the right graph. These are the same graph on the sphere.

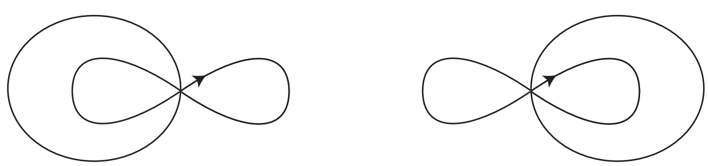

FIGURE 2.1. Two ways of representing the same arrowed graph on S^2 with one vertex. This graph corresponds to ((( ))).

Lemma 2.2. *The number of arrowed graphs on S^2 with one vertex of degree $2m$ is equal to the Catalan number C_m.*

Proof. We assign to each edge forming a loop a pair of parentheses. Their placement is nested according to the graph. The starting parenthesis '(' corresponds to the unique arrowed half-edge. We then examine all half-edges by the counter clock-wise order. When a new loop is started, we open a parenthesis '('. When it is closed to form a loop, we complete a pair of parentheses by placing a ')'. In this way we have a bijective correspondence between graphs on S^2 with one vertex of degree $2m$ and the nested pairs of $2m$ parentheses. $\square$

Now a higher-genus generalization is easy. A **cellular graph** of type (g,n) is the one-skeleton of a cell-decomposition of a connected, closed, oriented surface of genus g with n 0-cells labeled by the index set $[n]=\{1,2,\dots,n\}$. Two cellular graphs are identified if an orientation-preserving homeomorphism of a surface into another surface maps one cellular graph to another, honoring the labeling of each vertex. Let $D_{g,n}(\mu_1,\dots,\mu_n)$ denote the number of connected cellular graphs Γ of type (g,n) with n labeled vertices of degrees $(\mu_1,\dots,\mu_n)$, counted with the weight $1/|\mathrm{Aut}(\Gamma)|$. It is generally a rational number. The orientation of the surface induces a cyclic order of incident half-edges at each vertex of a cellular graph Γ. Since $\mathrm{Aut}(\Gamma)$ fixes each vertex, it is a subgroup of the Abelian group $\prod_{i=1}^n \mathbb{Z}/\mu_i\mathbb{Z}$ that rotates each vertex and the incident half-edges. Therefore,

$$C_{g,n}(\mu_1,\dots,\mu_n)=\mu_1\cdots\mu_n D_{g,n}(\mu_1,\dots,\mu_n) \tag{2.5}$$

is always an integer. The cellular graphs counted by (2.5) are connected graphs of genus g with n vertices of degrees $(\mu_1,\dots,\mu_n)$, and at the j-th vertex for every $j=1,\dots,n$, an arrow is placed on one of the incident μ_j half-edges (see Figure 2.2). The placement of n arrows corresponds to the factors $\mu_1\cdots\mu_n$ on the right-hand side. We call this integer the **Catalan number** of type (g,n). The reason for this naming comes from the fact that $C_{0,1}(2m)=C_m$, and the following theorem.

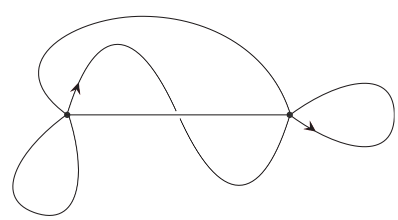

FIGURE 2.2. A cellular graph of type $(1,2)$.

Theorem 2.3. *The generalized Catalan numbers of* (2.5) *satisfy the following equation.*

$$\begin{aligned} C_{g,n}(\mu_1,\dots,\mu_n) &= \sum_{j=2}^n \mu_j C_{g,n-1}(\mu_1+\mu_j-2,\mu_2,\dots,\widehat{\mu_j},\dots,\mu_n) \\ &+ \sum_{\alpha+\beta=\mu_1-2}\left[C_{g-1,n+1}(\alpha,\beta,\mu_2,\cdots,\mu_n)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}} C_{g_1,|I|+1}(\alpha,\mu_I)C_{g_2,|J|+1}(\beta,\mu_J)\right], \end{aligned} \tag{2.6}$$

where $\mu_I=(\mu_i)_{i\in I}$ for an index set $I\subset[n]$, $|I|$ denotes the cardinality of I, and the third sum in the formula is for all partitions of g and set partitions of $\{2,\dots,n\}$.

Proof. Consider an arrowed cellular graph Γ counted by the left-hand side of (2.6), and let $\{p_1,\dots,p_n\}$ denote the set of labeled vertices of Γ. We look at the half-edge incident to p_1 that carries an arrow.

Case 1. The arrowed half-edge extends to an edge E that connects p_1 and p_j for some $j>1$.

We shrink the edge E and join the two vertices p_1 and p_j together. By this process we create a new vertex of degree $\mu_1+\mu_j-2$. To make the counting bijective, we need to be able to go back from the shrunken graph to the original, provided that we know μ_1 and μ_j.

Thus we place an arrow to the half-edge next to E around p_1 with respect to the counter-clockwise cyclic order that comes from the orientation of the surface. In this process we have μ_j different arrowed graphs that produce the same result, because we must remove the arrow placed around the vertex p_j in the original graph. This gives the right-hand side of the first line of (2.6). See Figure 2.3.

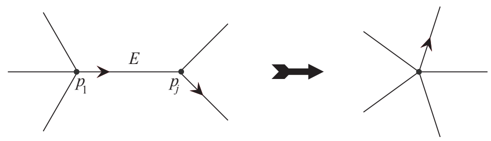

FIGURE 2.3. The process of shrinking the arrowed edge E that connects vertices p_1 and p_j, $j > 1$.

Case 2. The arrowed half-edge at p_1 is actually a loop E that goes out and comes back to p_1.

The process we apply is again shrinking the loop E. The loop E separates all other half-edges into two groups, one consisting of α of them placed on one side of the loop, and the other consisting of β half-edges placed on the other side. It can happen that $\alpha = 0$ or $\beta = 0$. Shrinking a loop on a surface causes pinching. Instead of creating a pinched (i.e., singular) surface, we separate the double point into two new vertices of degrees α and β. Here again we need to remember the position of the loop E. Thus we place an arrow to the half-edge next to the loop in each group. See Figure 2.4.

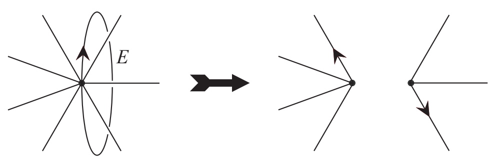

FIGURE 2.4. The process of shrinking the arrowed loop E that is attached to p_1.

After the pinching and separating the double point, the original surface of genus g with n vertices $\{p_1, \ldots, p_n\}$ may change its topology. It may have genus $g - 1$, or it splits into two pieces of genus g_1 and g_2. The second line of (2.6) records all such possibilities. This completes the proof. $\square$

Remark 2.4. For $(g, n) = (0, 1)$, the above formula reduces to

$$C_{0,1}(\mu_1) = \sum_{\alpha+\beta=\mu_1-2} C_{0,1}(\alpha)C_{0,1}(\beta), \tag{2.7}$$

which proves (2.2) since $C_{0,1}(2m) = C_m$.

Note that we *define* $C_{0,1}(0) = 1$. Only for the $(g, n) = (0, 1)$ case this irregularity of non-zero value happens for $\mu_1 = 0$. This is because a degree 0 single vertex is *connected*, and gives a cell-decomposition of S^2. We can imagine that a single vertex on S^2 has an infinite cyclic group as its automorphism, so that $C_{0,1}(0) = 1$ is consistent. In all other cases, if one of the vertices has degree 0, then the Catalan number $C_{g,n}$ is simply 0 because of the definition (2.5).

Following Kodama-Pierce [32], we introduce the generating function of the Catalan numbers by

$$z = z(x) = \sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}}. \tag{2.8}$$

Then by the quadratic recursion (2.7), we find that the inverse function of $z(x)$ that vanishes at $x = \infty$ is given by

$$x = z + \frac{1}{z},$$

which is exactly (2.4). We remark that solving the above equation as a quadratic equation for z yields

$$z = \frac{x - \sqrt{x^2-4}}{2} = \frac{x}{2}\left(1 - \sqrt{1 - \left(\frac{2}{x}\right)^2}\right) = \frac{x}{2}\sum_{m=1}^{\infty}(-1)^{m-1}\binom{\frac{1}{2}}{m}\left(\frac{2}{x}\right)^{2m},$$

from which the closed formula (2.3) follows.

3. The Laplace transform of the generalized Catalan numbers

Let us compute the Laplace transform of the generalized Catalan numbers. Why are we interested in the Laplace transform? The answer becomes clear only after we examine the result of computation.

So we define the discrete Laplace transform

$$F^C_{g,n}(t_1, \dots, t_n) = \sum_{(\mu_1,\dots,\mu_n)\in \mathbb{Z}^n_+} D_{g,n}(\mu_1, \dots, \mu_n)\, e^{-\langle w,\mu\rangle} \tag{3.1}$$

for (g, n) subject to $2g - 2 + n > 0$, where the Laplace dual coordinates $w = (w_1, \dots, w_n)$ of $(\mu_1, \dots, \mu_n)$ is related to the function coordinate $t = (t_1, \dots, t_n)$ by

$$e^{w_i} = x_i = z_i + \frac{1}{z_i} = \frac{t_i+1}{t_i-1} + \frac{t_i-1}{t_i+1}, \qquad i = 1, 2, \dots, n, \tag{3.2}$$

and $\langle w, \mu\rangle = w_1\mu_1 + \cdots + w_n\mu_n$. The **Eynard-Orantin differential form** of type (g, n) is given by

$$\begin{aligned} W^C_{g,n}(t_1, \dots, t_n) &= d_1 \cdots d_n F^C_{g,n}(t_1, \dots, t_n) \\ &= (-1)^n \sum_{(\mu_1,\dots,\mu_n)\in \mathbb{Z}^n_+} C_{g,n}(\mu_1, \dots, \mu_n)\, e^{-\langle w,\mu\rangle} dw_1 \cdots dw_n. \end{aligned} \tag{3.3}$$

Due to the irregularity that a single point is a connected cellular graph of type $(0, 1)$, we *define*

$$W^C_{0,1}(t) = -\sum_{\mu=0}^{\infty} C_{0,1}(\mu)\frac{1}{x^{\mu}} \cdot \frac{dx}{x} = -z(x)dx, \tag{3.4}$$

including the $\mu = 0$ term. Since $dF^C_{0,1} = W^C_{0,1}$, we find

$$F^C_{0,1}(t) = -\frac{1}{2}z^2 + \log z + \text{const.} \tag{3.5}$$

Using the value of Kodama and Pierce [32] for $D_{0,2}(\mu_1, \mu_2)$, we calculate (see [17])

$$F^C_{0,2}(t_1, t_2) = -\log(1 - z_1 z_2), \tag{3.6}$$

and hence

$$W^C_{0,2}(t_1,t_2) = \frac{dt_1\cdot dt_2}{(t_1-t_2)^2} - \frac{dx_1\cdot dx_2}{(x_1-x_2)^2} = \frac{dt_1\cdot dt_2}{(t_1+t_2)^2}. \tag{3.7}$$

The 2-form $\frac{dx_1\cdot dx_2}{(x_1-x_2)^2}$ is the local expression of the symmetric second derivative of the logarithm of Riemann's **prime form** on a Riemann surface. Thus $W^C_{0,2}$ is the difference of this quantity between the Riemann surface of $x = z+\frac{1}{z}$ and the x-coordinate plane. This relation is true for all known examples, and hence $W_{0,2}$ is *defined* as the second log derivative of the prime form of the spectral curve in [22]. It is important to note that in our definition, $W^C_{0,2}(t_1,t_2)$ is regular at the diagonal $t_1 = t_2$.

Note that the function $z(x)$ is absolutely convergent for $|x| > 2$. Since its inverse function is a rational function given by (2.4), the *Riemann surface* of the inverse function, i.e., the maximal domain of holomorphy of $x(z)$, is $\mathbb{P}^1\setminus\{0,\infty\}$. At $z=\pm 1$ the function $x = z+\frac{1}{z}$ is branched, and this is why $z(x)$ has the radius of convergence 2, measured from ∞. The coordinate change

$$z = \frac{t+1}{t-1}$$

brings the branch points to 0 and ∞.

Theorem 3.1 ([46])**.** *The Laplace transform $F^C_{g,n}(t_{[n]})$ satisfies the following differential recursion equation for every (g,n) subject to $2g-2+n>0$.*

$$\begin{aligned}
&\frac{\partial}{\partial t_1}F^C_{g,n}(t_{[n]}) \\
&= -\frac{1}{16}\sum_{j=2}^{n}\left[\frac{t_j}{t_1^2-t_j^2}\left(\frac{(t_1^2-1)^3}{t_1^2}\frac{\partial}{\partial t_1}F^C_{g,n-1}(t_{[\hat{j}]}) - \frac{(t_j^2-1)^3}{t_j^2}\frac{\partial}{\partial t_j}F^C_{g,n-1}(t_{[\hat{1}]})\right)\right] \\
&\qquad - \frac{1}{16}\sum_{j=2}^{n}\frac{(t_1^2-1)^2}{t_1^2}\frac{\partial}{\partial t_1}F^C_{g,n-1}(t_{[\hat{j}]}) \\
&\quad - \frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\left[\frac{\partial^2}{\partial u_1\partial u_2}F^C_{g-1,n+1}(u_1,u_2,t_2,t_3,\dots,t_n)\right]\Bigg|_{u_1=u_2=t_1} \\
&\qquad - \frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{\text{stable}}\frac{\partial}{\partial t_1}F^C_{g_1,|I|+1}(t_1,t_I)\frac{\partial}{\partial t_1}F^C_{g_2,|J|+1}(t_1,t_J).
\end{aligned} \tag{3.8}$$

Here we use the index convention $[n]=\{1,2,\dots,n\}$ and $[\hat{j}]=\{1,2,\dots,\hat{j},\dots,n\}$. The final sum is for partitions subject to the stability condition $2g_1-1+|I|>0$ and $2g_2-1+|J|>0$.

The proof follows from the Laplace transform of (2.6). Since the formula for the generalized Catalan numbers contain unstable geometries $(g,n)=(0,1)$ and $(0,2)$, we need to substitute the values (3.5) and (3.6) in the computation to derive the recursion in the form of (3.8).

Since the form of the equation (3.8) is identical to [41, Theorem 5.1], and since the initial values $F^C_{1,1}$ and $F^C_{0,3}$ of [46] agree with that of [41, (6.1), (6,2)], the same conclusion of [41] holds. Therefore,

Theorem 3.2. *The Laplace transform $F^C_{g,n}(t_1,\dots,t_n)$ in the stable range $2g-2+n>0$ satisfies the following properties.*

- *The reciprocity:* $F^C_{g,n}(1/t_1,\dots,1/t_n) = F^C_{g,n}(t_1,\dots,t_n)$.
- *The polynomiality:* $F^C_{g,n}(t_1,\dots,t_n)$ *is a Laurent polynomial of degree* $3(2g-2+n)$.
- *The highest degree asymptotics as the Virasoro condition: The leading terms of* $F^C_{g,n}(t_1,\dots,t_n)$ *form a homogeneous polynomial defined by*

$$F^{C\text{-}top}_{g,n}(t_1,\dots,t_n) = \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n\\=3g-3+n}} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{i=1}^n \left[(2d_i-1)!!\left(\frac{t_i}{2}\right)^{2d_i+1}\right], \tag{3.9}$$

where $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}$ *is the* ψ*-class intersection numbers of the Deligne-Mumford moduli stack* $\overline{\mathcal{M}}_{g,n}$*. The recursion Theorem 3.1 restricts to the highest degree terms and produces the DVV formulation* [16] *of the Witten-Kontsevich theorem* [33, 51]*, which is equivalent to the Virasoro constraint condition for the intersection numbers on* $\overline{\mathcal{M}}_{g,n}$*.*
- *The Poinaré polynomial: The principal specialization* $F^C_{g,n}(t,t,\dots,t)$ *is a polynomial in*

$$s = \frac{(t+1)^2}{4t}, \tag{3.10}$$

and coincides with the virtual Poincaré polynomial of $\mathcal{M}_{g,n}\times\mathbb{R}^n_+$*.*
- *The Euler characteristic: In particular, we have*

$$F^C_{g,n}(1,1\dots,1) = (-1)^n\chi(\mathcal{M}_{g,n}).$$

Remark 3.3. The above theorem explains why the Laplace transform of the generalized Catalan numbers is important. The function $F^C_{g,n}(t_1,\dots,t_n)$ knows a lot of topological information of both $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$.

Taking the n-fold differentiation of (3.8), we obtain a residue form of the recursion. The formula given in (3.12) is an example of the Eynard-Orantin topological recursion.

Theorem 3.4 ([17])**.** *The Laplace transform of the Catalan numbers of type* (g,n) *defined as a symmetric differential form*

$$W^C_{g,n}(t_1,\dots,t_n) = (-1)^n \sum_{(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n_+} C_{g,n}(\mu_1,\dots,\mu_n)\, e^{-\langle w,\mu\rangle} dw_1\cdots dw_n$$

satisfies the Eynard-Orantin recursion with respect to the Lagrangian immersion

$$\Sigma = \mathbb{C}\ni z \longmapsto (x(z),y(z))\in T^*\mathbb{C}, \qquad \begin{cases} x(z) = z+\frac{1}{z} \\ y(z) = -z \end{cases}. \tag{3.11}$$

The recursion formula is given by a residue transformation equation

$$\begin{aligned} W^C_{g,n}(t_1,\dots,t_n) = \frac{1}{2\pi i}\int_\gamma K^C(t,t_1)\Bigg[\sum_{j=2}^n \Big(& W^C_{0,2}(t,t_j)W^C_{g,n-1}(-t,t_2,\dots,\widehat{t_j},\dots,t_n) \\ & + W^D_{0,2}(-t,t_j)W^C_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\Big) \\ + W^C_{g-1,n+1}(t,-t,t_2,\dots,t_n) + & \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{stable} W^C_{g_1,|I|+1}(t,t_I)W^C_{g_2,|J|+1}(-t,t_J)\Bigg]. \end{aligned} \tag{3.12}$$

The kernel function is defined to be

$$K^C(t,t_1) = \frac{1}{2}\,\frac{\int_t^{-t} W_{0,2}(\,\cdot\,,t_1)}{W_{0,1}(-t)-W_{0,1}(t)} = -\frac{1}{64}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1, \tag{3.13}$$

which is an algebraic operator contracting dt, while multiplying dt_1. The contour integration is taken with respect to t on the curve defined in Figure 3.1.

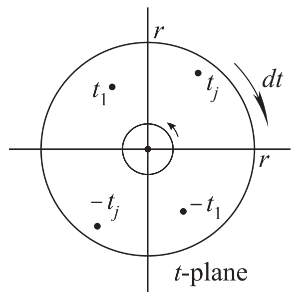

FIGURE 3.1. The integration contour γ.

Remark 3.5. The recursion (3.12) is a universal formula compared to (3.8), because the only input is the spectral curve Σ that is realized as a Lagrangian immersion, which determines $W_{0,1}$, and $W_{0,2}$ can be defined by taking the difference of the log of prime forms of Σ and $\mathbb{C}$.

4. THE PARTITION FUNCTION FOR THE GENERALIZED CATALAN NUMBERS AND THE SCHRÖDINGER EQUATION

Let us now consider the exponential generating function of the Poincaré polynomial $F^C_{g,n}(t,\dots,t)$. This function is called the **partition function** for the generalized Catalan numbers:

$$Z^C(t,\hbar) = \exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\,\hbar^{2g-2+n}F^C_{g,n}(t,t,\dots,t)\right). \tag{4.1}$$

The constant ambiguity in (3.5) makes the partition function well defined up an overall non-zero constant factor.

Theorem 4.1 ([44])**.** *The partition function satisfies the following Schrödinger equation*

$$\left(\hbar^2\frac{d^2}{dx^2}+\hbar x\frac{d}{dx}+1\right)Z^C(t,\hbar)=0, \tag{4.2}$$

where t is considered as a function in x by

$$t = t(x) = \frac{z(x)+1}{z(x)-1}$$

and (2.8)*. Moreover, the partition function has a matrix integral expression*

$$Z^C(z,\hbar) = \int_{\mathcal{H}_{N\times N}} \det(1-\sqrt{s}X)^N e^{-\frac{N}{2}\mathrm{trace}(X^2)}dX \tag{4.3}$$

with the identification (3.10) *and $\hbar = 1/N$. Here dX is the normalized Lebesgue measure on the space of $N\times N$ Hermitian matrices $\mathcal{H}_{N\times N}$. It is a well-known fact that this matrix integral is the principal specialization of a KP τ-function* [40].

The currently emerging picture [5, 12, 31] is the following. If we start with the A-polynomial of a knot K and consider the Lagrangian immersion it defines, like the one in (3.11), then the partition function Z of the Eynard-Orantin recursion, defined in a much similar way as in (4.1) but with a theta function correction factor of [5], *is* the colored Jones polynomial of K, and the corresponding Schrödinger equation like (4.2) is equivalent to the AJ-conjecture of [28].

Our example comes from an elementary enumeration problem, yet as Theorem 3.2 suggests, the geometric information contained in this example is quite non-trivial.

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Department of Mathematics, University of California, Davis, CA 95616–8633, U.S.A.
E-mail address: mulase@math.ucdavis.edu