HITCHIN INTEGRABLE SYSTEMS, DEFORMATIONS OF SPECTRAL CURVES, AND KP-TYPE EQUATIONS

ANDREW R. HODGE[1] AND MOTOHICO MULASE[2]

Abstract. An effective family of spectral curves appearing in Hitchin fibrations is determined. Using this family the moduli spaces of stable Higgs bundles on an algebraic curve are embedded into the Sato Grassmannian. We show that the Hitchin integrable system, the natural algebraically completely integrable Hamiltonian system defined on the Higgs moduli space, coincides with the KP equations. It is shown that the Serre duality on these moduli spaces corresponds to the formal adjoint of pseudo-differential operators acting on the Grassmannian. From this fact we then identify the Hitchin integrable system on the moduli space of Sp_{2m}-Higgs bundles in terms of a reduction of the KP equations. We also show that the dual Abelian fibration (the SYZ mirror dual) to the Sp_{2m}-Higgs moduli space is constructed by taking the symplectic quotient of a Lie *algebra* action on the moduli space of GL-Higgs bundles.

1. Introduction

The purpose of this paper is to determine the relation between the KP-type equations defined on the Sato Grassmannians and the Hitchin integrable systems defined on the moduli spaces of stable Higgs bundles. The results established are the following:

(1) We determine the *effective* family of spectral curves appearing in the Hitchin fibration of the moduli spaces of stable Higgs bundles.
(2) We embed the effective family of Jacobian varieties of the spectral curves into the Sato Grassmannian and show that the KP flows are tangent to each fiber of the Hitchin fibration.
(3) The moduli space of Higgs bundles of rank n and degree $n(g-1)$ on an algebraic curve of genus $g \geq 2$ is embedded into the *relative* Grassmannian of [2, 4, 23]. Using this embedding we show that the Hitchin integrable system is exactly the restriction of the KP equations on the Grassmannian to the image of this embedding.
(4) It is shown that the Krichever construction transforms the Serre duality of the geometric data consisting of algebraic curves and vector bundles on them to the formal adjoint of pseudo-differential operators acting on the Grassmannian. By identifying the fixed-point-set of the Serre duality and the formal adjoint operation we determine the KP-type equations that are

2000 *Mathematics Subject Classification.* 14H60, 14H70, 35Q53, 37J35.
Key words and phrases. Hitchin Integrable System, Spectral Curve, Higgs Bundle, Sato Grassmannian, Symplectic KP Equation.
[1]Research supported by NSF grants DMS-0135345 (VIGRE) and DMS-0406077 while at UC Davis.
[2]Research supported by NSF grant DMS-0406077 and UC Davis.

equivalent to the Hitchin integrable system defined on the moduli space of Sp_{2m}-Higgs bundles.

(5) There are two ways to *reduce* an algebraically completely integrable Hamiltonian system: one by restriction and the other by taking a quotient of a Lie algebra action that is similar to the symplectic quotient. When applied to the moduli spaces of Higgs bundles, these constructions yield SYZ-mirror pairs. We interpret the SL-PGL and Sp_{2m}-SO_{2m+1} dualities in this way.

Let $\mathcal{G}_{\mathbb{C}}$ be the category of complex Lie groups, and $\mathcal{CY}$ the category of Calabi-Yau spaces. For a compact oriented surface Σ of genus $g \geq 2$, the functor

$$\mathrm{Hom}(\hat{\pi}_1(\Sigma),\ \cdot\ )/\!\!/\ \cdot\ : \mathcal{G}_{\mathbb{C}} \longrightarrow \mathcal{CY}$$

assigns to each complex Lie group G its character variety

$$\mathrm{Hom}\big(\hat{\pi}_1(\Sigma), G\big)/\!\!/G,$$

where $\hat{\pi}_1(\Sigma)$ is the central extension of the fundamental group of Σ. The quotient by conjugation is the geometric invariant theory quotient of Mumford [21]. An amazing discovery of Hausel and Thaddeus [8], and its generalizations by [5, 14] and others, is that the character variety functor transforms the Langlands duality in $\mathcal{G}_{\mathbb{C}}$ to the mirror symmetry of Calabi-Yau spaces in the sense of Strominger-Yau-Zaslow [28]:

$$\begin{array}{ccc}
\mathcal{G}_{\mathbb{C}} & \xrightarrow{\mathrm{Hom}(\hat{\pi}_1(\Sigma),\ \cdot\ )/\!\!/\ \cdot} & \mathcal{CY} \\
{\scriptstyle \text{Langlands Dual}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Mirror Dual}} \\
\mathcal{G}_{\mathbb{C}} & \xrightarrow[\mathrm{Hom}(\hat{\pi}_1(\Sigma),\ \cdot\ )/\!\!/\ \cdot]{} & \mathcal{CY}
\end{array}$$

The character variety $\mathrm{Hom}\big(\hat{\pi}_1(\Sigma), G\big)/\!\!/G$ has many distinct complex structures [8, 14]. To understand the SYZ mirror symmetry among the character varieties, it is most convenient to realize them as *Hitchin integrable systems.* In his seminal papers [10, 11], Hitchin identifies the character variety with the moduli space of stable *G-Higgs bundles*, which has the structure of an *algebraically completely integrable Hamiltonian system.*

An algebraically completely integrable Hamiltonian system [4, 31] is a holomorphic symplectic manifold (X, ω) of dimension $2N$ together with a holomorphic map $H : X \to \mathfrak{g}^*$ such that

(1) a general fiber $H^{-1}(s)$, $s \in \mathfrak{g}^*$, is an Abelian variety of dimension N,
(2) $\mathfrak{g}^*$ is the dual Lie algebra of a general fiber $H^{-1}(s)$ considered as a Lie group, and
(3) the coordinate components of the map H are Poisson commutative with respect to the symplectic structure ω.

The notion corresponding to an algebraically completely integrable Hamiltonian system in *real* symplectic geometry is the cotangent bundle of a torus. The procedure of symplectic quotient is to remove the effect of this cotangent bundle from a given symplectic manifold. In the holomorphic context, it is often useful to take the quotient by a *family* of groups that have the same Lie algebra. Suppose we have a Lie algebra direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. If $\mathfrak{g}_1$-action on X is integrable to a group $G_{1,s}$-action in each fiber $H^{-1}(s)$ for $s \in \mathfrak{g}_2^*$, then we can define a *quotient* $X/\!\!/\mathfrak{g}_1$ as the family of quotients $H^{-1}(s)/G_{1,s}$ over $\mathfrak{g}_2^*$. We can also construct a *reduction* of (X, ω, H) by restricting the fibration to $\mathfrak{g}_2^*$ and considering the family of

$\mathfrak{g}_2$-orbits in $H^{-1}(s)$, if the $\mathfrak{g}_2$-action is integrated to a group action over $\mathfrak{g}_2^*$. When applied to the moduli space of Higgs bundles, these two constructions yield Abelian fibrations that are dual to one another, producing an SYZ mirror pair. We examine these constructions for the SL-PGL and Sp_{2m}-SO_{2m+1} dualities.

From the results established in [4, 16, 17, 19], we know that linear integrable evolution equations on the Jacobians or Prym varieties are realized as the restriction of KP-type equations defined on the Sato Grassmannians through a generalization of Krichever construction. Since the Hitchin integrable systems are defined on a *family* of Jacobian varieties or Prym varieties, we need to embed the whole family into the Sato Grassmannian to compare the Hitchin systems and the KP equations. To deal with families, we use two different approaches in this article. One approach is to utilize the theory of Sato Grassmannians defined over an arbitrary scheme developed in [2, 4, 22, 23]. In this way we can directly compare the integrable Hamiltonian systems on the Higgs moduli spaces and the KP equations. The other approach is to examine the deformations of spectral curves that appear in the Hitchin Hamiltonian systems. Once we identify the effective family of spectral curves, we can embed the whole family into a single Sato Grassmannian over $\mathbb{C}$, using the method developed in [16].

The second approach has an unexpected application: we can identify the effect of Serre duality operation on the algebro-geometric data in terms of the language of Grassmannians. Note that Sato Grassmannians are constructed from pseudo-differential operators [24, 25]. We will show, using Abel's theorem, that the Serre duality is simply the formal adjoint operation on the pseudo-differential operators. Since the Sp-Hitchin system is the fixed-point-set of the Serre duality on the GL-Hitchin system, we can determine the integrable equations corresponding to the Sp case as a reduction of the KP equations on the fixed-point-set of the formal adjoint action on pseudo-differential operators. Since our formal adjoint is slightly different from what has been studied in the literature [13, 27, 29], the equations coming up for the symplectic groups are *not* BKP or CKP equations. Let

$$P = \sum a_i(x) \left(\frac{\partial}{\partial x}\right)^i$$

be a formal pseudo-differential operator, where $a_i(x)$ is a matrix valued functions. We define the *formal adjoint* by

$$P^* = \sum \left(\frac{\partial}{\partial x}\right)^i \cdot a_i(-x)^t.$$

The reduction of the KP equations that corresponds to the Sp-Hitchin system is the $2m$-component KP equations that preserve the algebraic condition

$$\mathbf{L}^* = \begin{bmatrix} & I_m \\ I_m & \end{bmatrix} \cdot \mathbf{L} \cdot \begin{bmatrix} & I_m \\ I_m & \end{bmatrix} \tag{1.1}$$

for a $2m \times 2m$ matrix Lax operator $\mathbf{L}$ with the leading term $I_{2m} \cdot \partial/\partial x$. Several authors have proposed integrable systems with Sp_{2m}-symmetry (cf. [30]). It would be interesting to study our reduction (1.1) from the point of view of integrable systems and to investigate the relation with the other Sp integrable systems.

The fundamental literature of the algebro-geometric study of the Hitchin integrable systems and related topics is the book [4] by Donagi and Markman. Our present paper employs a slightly different perspective, that leads to the discovery of the KP-type equations corresponding to the Sp Hitchin system.

The relation between the Hitchin integrable systems and the KP equations was also studied in [15]. The treatment there was limited to the study of the Hitchin system on a single fiber. The present article extends the result therein.

We also note that many topics of this paper have been studied by the Salamanca school of algebraic geometers from yet another point of view [2, 7, 9, 22].

The paper is organized as follows. The first section is devoted to reviewing the Hitchin integrable systems of [3, 11]. We then determine an effective family of spectral curves in Section 3. Section 4 is devoted to giving two constructions of reduced integrable systems from a Hitchin system: one is a straightforward specialization, and the other is a kind of symplectic reduction by a Lie subalgebra. These two constructions give rise to an Abelian fibration and its dual Abelian fibration. We show that the GL-Hitchin integrable system is equivalent to the KP equations in Section 5. The identification of the Serre duality in terms of Grassmannians and pseudo-differential operators is carried out in Section 6. Finally we determine the KP-type equations for the Sp Hitchin system.

2. Hitchin integrable systems

In this section we review the algebraically completely integrable Hamiltonian systems defined on the moduli spaces of Higgs bundles, following [3, 4, 10, 11].

Throughout the paper we denote by C a non-singular algebraic curve of genus $g \geq 2$. The moduli space of semistable algebraic vector bundles on C of rank n and degree d is denoted by $\mathcal{U}_C(n,d)$. When n and d are relatively prime, a semistable bundle is automatically stable, and the moduli space is a non-singular projective algebraic variety of dimension $n^2(g-1)+1$. We denote by

$$\mathcal{U}_C(n) = \coprod_{d\in\mathbb{Z}} \mathcal{U}_C(n,d) \tag{2.1}$$

the space of all stable vector bundles. A *Higgs bundle* is a pair (E,ϕ) consisting of an algebraic vector bundle E on C and a global section

$$\phi \in H^0(C, \mathrm{End}(E)\otimes K_C) \tag{2.2}$$

of the endomorphism sheaf of E twisted by the canonical sheaf K_C of C. A Higgs bundle is *stable* if $\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}$ for every ϕ-invariant proper holomorphic vector subbundle F. An endomorphism of a Higgs bundle (E,ϕ) is an endomorphism ψ of E that commutes with ϕ:

$$\begin{array}{ccc} E & \xrightarrow{\ \psi\ } & E \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ E\otimes K_C & \xrightarrow[\psi\otimes 1]{} & E\otimes K_C \end{array}$$

It is known that $H^0(C, \mathrm{End}(E,\phi)) = \mathbb{C}$ for a stable Higgs bundle, and one can define the moduli space of stable objects. We denote by $\mathcal{H}_C(n,d)$ the moduli space of stable Higgs bundles of rank n and degree d on C, and

$$\mathcal{H}_C(n) = \coprod_{d\in\mathbb{Z}} \mathcal{H}_C(n,d). \tag{2.3}$$

Note that the *Serre duality* induces an involution on $\mathcal{H}_C(n)$ defined by

$$\mathcal{H}_C(n,d) \ni (E,\phi) \longmapsto (E^* \otimes K_C, -\phi^*) \in \mathcal{H}_C(n, -d + 2n(g-1)). \tag{2.4}$$

The dual of the Higgs field $\phi : E \to E \otimes K_C$ is a homomorphism $\phi^* : E^* \otimes K_C^{-1} \to E^*$. We use the same notation for the homomorphism $E^* \otimes K_C \to E^* \otimes K_C^{\otimes 2}$ induced by ϕ^*.

If E is stable, then (E,ϕ) is stable for any ϕ of (2.2). And if $\phi = 0$, then the stability of (E,ϕ) simply means E is stable. Therefore, the Higgs moduli space contains the total space of the holomorphic cotangent bundle

$$T^*\mathcal{U}_C(n,d) \subset \mathcal{H}_C(n,d), \tag{2.5}$$

since

$$H^0(C, \mathrm{End}(E) \otimes K_C) \cong H^1(C, \mathrm{End}(E))^* \cong T_E^*\mathcal{U}_C(n,d).$$

Note that $p^*\Lambda^1(\mathcal{U}_C(n,d)) \subset \Lambda^1(T^*\mathcal{U}_C(n,d))$ has a tautological section

$$\eta \in H^0(T^*\mathcal{U}_C(n,d), p^*\Lambda^1(\mathcal{U}_C(n,d))), \tag{2.6}$$

where $p : T^*\mathcal{U}_C(n,d) \to \mathcal{U}_C(n,d)$ is the projection, and Λ^r denotes the sheaf of holomorphic r-forms. The differential $\omega = -d\eta$ of the tautological section defines the canonical holomorphic symplectic form on $T^*\mathcal{U}_C(n,d)$. The restriction of ω on $\mathcal{U}_C(n,d)$, which is the 0-section of the cotangent bundle, is identically 0. Therefore the 0-section is a Lagrangian submanifold of this holomorphic symplectic space.

Let us denote by

$$V_{GL}^* = V_{GL_n(\mathbb{C})}^* = \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}). \tag{2.7}$$

As a vector space V_{GL}^* has the same dimension of $H^0(C, \mathrm{End}(E) \otimes K_C)$. The Higgs field $\phi : E \to E \otimes K_C$ induces a homomorphism of the i-th anti-symmetric tensor product spaces

$$\wedge^i(\phi) : \wedge^i(E) \longrightarrow \wedge^i(E \otimes K_C) = \wedge^i(E) \otimes K_C^{\otimes i},$$

or equivalently $\wedge^i(\phi) \in H^0(C, \mathrm{End}(\wedge^i(E)) \otimes K_C^{\otimes i})$. Taking its natural trace, we obtain

$$\mathrm{tr} \wedge^i (\phi) \in H^0(C, K_C^{\otimes i}).$$

This is exactly the i-th characteristic coefficient of the twisted endomorphism ϕ:

$$\det(x - \phi) = x^n + \sum_{i=1}^n (-1)^i \mathrm{tr} \wedge^i (\phi) \cdot x^{n-i}. \tag{2.8}$$

By assigning the coefficients of (2.8), Hitchin [11] defines a holomorphic map, now known as the *Hitchin fibration* or *Hitchin map*,

$$H : \mathcal{H}_C(n,d) \ni (E,\phi) \longmapsto \det(x - \phi) \in \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}) = V_{GL}^*. \tag{2.9}$$

The map H to the vector space V_{GL}^* is a collection of $N = n^2(g-1) + 1$ globally defined holomorphic functions on $\mathcal{H}_C(n,d)$. The 0-fiber of the Hitchin fibration is the moduli space $\mathcal{U}_C(n,d)$.

To determine the generic fiber of H, the notion of *spectral curves* is introduced in [11]. The total space of the canonical sheaf $K_C = \Lambda^1(C)$ on C is the cotangent bundle T^*C. Let

$$\pi : T^*C \longrightarrow C \tag{2.10}$$

be the projection, and

$$\tau \in H^0(T^*C, \pi^*K_C) \subset H^0(T^*C, \Lambda^1(T^*C))$$

be the tautological section of π^*K_C on T^*C. Here again $\omega_{T^*C} = -d\tau$ is the holomorphic symplectic form on T^*C. The tautological section τ satisfies that $\sigma^*\tau = \sigma$ for every section $\sigma \in H^0(C, K_C)$ viewed as a holomorphic map $\sigma : C \to T^*C$. The characteristic coefficients (2.8) of ϕ give a section

$$s = \det(\tau - \phi) = \tau^{\otimes n} + \sum_{i=1}^{n} (-1)^i \mathrm{tr} \wedge^i (\phi) \cdot \tau^{\otimes n-1} \in H^0(T^*C, \pi^*K_C^{\otimes n}). \tag{2.11}$$

We define the spectral curve C_s associated with a Higgs pair (E, ϕ) as the divisor of 0-points of the section $s = \det(\tau - \phi)$ of the line bundle $\pi^*K_C^{\otimes n}$:

$$C_s = (s)_0 \subset T^*C. \tag{2.12}$$

The projection π of (2.10) defines a ramified covering map $\pi : C_s \to C$ of degree n.

There is yet another construction of the spectral curve C_s. Since the section $s = \det(\tau - \phi)$ is determined by the characteristic coefficients of ϕ, by abuse of notation we identify s as an element of V^*_{GL}:

$$s = (s_1, s_2, \dots, s_n) = (-\mathrm{tr}\,\phi, \mathrm{tr} \wedge^2 (\phi), \dots, (-1)^n \mathrm{tr} \wedge^n (\phi)) \in \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}).$$

It defines an $\mathcal{O}_C$-module $(s_1 + s_2 + \cdots + s_n) \cdot K_C^{\otimes -n}$. Let $\mathcal{I}_s$ denote the ideal generated by this module in the symmetric tensor algebra $\mathrm{Sym}(K_C^{-1})$. Since K_C^{-1} is the sheaf of linear functions on T^*C, the scheme associated to this tensor algebra is $\mathrm{Spec}\big(\mathrm{Sym}(K_C^{-1})\big) = T^*C$. The spectral curve as the divisor of 0-points of s is then defined by

$$C_s = \mathrm{Spec}\left(\frac{\mathrm{Sym}(K_C^{-1})}{\mathcal{I}_s}\right) \subset \mathrm{Spec}\big(\mathrm{Sym}(K_C^{-1})\big) = T^*C. \tag{2.13}$$

We denote by U_{reg} the set consisting of points s for which C_s is non-singular. It is an open dense subset of V^*_{GL} [3]. We note that since $\deg(K_C) = 2g - 2 > 0$, every divisor of T^*C intersects with the 0-section C. Therefore, if C_s is non-singular, then it has only one component, and is therefore irreducible. The genus of C_s is $g(C_s) = n^2(g-1) + 1$, which follows from an isomorphism

$$\pi_*\mathcal{O}_{C_s} = \mathrm{Sym}(K_C^{-1})/\mathcal{I}_s \cong \bigoplus_{i=0}^{n-1} K_C^{\otimes -i}$$

as an $\mathcal{O}_C$-module.

The Higgs field $\phi \in H^0(C, \mathrm{End}(E) \otimes K_C)$ gives a homomorphism

$$\varphi : K_C^{-1} \longrightarrow \mathrm{End}(E),$$

which induces an algebra homomorphism, still denoted by the same letter,

$$\varphi : \mathrm{Sym}(K_C^{-1}) \longrightarrow \mathrm{End}(E).$$

Since $s \in V_{GL}^*$ is the characteristic coefficients of φ, by the Cayley-Hamilton theorem, the homomorphism φ factors through

$$\mathrm{Sym}(K_C^{-1}) \longrightarrow \mathrm{Sym}(K_C^{-1})/\mathcal{I}_s \longrightarrow \mathrm{End}(E).$$

Hence E is a module over $\mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$ of rank 1. The rank is 1 because the ranks of E and $\mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$ are the same as $\mathcal{O}_C$-modules. In this way a Higgs pair (E, ϕ) gives rise to a line bundle $\mathcal{L}_E$ on the spectral curve C_s, if it is nonsingular. Since $\mathcal{L}_E$ being an $\mathcal{O}_{C_s}$-module is equivalent to E being a $\mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$-module, we recover E from $\mathcal{L}_E$ simply by $E = \pi_*\mathcal{L}_E$, which has rank n because π is a covering of degree n. From the equality $\chi(C, E) = \chi(C_s, \mathcal{L}_E)$ and Riemann-Roch, we find that $\deg \mathcal{L}_E = d + n(n-1)(g-1)$. To summarize, the above construction defines an inclusion map

$$H^{-1}(s) \subset \mathrm{Pic}^{d+n(n-1)(g-1)}(C_s) \cong \mathrm{Jac}(C_s),$$

if C_s is non-singular.

Conversely, suppose we have a line bundle $\mathcal{L}$ of degree $d+n(n-1)(g-1)$ on a non-singular spectral curve C_s. Then $\pi_*\mathcal{L}$ is a module over $\pi_*\mathcal{O}_{C_s} = \mathrm{Sym}(K_C^{-1})/\mathcal{I}_s$, which defines a homomorphism $K_C^{-1} \to \mathrm{End}(\pi_*\mathcal{L})$, or equivalently, $\psi : \pi_*\mathcal{L} \to \pi_*\mathcal{L} \otimes K_C$. It is easy to see that the Higgs pair $(\pi_*\mathcal{L}, \psi)$ is stable. Suppose we had a ψ-invariant subbundle $F \subset \pi_*\mathcal{L}$ of rank $k < n$. Since $(F, \psi|_F)$ is a Higgs pair, it gives rise to a spectral curve $C_{s'}$ that is a degree k covering of C. Note that the characteristic polynomial $s' = \det(x - \psi)$ is a factor of the full characteristic polynomial $s = \det(x - \phi)$. Therefore, $C_{s'}$ is a component of C_s, which contradicts to our assumption that C_s is irreducible. Therefore, $\pi_*\mathcal{L}$ has no ψ-invariant proper subbundle. Thus we have established that

$$H^{-1}(s) = \mathrm{Pic}^{d+n(n-1)(g-1)}(C_s) \cong \mathrm{Jac}(C_s), \qquad s \in U_{\mathrm{reg}} \subset V_{GL}^*. \tag{2.14}$$

The vector bundle $\pi_*\mathcal{L}$ itself is not necessarily stable. It is proved in [3] that the locus of $\mathcal{L}$ in $\mathrm{Pic}^{d+n(n-1)(g-1)}(C_s)$ that gives unstable $\pi_*\mathcal{L}$ has codimension two or more.

Recall that the tautological section

$$\eta \in H^0(T^*\mathcal{U}_C(n,d), p^*\Lambda^1(\mathcal{U}_C(n,d)))$$

is a holomorphic 1-form on $T^*\mathcal{U}_C(n,d) \subset \mathcal{H}_C(n,d)$. Its restriction to the fiber $H^{-1}(s)$ of $s \in U_{\mathrm{reg}}$ for which C_s is non-singular extends to a holomorphic 1-form on the whole fiber $H^{-1}(s) \cong \mathrm{Jac}(C_s)$ since η is undefined only on a codimension 2 subset. Hence η extends as a holomorphic 1-form on $H^{-1}(U_{\mathrm{reg}})$. Thus η is well defined on $T^*\mathcal{U}_C(n,d) \cup H^{-1}(U_{\mathrm{reg}})$. The complement of this open subset in $\mathcal{H}_C(n,d)$ consists of such Higgs pairs (E, ϕ) that E is unstable *and* C_s is singular. Since the stability of E and the non-singular condition for C_s are both open conditions, this complement has codimension at least two. Consequently, both the tautological section η and the holomorphic symplectic form $\omega = -d\eta$ extend holomorphically to the whole Higgs moduli space $\mathcal{H}_C(n,d)$.

We note that there are no holomorphic 1-forms other than constants on a Jacobian variety since its cotangent bundle is trivial. It implies that

$$\omega|_{H^{-1}(s)} = -d(\eta|_{H^{-1}(s)}) \equiv 0$$

for $s \in U_{\mathrm{reg}}$. Therefore, a generic fiber of the Hitchin fibration is a Lagrangian subvariety of the holomorphic symplectic variety $(\mathcal{H}_C(n,d),\omega)$. The *Poisson structure* on $H^0(\mathcal{H}_C(n,d),\mathcal{O}_{\mathcal{H}_C(n,d)})$ is defined by

$$\{f,g\} = \omega(X_f, X_g), \qquad f,g \in H^0(\mathcal{H}_C(n,d),\mathcal{O}_{\mathcal{H}_C(n,d)}),$$

where X_f denotes the Hamiltonian vector field defined by the relation $df = \omega(X_f,\cdot)$. Since ω vanishes on a generic fiber of H, the holomorphic functions on $\mathcal{H}_C(n,d)$ coming from the coordinates of the Hitchin fibration are *Poisson commutative* with respect to the holomorphic symplectic structure ω. Therefore, $(\mathcal{H}_C(n,d),\omega,H)$ is an algebraically completely integrable Hamiltonian system [4, 31], called the *Hitchin integrable system.*

Theorem 2.1 (Hitchin [11])**.** *The Hitchin fibration*

$$H : \mathcal{H}_C(n,d) \longrightarrow V^*_{GL}$$

is a generically Lagrangian Jacobian fibration that defines an algebraically completely integrable Hamiltonian system $(\mathcal{H}_C(n,d),\omega,H)$. A generic fiber $H^{-1}(s)$ is a Lagrangian with respect to the holomorphic symplectic structure ω and is isomorphic to the Jacobian variety of a spectral curve C_s.

The Hitchin map H restricted to $T^*_E\mathcal{U}_C(n,d)$ for a stable E is a polynomial map

$$T^*_E\mathcal{U}_C(n,d) = H^0(C,\mathrm{End}(E)\otimes K_C) \ni \phi \longmapsto \det(x-\phi) \in \bigoplus_{i=1}^n H^0(C,K_C^{\otimes i})$$

consisting of g linear components, $3g-3$ quadratic components, $5g-5$ cubic components, etc., and $(2n-1)(g-1)$ components of degree n. Thus the inverse image

$$H^{-1}(s)\cap T^*_E\mathcal{U}_C(n,d)$$

for a generic (E,ϕ) consists of

$$\delta = \prod_{i=1}^n i^{(2i-1)(g-1)} \tag{2.15}$$

points [3]. Consequently, the map

$$p : \mathrm{Jac}(C_s) \cong H^{-1}(s) \longrightarrow H^{-1}(0) = \mathcal{U}_C(n,d) \tag{2.16}$$

is a covering morphism of degree δ.

Let us now determine the Hamiltonian vector field associated to each coordinate function of H. Take a point $(E,\phi)\in T^*\mathcal{U}_C(n,d)\cap H^{-1}(U_{\mathrm{reg}})$. The tangent space of the Higgs moduli space at this point is given by

$$T_{(E,\phi)}\mathcal{H}_C(n,d) = H^1(C,\mathrm{End}(E))\oplus H^0(C,\mathrm{End}(E)\otimes K_C). \tag{2.17}$$

Since the symplectic form $\omega = -d\eta$ is the exterior derivative of the tautological 1-form η of (2.6) on $T^*\mathcal{U}_C(n,d)$, the evaluation of ω at (E,ϕ) is given by

$$\omega\big((a_1,b_1),(a_2,b_2)\big) = \langle a_1,b_2\rangle - \langle a_2,b_1\rangle, \tag{2.18}$$

where $(a_1,b_1),(a_2,b_2)\in H^1(C,\mathrm{End}(E))\oplus H^0(C,\mathrm{End}(E)\otimes K_C)$, and

$$\langle\cdot,\cdot\rangle : H^1(C,\mathrm{End}(E))\oplus H^0(C,\mathrm{End}(E)\otimes K_C)\longrightarrow \mathbb{C}$$

is the Serre duality pairing. This expression is the standard Darboux form of the symplectic form ω. Choose an open neighborhood Y_1 of E in $\mathcal{U}_C(n,d)$ on which the cotangent bundle $T^*\mathcal{U}_C(n,d)$ is trivial. Then H is a polynomial map

$$H : Y_1 \times H^0(C, \mathrm{End}(E) \otimes K_C) \ni (E', \phi) \longmapsto s = \det(x - \phi) \in \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i})$$

that depends only on the second factor. The differential $dH_{(E,\phi)}$ at the point (E, ϕ) gives a linear isomorphism

$$dH_{(E,\phi)} : H^0(C, \mathrm{End}(E) \otimes K_C) \xrightarrow{\sim} \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}). \tag{2.19}$$

As to the first factor of the tangent space of (2.17), we use the differential of the covering map p of (2.16):

$$dp_{(E,\phi)} : H^1(C_s, \mathcal{O}_{C_s}) \xrightarrow{\sim} H^1(C, \mathrm{End}(E)), \tag{2.20}$$

where $s = H(E, \phi)$. The dual of (2.19), together with (2.20), gives an isomorphism

$$V_{GL} \stackrel{\mathrm{def}}{=} \bigoplus_{i=0}^{n-1} H^1(C, K_C^{-\otimes i}) = H^1(C, \pi_* \mathcal{O}_{C_s}) \xrightarrow{\sim} H^1(C, \mathrm{End}(E)) \cong H^1(C_s, \mathcal{O}_{C_s}). \tag{2.21}$$

From (2.19) and (2.21), we see that the tangent space to the Higgs moduli is given by

$$\begin{aligned} T_{(E,\phi)} \mathcal{H}_C(n, d) &\cong \bigoplus_{i=1}^{n} H^1(C, K_C^{-\otimes(i-1)}) \oplus \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}) \\ &= V_{GL} \oplus V_{GL}^*. \end{aligned} \tag{2.22}$$

Therefore, the symplectic form has a decomposition into n pieces $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$, and in each factor $H^1(C, K_C^{-\otimes(i-1)}) \oplus H^0(C, K_C^{\otimes i})$, ω_i takes the standard Darboux form

$$\omega_i\big((a_1^i, b_1^i), (a_2^i, b_2^i)\big) = \langle a_1^i, b_2^i \rangle_i - \langle a_2^i, b_1^i \rangle_i, \tag{2.23}$$

where $(a_1^i, b_1^i), (a_2^i, b_2^i) \in H^1(C, K_C^{-\otimes(i-1)}) \oplus H^0(C, K_C^{\otimes i})$, and $\langle \cdot, \cdot \rangle_i$ is the duality pairing of $H^1(C, K_C^{-\otimes(i-1)})$ and $H^0(C, K_C^{\otimes i})$.

Since $s = H(E, \phi) \in U_{\mathrm{reg}}$ is a regular value of H, there is an open subset $Y_2 \subset U_{\mathrm{reg}} \subset V_{GL}^*$ around s such that $H^{-1}(Y_2)$ is locally the product of the fiber $H^{-1}(s)$ and Y_2. By taking Y_1 smaller if necessary, we thus obtain a local product neighborhood

$$p^{-1}(Y_1) \cap H^{-1}(Y_2) \cong Y_1 \times Y_2 = Y$$

of (E, ϕ) in $\mathcal{H}_C(n, d)$. By construction, the projections to the first and the second factors coincide with the projection $p : T^*\mathcal{U}_C(n, d) \to \mathcal{U}_C(n, d)$ and the Hitchin map H:

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "H"] & \\ Y_1 & & Y_2. \end{tikzcd}$$

The neighborhood Y and these projections provide the Darboux coordinate system for ω, and its expression (2.18, 2.23) globally holds on Y. In particular, the Hamiltonian vector fields corresponding to the components of H on Y are constant vector fields that are determined by elements of $H^1(C, K_C^{-\otimes(i-1)})$ for each $i = 1, \dots, n$. Let $(h_1, \dots, h_N)$ be a linear coordinate system of V_{GL}^*, where $N = n^2(g-1)+1$. Since $V_{GL}^* = \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i})$, each coordinate function is an element of V_{GL}:

$$h_1, \dots, h_N \in V_{GL} = \bigoplus_{i=0}^{n-1} H^1(C, K_C^{-\otimes i}). \tag{2.24}$$

Since $H^0(C_s, \mathcal{O}_{C_s}) = \mathbb{C}$, the Hitchin map $H : \mathcal{H}_C(n,d) \to V_{GL}^*$ is actually the pull back of the coordinate functions $H^{-1}(h_1), \dots, H^{-1}(h_N)$ on $H^{-1}(U_{\text{reg}}) \subset \mathcal{H}_C(n,d)$. The identification of V_{GL} as the first factor of the tangent space in (2.22) gives the Hamiltonian vector fields corresponding to the coordinate components of the Hitchin map. We have therefore established that the Hamiltonian vector fields are linear flows with respect to the linear coordinate of the Jacobian $\text{Jac}(C_s)$ determined by (2.21). Of course each of these linear flows extends globally on $\text{Jac}(C_s)$ since the tangent bundle of a Jacobian is trivial and a local linear function extends globally as a linear function.

Theorem 2.2 (Hitchin [11]). *The Hamiltonian vector fields corresponding to the Hitchin fibration are linear Jacobian flows on a generic fiber $H^{-1}(s) \cong \text{Jac}(C_s)$.*

How does the construction of the *spectral data* $(\mathcal{L}_E, C_s)$ from a Higgs pair (E, ϕ) change when we consider the Serre dual $(E^* \otimes K_C, -\phi^*)$? To answer this question, first we note

$$\begin{aligned} \text{tr} \wedge^i (\phi) = \text{tr} \wedge^i (\phi^*) &\in H^0(C, \text{End}(\wedge^i E) \otimes K_C^i) \\ &= H^0(C, \text{End}(\wedge^i (E^* \otimes K_C)) \otimes K_C^i). \end{aligned} \tag{2.25}$$

For $s = (s_1, s_2, \dots, s_n) \in V_{GL}^*$, we write $s^* = (-s_1, s_2, \dots, (-1)^n s_n)$. By definition, the spectral curves C_s and C_{s^*} are isomorphic. As divisors of T^*C, their isomorphism $C_s \cong \epsilon(C_{s^*})$ is induced by the involution of T^*C

$$\epsilon : T^*C \ni (p, x) \longmapsto (p, -x) \in T^*C, \tag{2.26}$$

where $p \in C$ and $x \in T_p^*C$.

Proposition 2.3 (Hitchin [12]). *The spectral data $(\mathcal{L}_{E^* \otimes K_C}, C_{s^*})$ corresponding to the Serre dual $(E^* \otimes K_C, -\phi^*)$ of the Higgs pair (E, ϕ) is given by*

$$\begin{cases} \mathcal{L}_{E^* \otimes K_C} = \epsilon^*(\mathcal{L}_E^* \otimes K_{C_s}) \\ C_{s^*} = \epsilon(C_s). \end{cases}$$

The degree of these isomorphic line bundles is $-\deg(E) + (n^2+n)(g-1)$.

Proof. We use the exact sequence of [3, 12] that characterizes $\mathcal{L}_E$:

$$0 \longrightarrow \mathcal{L}_E \otimes K_{C_s}^{-1} \otimes \pi^* K_C \longrightarrow \pi^* E \longrightarrow \pi^*(E \otimes K_C) \longrightarrow \mathcal{L}_E \otimes \pi^* K_C \longrightarrow 0.$$

The dual of this sequence is then

$$\begin{aligned} 0 \longrightarrow \mathcal{L}_E^* \otimes \pi^* K_C \longrightarrow \pi^*(E^* \otimes K_C) \longrightarrow \pi^*(E^* \otimes K_C^{\otimes 2}) \\ \longrightarrow \mathcal{L}_E^* \otimes K_{C_s} \otimes \pi^* K_C \longrightarrow 0. \end{aligned}$$

Thus the spectral data of the Higgs pair $(E^* \otimes K_C, \phi^*)$ is $(\mathcal{L}_E^* \otimes K_{C_s}, C_s)$. The involution ϵ comes in here when we consider the Higgs pair $(E^* \otimes K_C, -\phi^*)$. $\square$

Proposition 2.4. *Hitchin's integrable system for the group $Sp_{2m}(\mathbb{C})$ is realized as the fixed-point-set of the Serre duality involution $(E, \phi) \mapsto (E^* \otimes K_C, -\phi^*)$ defined on the Higgs moduli space $\mathcal{H}_C(2m, 2m(g-1))$.*

Proof. The fixed-point-set consists of Higgs pairs (E, ϕ) such that $E \cong E^* \otimes K_C$ and $\phi = -\phi^*$. Choose a square root of K_C and define $F = E \otimes K_C^{-1/2}$. The Higgs field $\phi = -\phi^*$ naturally acts on this bundle, and the pair (F, ϕ) forms the moduli space of Sp_{2m}-Higgs bundles [12]. The characteristic coefficients satisfy the relation $s = s^*$, hence

$$s \in V_{Sp}^* \overset{\text{def}}{=} \bigoplus_{i=1}^{m} H^0(C, K_C^{\otimes 2i}). \tag{2.27}$$

The spectral curve C_s has a non-trivial involution $\epsilon : C_s \to C_s$. From the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{L}_E \otimes K_{C_s}^{-1} \otimes \pi^* K_C^{1/2} \longrightarrow \pi^*(E \otimes K_C^{-1/2}) \longrightarrow \pi^*(E \otimes K_C^{1/2}) \\ \longrightarrow \mathcal{L}_E \otimes \pi^* K_C^{1/2} \longrightarrow 0, \end{aligned}$$

we see that $\mathcal{L}_F \cong \mathcal{L}_E \otimes \pi^* K_C^{-1/2}$ and $\mathcal{L}_{F^*} \cong \epsilon^* \mathcal{L}_F^* \otimes K_{C_s} \otimes \pi^* K_C^{-1/2}$. If we define $\mathcal{L}_0 = \mathcal{L}_F \otimes \pi^* K_C^{-m+1/2}$ following [12], then $\mathcal{L}_0^* \cong \epsilon^* \mathcal{L}_0$. □

3. Deformations of spectral curves

The Hitchin fibration (2.9) is a family of deformations of Jacobians, but it is not *effective*. In this section we determine the natural effective family associated with the Hitchin fibration.

An obvious action on $\mathcal{H}_C(n, d)$ that preserves the spectral curves is the scalar multiplication of Higgs fields $\phi \mapsto \lambda \cdot \phi$ by $\lambda \in \mathbb{C}^*$. Although this $\mathbb{C}^*$-action is *not* symplectomorphic because it changes the symplectic form $\omega \mapsto \lambda \cdot \omega$, from the point of view of constructing an effective family of Jacobians, we need to quotient it out. We note that the $\mathbb{C}^*$-action on V_{GL}^* defined by

$$\begin{aligned} V_{GL}^* = \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}) \ni s = (s_1, s_2, \dots, s_n) \\ \longmapsto \lambda \cdot s = (\lambda s_1, \lambda^2 s_2, \dots, \lambda^n s_n) \in V_{GL}^* \end{aligned} \tag{3.1}$$

makes the Hitchin fibration $\mathbb{C}^*$-equivariant:

$$\begin{array}{ccc} \mathcal{H}_C(n,d) & \xrightarrow{\quad\lambda\quad} & \mathcal{H}_C(n,d) \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}) & \xrightarrow[\operatorname{diag}(\lambda,\dots,\lambda^n)]{} & \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}), \end{array} \qquad \lambda \in \mathbb{C}^*. \tag{3.2}$$

Since C_s is defined by the equation (2.11), the natural $\mathbb{C}^*$-action on the cotangent bundle T^*C gives the isomorphism $\lambda : C_s \to C_{\lambda \cdot s}$ that commutes with the projection $\pi : T^*C \to C$. The line bundles $\mathcal{L}_{(E,\phi)}$ on C_s and $\mathcal{L}_{(E,\lambda\phi)}$ on $C_{\lambda \cdot s}$ corresponding to E are related by this isomorphism by

$$\lambda^* \mathcal{L}_{(E,\lambda\phi)} \xrightarrow{\sim} \mathcal{L}_{(E,\phi)}.$$

Another group action on the Higgs moduli space that leads to trivial deformations of the spectral curves is the Jacobian action. The $\mathrm{Jac}(C)$-action on $\mathcal{H}_C(n,d)$ is defined by $(E,\phi) \longmapsto (E\otimes L, \phi)$, where $L \in \mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ is a line bundle on C of degree 0. The Higgs field is preserved in this action because $(E\otimes L)^* \otimes (E\otimes L) = E^*\otimes E$ is unchanged, hence

$$\phi \in H^0(C, \mathrm{End}(E)\otimes K_C) = H^0(C, \mathrm{End}(E\otimes L)\otimes K_C).$$

Thus the cotangent bundle $T^*\mathcal{U}_C(n,d)$ is trivial along every orbit of the $\mathrm{Jac}(C)$-action on $\mathcal{U}_C(n,d)$, and each $\mathrm{Jac}(C)$-orbit in $\mathcal{H}_C(n,d)$ lies in the same fiber of the Hitchin fibration. Let us identify the $\mathrm{Jac}(C)$-action on a generic fiber $H^{-1}(s) \cong \mathrm{Jac}(C_s)$. The covering map $\pi : C_s \to C$ induces an injective homomorphism $\pi^* : \mathrm{Jac}(C) \ni L \longmapsto \pi^*L \in \mathrm{Jac}(C_s)$. This is injective because if $\pi^*L \cong \mathcal{O}_{C_s}$, then by the projection formula we have

$$\pi_*(\pi^*L) \cong \pi_*\mathcal{O}_{C_s} \otimes L \cong \bigoplus_{i=0}^{n-1} L \otimes K_C^{\otimes -i},$$

which has a nowhere vanishing section. Hence $L \cong \mathcal{O}_C$. Take a point $(E,\phi) \in H^{-1}(s)$ and let $\mathcal{L}_E$ be the corresponding line bundle on C_s. Since $\pi_*(\mathcal{L}_E \otimes \pi^*L) \cong E\otimes L$, the action of $\mathrm{Jac}(C)$ on $H^{-1}(s) \cong \mathrm{Jac}(C_s)$ is through the canonical action of the subgroup

$$\mathrm{Jac}(C) \cong \pi^*\mathrm{Jac}(C) \subset \mathrm{Jac}(C_s)$$

on $\mathrm{Jac}(C_s)$.

On the open subset $T^*\mathcal{U}_C(n,d)$ of $\mathcal{H}_C(n,d)$, the $\mathrm{Jac}(C)$-action is symplectomorphic because it is induced by the action on the base space $\mathcal{U}_C(n,d)$. On the other open subset $H^{-1}(U_{\mathrm{reg}})$ the action is also symplectomorphic because it preserves each fiber which is a Lagrangian. Since the symplectic form ω is defined by extending the canonical form $\omega = -d\eta$ to $\mathcal{H}_C(n,d)$, the $\mathrm{Jac}(C)$-action is globally holomorphic symplectomorphic on $\mathcal{H}_C(n,d)$. This action is actually a Hamiltonian action and the first component of the Hitchin map

$$H_1 : \mathcal{H}_C(n,d) \ni (E,\phi) \longmapsto \mathrm{tr}(\phi) \in H^0(C, K_C) \tag{3.3}$$

is the moment map. Note that $H^1(C,\mathcal{O}_C)$ is the Lie algebra of the Abelian group $\mathrm{Jac}(C)$, and $H^0(C,K_C)$ is its dual Lie algebra. Since the infinitesimal action of $H^1(C,\mathcal{O}_C)$ on the Higgs moduli space defines a vector field which is obtained by identifying $H^1(C,\mathcal{O}_C)$ with the first component of (2.21), the symplectic dual to this vector field is the map to the first component of second factor in (2.22), i.e, H_1.

We can therefore construct the symplectic quotient $\mathcal{H}_C(n,d)/\!\!/\mathrm{Jac}(C)$, which we will do in Section 4. Here our interest is to determine an effective family of spectral curves. The moment map H_1 of (3.3) being the trace of ϕ, it is natural to define

$$V^*_{SL} = V^*_{SL_n(\mathbb{C})} = \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i}) \subset V^*_{GL}. \tag{3.4}$$

This is a vector space of dimension $(n^2-1)(g-1)$. Now consider a partial projective space

$$\mathbb{P}(\mathcal{H}_C^{SL}(n,d)) = \big(H^{-1}(V^*_{SL}) \setminus H^{-1}(0)\big)/\mathbb{C}^*. \tag{3.5}$$

This is no longer a holomorphic symplectic manifold, yet the Hitchin fibration naturally descends to a generically Jacobian fibration

$$PH : \mathbb{P}(\mathcal{H}_C^{SL}(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*) \tag{3.6}$$

over the weighted projective space of V_{SL}^* defined by the restriction of (3.1) on V_{SL}^*. We now claim

Theorem 3.1 (Effective Jacobian fibration)**.** *The Jacobian fibration*

$$PH : \mathbb{P}(\mathcal{H}_C^{SL}(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*)$$

is generically effective.

The rest of this section is devoted to the proof of this theorem. The statement is equivalent to the claim that the family of deformations of spectral curves parametrized by $\mathbb{P}_w(V_{SL}^*)$ is effective. Let $s \in V_{SL}^*$ be a point for which the spectral curve C_s is non-singular, and $\bar{s}$ the corresponding point of $\mathbb{P}_w(V_{SL}^*)$. We wish to show that the Kodaira-Spencer map

$$T_{\bar{s}}\,\mathbb{P}_w(V_{SL}^*) \longrightarrow H^1(C_s, \mathcal{T}C_s)$$

is injective, where $\mathcal{T}X$ denotes the tangent sheaf of X.

The spectral curve C_s of (2.11) is the divisor of T^*C defined by the section

$$\det(\tau - \phi) \in H^0(T^*C, \pi^* K_C^{\otimes n}),$$

where τ is the tautological section of $\pi^* K_C$ on T^*C. Let $\mathcal{N}_s$ denote the normal sheaf of C_s in T^*C. Since $K_{T^*C} = \Lambda^2(T^*C) \cong \mathcal{O}_{T^*C}$ by the holomorphic symplectic form $\omega_C = -d\tau$, we have

$$\mathcal{N}_s \cong K_{C_s} \cong \pi^* K_C^{\otimes n}.$$

From

$$0 \longrightarrow \mathcal{T}C_s \longrightarrow \mathcal{T}T^*C|_{C_s} \longrightarrow \mathcal{N}_s \longrightarrow 0,$$

we obtain

$$\begin{aligned} 0 \longrightarrow H^0(C_s, \mathcal{T}T^*C) \xrightarrow{\ \iota\ } H^0(C_s, \mathcal{N}_s) \xrightarrow{\ \kappa\ } H^1(C_s, \mathcal{T}C_s) \\ \longrightarrow H^1(C_s, \mathcal{T}T^*C) \longrightarrow H^1(C_s, \mathcal{N}_s). \end{aligned} \tag{3.7}$$

Since

$$H^0(C_s, \mathcal{N}_s) \cong H^0(C, \pi_* K_{C_s}) \cong \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}) = V_{GL}^*,$$

the homomorphism κ is the Kodaira-Spencer map for the deformations of spectral curves on V_{GL}^*. We thus need to identify the image of ι. We note that the tangent sheaf and the cotangent sheaf are isomorphic on the total space of the cotangent bundle T^*C, i.e., $\mathcal{T}T^*C \cong \Lambda^1(T^*C)$.

Proposition 3.2. *We have the following isomorphisms:*

$$\begin{aligned} &H^0(T^*C, \Lambda^0(T^*C)) \cong \mathbb{C} \\ &H^0(T^*C, \Lambda^1(T^*C)) \cong H^0(C, K_C) \oplus \mathbb{C} \cdot \tau \\ &H^0(T^*C, \Lambda^2(T^*C)) \cong \mathbb{C}. \end{aligned} \tag{3.8}$$

Proof. The first isomorphism of (3.8) asserts that every globally defined holomorphic function f on T^*C is a constant. A section $\sigma \in H^0(C, K_C)$ is a map $\sigma : C \to T^*C$. Take an arbitrary pair of points (x, y) in T^*C. If they are not on the same fiber, then there is a section σ such that both x and y are on the image $\sigma(C)$. Since f is constant on $\sigma(C)$, $f(x) = f(y)$. If they are on the same fiber, then choose a point $z \in T^*C$ not on this fiber and use the same argument.

Since $\Lambda^2(T^*C) \cong \mathcal{O}_{T^*C}$, the third isomorphism follows from the first one.

The second isomorphism of (3.8) asserts that every holomorphic 1-form α on T^*C is either the pull-back of a holomorphic 1-form on C via $\pi : T^*C \to C$, the tautological 1-form τ, or a linear combination of them. Let $\sigma : C \to T^*C$ be an arbitrary section. Since the normal sheaf of $\sigma(C)$ in T^*C is $K_{\sigma(C)}$, we have on $\sigma(C)$

$$0 \longrightarrow K_{\sigma(C)}^{-1} \longrightarrow \Lambda^1(T^*C) \otimes \mathcal{O}_{\sigma(C)} \longrightarrow K_{\sigma(C)} \longrightarrow 0.$$

Therefore, identifying $C \cong \sigma(C)$, we obtain

$$\begin{aligned} 0 \longrightarrow H^0(C, \Lambda^1(T^*C) \otimes \mathcal{O}_C) \longrightarrow H^0(C, K_C) \xrightarrow{\kappa_0} H^1(C, K_C^{-1}) \\ \longrightarrow H^1(C, \Lambda^1(T^*C) \otimes \mathcal{O}_C) \longrightarrow H^1(C, K_C) \longrightarrow 0. \end{aligned}$$

Here κ_0 is the Kodaira-Spencer map assigning a deformation of C to a *displacement* of C in T^*C through a section of K_C. But since $\sigma(C)$ is always isomorphic to C, we do not obtain any deformation of C in this way. Hence κ_0 is the 0-map. Therefore,

$$H^0(\sigma(C), \Lambda^1(T^*C) \otimes \mathcal{O}_{\sigma(C)}) \cong H^0(\sigma(C), K_{\sigma(C)}) \cong H^0(C, K_C). \tag{3.9}$$

Now consider an exact sequence on T^*C

$$\begin{aligned} 0 \longrightarrow \Lambda^1(T^*C) \otimes \mathcal{O}_{T^*C}(-\sigma(C)) \longrightarrow \Lambda^1(T^*C) \\ \longrightarrow \Lambda^1(T^*C) \otimes \mathcal{O}_{\sigma(C)} \longrightarrow 0, \end{aligned}$$

which produces

$$\begin{aligned} 0 \longrightarrow H^0(T^*C, \Lambda^1(T^*C) \otimes \mathcal{O}_{T^*C}(-\sigma(C))) \longrightarrow H^0(T^*C, \Lambda^1(T^*C)) \\ \xrightarrow{r} H^0(\sigma(C), \Lambda^1(T^*C) \otimes \mathcal{O}_{\sigma(C)}). \end{aligned}$$

Because of (3.9), the homomorphism r is equal to the pull-back

$$r = \sigma^* : H^0(T^*C, \Lambda^1(T^*C)) \longrightarrow H^0(C, K_C)$$

by the section $\sigma : C \to T^*C$. It is surjective because $\sigma^* \circ \pi^* = id_{H^0(C,K_C)}$. Therefore, we have a splitting exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(T^*C, \Lambda^1(T^*C) \otimes \mathcal{O}(-\sigma(C))) \longrightarrow H^0(T^*C, \Lambda^1(T^*C)) \\ \xrightarrow{\sigma^*} H^0(C, K_C) \longrightarrow 0. \end{aligned} \tag{3.10}$$

The tautological 1-form $\tau \in H^0(T^*C, \pi^*K_C)$ defines

$$0 \longrightarrow \pi^*K_C \longrightarrow \Lambda^1(T^*C) \xrightarrow{\wedge\tau} \Lambda^2(T^*C) \otimes \mathcal{O}_{T^*C}(-C) \longrightarrow 0,$$

noting that τ vanishes along the divisor $C \subset T^*C$. Since

$$H^0(T^*C, \Lambda^2(T^*C) \otimes \mathcal{O}_{T^*C}(-C)) \cong H^0(T^*C, \mathcal{O}_{T^*C}(-C)) = 0,$$

we obtain

$$H^0(T^*C, \pi^*K_C) \cong H^0(T^*C, \Lambda^1(T^*C)).$$

Take $\alpha \in H^0(T^*C, \Lambda^1(T^*C) \otimes \mathcal{O}_{T^*C}(-C)) \cong H^0(T^*C, \pi^* K_C(-C))$. Then

$$\alpha/\tau \in H^0(T^*C, \pi^* K_C(-C) \otimes \pi^* K_C^{-1}(C)) \cong H^0(T^*C, \mathcal{O}_{T^*C}) \cong \mathbb{C}.$$

Therefore, α is a constant multiple of τ, and we have obtained

$$H^0(T^*C, \Lambda^1(T^*C) \otimes \mathcal{O}_{T^*C}(-C)) \cong \mathbb{C}. \tag{3.11}$$

From (3.10) and (3.11), we conclude that

$$H^0(T^*C, \Lambda^1(T^*C)) \cong H^0(C, K_C) \oplus \mathbb{C} \cdot \tau.$$

$\square$

Lemma 3.3. *Let $s \in V^*_{GL}$ be a point such that the spectral curve C_s is nonsingular. Then we have an isomorphism*

$$H^0(C_s, \mathcal{T}T^*C) \cong H^0(C_s, \Lambda^1(T^*C)) \cong H^0(T^*C, \Lambda^1(T^*C)). \tag{3.12}$$

Proof. The line bundle on T^*C that corresponds to the divisor $C_s \subset T^*C$ is $\pi^* K_C^{\otimes n}$. Thus

$$\mathcal{O}_{T^*C}(-C_s) \cong \pi^* K_C^{\otimes(-n)} \cong \mathcal{O}_{T^*C}(-nC).$$

As above, let us consider an exact sequence

$$0 \longrightarrow \Lambda^1(T^*C) \otimes \mathcal{O}(-nC) \longrightarrow \Lambda^1(T^*C) \longrightarrow \Lambda^1(T^*C) \otimes \mathcal{O}_{C_s} \longrightarrow 0$$

and its cohomology sequence

$$\begin{aligned} 0 \longrightarrow H^0(T^*C, \Lambda^1(T^*C) \otimes \mathcal{O}(-nC)) \longrightarrow H^0(T^*C, \Lambda^1(T^*C)) \\ \xrightarrow{q} H^0(C_s, \Lambda^1(T^*C) \otimes \mathcal{O}_{C_s}). \end{aligned}$$

From (3.11), we have

$$H^0(T^*C, \Lambda^1(T^*C) \otimes \mathcal{O}_{T^*C}(-nC)) = 0$$

for $n \geq 2$, hence q is injective.

Take $\beta \in H^0(C_s, \Lambda^1(T^*C) \otimes \mathcal{O}_{C_s})$, and extend it as a meromorphic 1-form on T^*C. Since $\deg K_C = 2g-2 > 0$, every divisor of T^*C intersects with the 0-section C, and since $C_s \sim nC$ as a divisor, it also intersects with C_s. If $D \subset T^*C$ is the pole divisor of β, then it cannot intersect with C_s, hence $D = \emptyset$. Therefore, q is surjective. $\square$

Let us go back to the Kodaira-Spencer map (3.7). We now know from (3.8) and (3.12) that

$$0 \longrightarrow H^0(C, K_C) \oplus \mathbb{C} \cdot \tau \xrightarrow{\iota} \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}) \xrightarrow{\kappa} H^1(C_s, K_{C_s}^{-1}). \tag{3.13}$$

The $H^0(C, K_C)$-factor of the second term of (3.13) maps to the first component of the third term via the injective homomorphism ι. The tautological 1-form τ on T^*C that appears as the second factor of the second term is mapped to a diagonal ray in the third term.

To see this fact, we recall that the tangent and cotangent sheaves are isomorphic on T^*C through the symplectic form $\omega_C = -d\tau$. Let v be the vector field on T^*C corresponding to τ through this isomorphism, i.e., $\tau = \omega_C(\cdot, v)$. This vector field v represents the $\mathbb{C}^*$-action on T^*C along fiber. In terms of a local coordinate system, these correspondences are clearly described. Choose a local coordinate z on the algebraic curve C around a point $p \in C$, and denote by x the linear coordinate on

T_p^*C with respect to the basis dz_p. Then at the point $(p, xdz_p) \in T^*C$ we have the following expressions:

$$\begin{cases} \tau = x\,dz \\ \omega_C = dz \wedge dx \\ v = x\,\frac{\partial}{\partial x}\,. \end{cases}$$

The $\lambda \in \mathbb{C}^*$ action on T^*C generated by the vector fields v produces a displacement of $C_s \subset T^*C$ to $C_{\lambda \cdot s}$, which corresponds to the equivariant action of λ on $\mathcal{H}_C(n,d)$ as described in (3.2). In terms of the holomorphic 1-form τ, its restriction on C_s gives an element

$$\begin{aligned} \tau|_{C_s} \in H^0(C_s, K_{C_s}) \cong H^0(C, \pi_*\mathcal{O}_{C_s} \otimes K_C^{\otimes n}) \cong H^0(C, \oplus_{i=1}^n K_C^{\otimes i}) \\ = \bigoplus_{i=1}^n H^0(C, K_C^{\otimes i}). \end{aligned}$$

This is the image $\iota(\tau)$ of (3.13).

Summing up, we have constructed an injective homomorphism

$$V_{SL}^*/\mathbb{C} \cong \big(\bigoplus_{i=1}^n H^0(C, K_C^{\otimes i})\big)/\iota\big(H^0(C, K_C) \oplus \mathbb{C}\cdot\tau\big) \xrightarrow{\bar\kappa} H^1(C_s, K_{C_s}^{-1}). \tag{3.14}$$

The image of $\bar\kappa$ represents the generic Kodaira-Spencer class of the Hitchin fibration PH of (3.6). This completes the proof of Theorem 3.1.

4. Symplectic quotient of the Higgs moduli space and Prym fibrations

The Hamiltonian vector fields corresponding to the coordinate components of the Hitchin map $\mathcal{H}_C(n,d) \to V_{GL}^*$ are constant Jacobian flows along each fiber of the map. Suppose we have a direct sum decomposition of V_{GL} into two Lie subalgebras

$$V_{GL} \overset{\text{def}}{=} \bigoplus_{i=0}^{n-1} H^1(C, K_C^{\otimes -i}) = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Then there are two possible ways to construct new algebraically completely integrable Hamiltonian systems. If the $\mathfrak{g}_1$-action on $\mathcal{H}_C(n,d)$ is integrated to a group action, then it is Hamiltonian by definition and we can construct the symplectic quotient. Or if the $\mathfrak{g}_2$-action is integrated to a group action instead, then we may find a family of $\mathfrak{g}_2$-orbits in $\mathcal{H}_C(n,d)$ fibered over the dual Lie algebra $\mathfrak{g}_2^*$. An important difference between real symplectic geometry and holomorphic symplectic geometry is that in the latter case integrations of the same Lie algebra may generate different (non-isomorphic) Lie groups. Consequently, the idea of symplectic quotient has to be generalized so that we can allow a *family* of groups acting on a symplectic manifold. The discovery of Hausel and Thaddeus in [8] is that the above two constructions lead to *mirror symmetric* pairs of Calabi-Yau spaces in the sense of Strominger-Yau-Zaslow [28]. In this section we consider two cases, the SL-PGL duality and the Sp_{2m}-SO_{2m+1} duality.

The SL-PGL duality comes from the decomposition

$$V_{GL} = H^1(C, \mathcal{O}_C) \oplus \left(\bigoplus_{i=i}^{n-1} H^1(C, K_C^{\otimes -i})\right).$$

Obviously, the vector fields generated by the $H^1(C, \mathcal{O}_C)$-action are integrable to the $\mathrm{Jac}(C)$-action everywhere on $\mathcal{H}_C(n,d)$. Therefore, we do have the usual symplectic quotient mod $\mathrm{Jac}(C)$. On the other hand, the integration of the other Lie algebra

$$V_{SL} \overset{\text{def}}{=} \bigoplus_{i=i}^{n-1} H^1(C, K_C^{\otimes -i})$$

produces different Lie groups, called *Prym varieties*, along each fiber of the Hitchin fibration.

The spectral covering $\pi : C_s \to C$ induces two group homomorphisms dual to one another, the pull-back π^* and the *norm map* Nm_π defined by

$$\begin{aligned} \pi^* &: \mathrm{Jac}(C) \ni L \longmapsto \pi^* L \in \mathrm{Jac}(C_s) \\ \mathrm{Nm}_\pi &: \mathrm{Jac}(C_s) \ni \mathcal{L} \longmapsto \det(\pi_* \mathcal{L}) \otimes \det(\pi_* \mathcal{O}_{C_s})^{-1} \in \mathrm{Jac}(C). \end{aligned} \tag{4.1}$$

In terms of divisors the norm map can be defined alternatively by

$$\mathrm{Pic}(C_s) \ni \sum_{p \in C_s} m(p) \cdot p \longmapsto \sum_{p \in C_s} m(p) \cdot \pi(p) \in \mathrm{Jac}(C).$$

The *Prym variety* and the *dual Prym variety*

$$\begin{aligned} \mathrm{Prym}(C_s/C) &\overset{\text{def}}{=} \mathrm{Ker}(\mathrm{Nm}_\pi) \\ \mathrm{Prym}^*(C_s/C) &\overset{\text{def}}{=} \mathrm{Jac}(C_s)/\pi^* \mathrm{Jac}(C) \end{aligned} \tag{4.2}$$

constructed by using these homomorphisms are Abelian varieties of dimension $g(C_s) - g(C)$ and are *dual* to one another. The algebraically completely integrable Hamiltonian systems with these Abelian fibrations, that are naturally constructed from $\mathcal{H}_C(n,d)$, then become SYZ-mirror symmetric.

We have shown in Section 3 that the $\mathrm{Jac}(C)$-action on $\mathcal{H}_C(n,d)$ is Hamiltonian. So we can define the symplectic quotient

$$\mathcal{PH}_C(n,d) \overset{\text{def}}{=} \mathcal{H}_C(n,d) /\!\!/ \mathrm{Jac}(C) = H_1^{-1}(0)/\mathrm{Jac}(C). \tag{4.3}$$

This is a symplectic space of dimension $2(n^2-1)(g-1)$ modeled by the moduli space of stable principal $PGL_n(\mathbb{C})$-bundles on C [10, 11]. Since the $\mathrm{Jac}(C)$-action on $\mathcal{H}_C(n,d)$ preserves the Hitchin fibration, we have an induced Lagrangian fibration

$$H_{PGL} : \mathcal{PH}_C(n,d) \longrightarrow V_{SL}^* = \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i}). \tag{4.4}$$

It's 0-fiber is $H_{PGL}^{-1}(0) = \mathcal{U}_C(n,d)/\mathrm{Jac}(C)$. Following [21] we denote by $\mathcal{SU}_C(n,d)$ the moduli space of stable vector bundles with a fixed determinant line bundle. This is a fiber of the determinant map

$$\mathcal{U}_C(n,d) \ni E \longmapsto \det E \in \mathrm{Pic}^d(C), \tag{4.5}$$

and is independent of the choice of the value of the determinant. Note that (4.5) is a non-trivial fiber bundle. The equivariant $\mathrm{Jac}(C)$-action on (4.5) is given by

$$\begin{CD} \mathcal{U}_C(n,d) @>{\otimes L}>> \mathcal{U}_C(n,d) \\ @V{\det}VV @VV{\det}V \\ \mathrm{Pic}^d(C) @>>{\otimes L^{\otimes n}}> \mathrm{Pic}^d(C) \end{CD} \qquad L \in \mathrm{Jac}(C). \tag{4.6}$$

The isotropy subgroup of the $\mathrm{Jac}(C)$-action on $\mathcal{U}_C(n,d)$ is the group of n-torsion points

$$J_n(C) \overset{\text{def}}{=} \{L \in \mathrm{Jac}(C) \mid L^{\otimes n} = \mathcal{O}_C\} \cong H^1(C, \mathbb{Z}/n\mathbb{Z}), \tag{4.7}$$

since $E \otimes L \cong E$ implies $\det(E) \otimes L^{\otimes n} \cong \det(E)$. Choose a reference line bundle $L_0 \in \mathrm{Pic}^d(C)$ and consider a degree n covering

$$\nu : \mathrm{Pic}^d(C) \ni L \otimes L_0 \longmapsto L^{\otimes n} \otimes L_0 \in \mathrm{Pic}^d(C), \qquad L \in \mathrm{Jac}(C).$$

Then the pull-back bundle $\nu^*\mathcal{U}_C(n,d)$ on $\mathrm{Pic}^d(C)$ becomes trivial:

$$\nu^*\mathcal{U}_C(n,d) = \mathrm{Pic}^d(C) \times \mathcal{SU}_C(n,d).$$

The quotient of this product by the diagonal action of $J_n(C)$ is the original moduli space:

$$\big(\mathrm{Pic}^d(C) \times \mathcal{SU}_C(n,d)\big)\big/ J_n(C) \cong \mathcal{U}_C(n,d). \tag{4.8}$$

It is now clear that

$$\mathcal{U}_C(n,d)/\mathrm{Jac}(C) \cong \mathcal{SU}_C(n,d)/J_n(C).$$

What are the other fibers of (4.4)? Let $s \in V^*_{SL} \cap U_{\mathrm{reg}}$ be a point such that C_s is non-singular. We have already noted that the covering map $\pi : C_s \to C$ induces an injective homomorphism $\pi^* : \mathrm{Jac}(C) \ni L \longmapsto \pi^* L \in \mathrm{Jac}(C_s)$. Therefore, the fiber $H^{-1}_{PGL}(s)$ is isomorphic to the dual Prym variety $\mathrm{Prym}^*(C_s/C)$. Similarly to the equivariant action (4.6), we have

$$\begin{array}{ccc} \mathrm{Jac}(C_s) & \xrightarrow{\otimes L} & \mathrm{Jac}(C_s) \\ {\scriptstyle \mathrm{Nm}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Nm}} \\ \mathrm{Jac}(C) & \xrightarrow[\otimes L^{\otimes n}]{} & \mathrm{Jac}(C) \end{array} \qquad L \in \mathrm{Jac}(C). \tag{4.9}$$

By the same argument of (4.8), we obtain

$$\big(\mathrm{Prym}(C_s/C) \times \mathrm{Jac}(C)\big)\big/ J_n(C) \cong \mathrm{Jac}(C_s). \tag{4.10}$$

From (4.2) and (4.10), it follows that

$$\mathrm{Prym}^*(C_s/C) = \mathrm{Prym}(C_s/C)/J_n(C).$$

We have thus established

Theorem 4.1 ([11, 12])**.** *The natural fibration*

$$H_{PGL} : \mathcal{PH}_C(n,d) \longrightarrow V^*_{SL} = \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i})$$

of (4.4) *is a Lagrangian dual Prym fibration with respect to the canonical holomorphic symplectic form $\bar{\omega}$ on $\mathcal{PH}_C(n,d)$.*

We recall that the Higgs moduli space $\mathcal{H}_C(n,d)$ contains the cotangent bundle $T^*\mathcal{U}_C(n,d)$ as an open dense subspace, and that the holomorphic symplectic form ω is the canonical symplectic form on this cotangent bundle. Similarly, we can show the following

Proposition 4.2. *The symplectic form $\bar{\omega}$ on $\mathcal{PH}_C(n,d)$ given by the symplectic quotient is the canonical cotangent symplectic form on the cotangent bundle*

$$T^*\big(\mathcal{SU}_C(n,d)/J_n(C)\big) \subset \mathcal{PH}_C(n,d).$$

Proof. Let E be a stable vector bundle on C. The exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathrm{End}(E) \longrightarrow \mathcal{Q} = \mathrm{End}(E)/\mathcal{O}_C \longrightarrow 0$$

induces a cohomology sequence

$$0 \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathrm{End}(E)).$$

Therefore,

$$T_E\big(\mathcal{SU}_C(n,d)/J_n(C)\big) \cong H^1(C, \mathrm{End}(E))/H^1(C, \mathcal{O}_C).$$

Dualizing the situation, we have

$$0 \longrightarrow \mathrm{End}_0(E) \otimes K_C \longrightarrow \mathrm{End}(E) \otimes K_C \xrightarrow{\mathrm{tr}} K_C \longrightarrow 0,$$

where $\mathrm{End}_0(E)$ is the sheaf of traceless endomorphisms of E. We then have

$$0 \longrightarrow H^0(C, \mathrm{End}_0(E) \otimes K_C) \longrightarrow H^0(C, \mathrm{End}(E) \otimes K_C) \xrightarrow{\mathrm{tr}} H^0(C, K_C) \longrightarrow 0.$$

The trace homomorphism is globally surjective because $\mathcal{O}_C$ is contained in $\mathrm{End}(E)$. Hence

$$T_E^*\big(\mathcal{SU}_C(n,d)/J_n(C)\big) \cong H^0(C, \mathrm{End}_0(E) \otimes K_C).$$

Since the moment map H_1 of (3.3) is the trace map, we conclude that the symplectic form $\bar{\omega}$ on the symplectic quotient $\mathcal{PH}_C(n,d)$ is the canonical cotangent symplectic form on the cotangent bundle

$$T^*\big(\mathcal{SU}_C(n,d)/J_n(C)\big).$$

□

The other reduction of $\mathcal{H}_C(n,d)$ consisting of the V_{SL}-orbits is the moduli space $\mathcal{SH}_C(n,d)$ of stable Higgs bundles (E,ϕ) with a fixed determinant $\det(E) \cong L$ and traceless Higgs fields

$$\phi \in H^0(C, \mathrm{End}_0(E) \otimes K_C).$$

This moduli space is modeled by the moduli space of stable principal $SL_n(\mathbb{C})$-bundles on C and has dimension $2(n^2-1)(g-1)$. The cotangent bundle $T^*\mathcal{SU}_C(n,d)$ is an open dense subspace of $\mathcal{SH}_C(n,d)$. Since $\mathrm{tr}(\phi) = 0$, the Hitchin fibration H of (2.9) naturally restricts to

$$H_{SL} : \mathcal{SH}_C(n,d) \longrightarrow V_{SL}^* = \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i}). \tag{4.11}$$

The 0-fiber is $H_{SL}^{-1}(0) = \mathcal{SH}_C(n,d)$. For a generic $s \in V_{SL}^*$ such that C_s is non-singular, the fiber $H_{SL}^{-1}(s)$ is the subset of $H^{-1}(s) \cong \mathrm{Jac}(C_s)$ consisting of Higgs bundles (E,ϕ) such that $\det(E) = L$ is fixed and $\det(x - \phi) = s$. Therefore, $H_{SL}^{-1}(s) \cong \mathrm{Prym}(C_s/C)$.

By comparing $\mathcal{PH}_C(n,d)$ and $\mathcal{SH}_C(n,d)$, we find that

$$\mathcal{SH}_C(n,d)\big/J_n(C) \cong \mathcal{PH}_C(n,d), \tag{4.12}$$

where the $J_n(C)$-action on $\mathcal{SH}_C(n,d)$ is defined by $E \mapsto E \otimes L$ for $L \in J_n(C)$. Since $J_n(C)$ is a finite group and $\mathcal{PH}_C(n,d)$ is a holomorphic symplectic variety, we can define a holomorphic symplectic form $\hat{\omega}$ on $\mathcal{SH}_C(n,d)$ via the pull-back of the projection

$$\mathcal{SH}_C(n,d) \longrightarrow \mathcal{SH}_C(n,d)\big/J_n(C).$$

Obviously $\hat{\omega}$ agrees with the canonical cotangent symplectic form on $T^*\mathcal{SU}_C(n,d)$. Therefore,

Theorem 4.3 ([11, 12])**.** *The fibration*

$$H_{SL} : \mathcal{SH}_C(n,d) \longrightarrow V_{SL}^* = \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i})$$

is a Lagrangian Prym fibration with respect to the canonical holomorphic symplectic form $\hat{\omega}$ *on* $\mathcal{SH}_C(n,d)$*.*

Hausel and Thaddeus [8] shows that

Theorem 4.4 (Theorem (3.7) in [8])**.** *The two fibrations*

$$\begin{tikzcd} \mathcal{SH}_C(n,d) \arrow[dr, "H_{SL}"] & & \mathcal{PH}_C(n,d) \arrow[dl, "H_{PGL}"'] \\ & V_{SL}^* & \end{tikzcd}$$

are mirror symmetric in the sense of Strominger-Yau-Zaslow [28]*.*

The effectiveness of the family of Prym and dual Prym varieties can be established by the same method of Section 3. Let us define partial projective moduli spaces

$$\mathbb{P}(\mathcal{PH}_C(n,d)) \overset{\text{def}}{=} \big(\mathcal{PH}_C(n,d) \setminus H_{PGL}^{-1}(0)\big)\big/\mathbb{C}^*$$

and

$$\mathbb{P}(\mathcal{SH}_C(n,d)) \overset{\text{def}}{=} \big(\mathcal{SH}_C(n,d) \setminus H_{SL}^{-1}(0)\big)\big/\mathbb{C}^*.$$

The induced Hitchin maps are denoted by

$$\begin{aligned} PH_{PGL} &: \mathbb{P}(\mathcal{PH}_C(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*) \\ PH_{SL} &: \mathbb{P}(\mathcal{SH}_C(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*). \end{aligned} \tag{4.13}$$

We have the following

Theorem 4.5. *The Prym and dual Prym fibrations*

$$\begin{aligned} PH_{SL} &: \mathbb{P}(\mathcal{SH}_C(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*) \\ PH_{PGL} &: \mathbb{P}(\mathcal{PH}_C(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*) \end{aligned}$$

are generically effective.

For the case of $Sp_{2m}(\mathbb{C})$-Hitchin systems, we consider Lie subalgebras

$$\begin{aligned} \mathfrak{g} &= \bigoplus_{i=0}^{m-1} H^1\big(C, K_C^{\otimes -2i}\big) \\ V_{Sp} &= \bigoplus_{i=0}^{m-1} H^1\big(C, K_C^{\otimes -2i-1}\big)\,, \end{aligned} \tag{4.14}$$

and a direct sum decomposition

$$V_{GL_{2m}} = \mathfrak{g} \oplus V_{Sp}. \tag{4.15}$$

This time the Hamiltonian flows on $\mathcal{H}_C(n, d)$ generated by elements of $\mathfrak{g}$ do not form a group action. However, if we restrict our attention to points of $U_{\text{reg}} \cap V^*_{Sp}$ as in (2.27), then the integral of the Lie algebra action becomes a group action. Recall that the spectral curve C_s has an involution induced by ϵ of (2.26). We denote by

$$r : C_s \to C'_s = C_s/\langle\epsilon\rangle \tag{4.16}$$

the natural projection. It is ramified at the intersection of C_s with the 0-section of T^*C, which is the divisor of π^*K_C on C_s of degree $4m(g-1)$. Therefore, we find the genus of C'_s by the Riemann-Hurwitz formula:

$$g(C'_s) = m(2m-1)(g-1) + 1 = \dim \mathfrak{g}. \tag{4.17}$$

Since $H^0(C'_s, r_*\mathcal{O}_{C_s})$ has a nowhere vanishing section, we have

$$0 \longrightarrow \mathcal{O}_{C'_s} \longrightarrow r_*\mathcal{O}_{C_s} \longrightarrow N^{-1} \longrightarrow 0 \tag{4.18}$$

with a line bundle N on C'_s of degree $2m(g-1)$. Note that

$$N^{\otimes 2} = \det(r_*\mathcal{O}_{C_s})^{\otimes -2} \cong \mathrm{Nm}_r(\pi^*K_C),$$

hence N is a square root of the *branch divisor* $\mathrm{Nm}_r(\pi^*K_C)$ of the covering r. The exact sequence (4.18) gives

$$H^1(C_s, \mathcal{O}_{C_s}) \cong H^1(C'_s, \mathcal{O}_{C'_s}) \oplus H^1(C'_s, N^{-1}), \tag{4.19}$$

and the projection to the first factor is the differential of the norm map

$$\mathrm{Nm}_r : \mathrm{Jac}(C_s) \ni \mathcal{L} \longmapsto \det(r_*\mathcal{L}) \otimes \det(r_*\mathcal{O}_{C_s})^{-1} \in \mathrm{Jac}(C'_s).$$

The construction of the Sp-Hitchin system of Proposition 2.4 is to reduce the GL-Hitchin system $\mathcal{H}_C(2m, 2m(g-1))$ by finding the right fibration of groups. Along the fixed-point-set of the Serre duality on this Higgs moduli space, the action of V_{Sp} generates the Prym fibration $\{\mathrm{Prym}(C_s/C'_s)\}_{s \in V^*_{Sp}}$ since the condition $\mathcal{L}_0^* \cong \epsilon^*\mathcal{L}_0$ on C_s is the same as $\mathcal{L}_0 \in \mathrm{Prym}(C_s/C'_s)$. Comparing (4.15) and (4.19), we have

$$H^1(C'_s, \mathcal{O}_{C'_s}) \cong \mathfrak{g} \qquad \text{and} \qquad H^1(C'_s, N^{-1}) \cong V_{Sp}.$$

The *dual* fibration is the result of a kind of symplectic quotient of $\mathcal{H}_C(2m, 2m(g-1))$ by $\mathfrak{g}$. In this quotient we restrict the Hitchin fibration to the 0-fiber of

$$\mathfrak{g}^* = \bigoplus_{i=1}^{m} H^0\left(C, K_C^{\otimes 2i-1}\right),$$

and then take the quotient of each fiber by the Lie group $\mathrm{Jac}(C'_s)$ of $\mathfrak{g}$. The result is the dual Prym fibration with a fiber

$$\mathrm{Prym}^*(Cs/C'_s) = \mathrm{Jac}(C_s)/\mathrm{Jac}(C'_s) = \mathrm{Prym}(Cs/C'_s)/J_2(C'_s)$$

over each $s \in V^*_{Sp}$, where $J_2(C'_s)$ denotes the group of 2-torsion points of $\mathrm{Jac}(C'_s)$.

5. Hitchin's integrable systems and the KP equations

The partial projective Hitchin fibration

$$PH : \mathbb{P}(\mathcal{H}_C^{SL}(n,d)) \to \mathbb{P}_w(V_{SL}^*)$$

is a generically effective Jacobian fibration. In this section we embed $\mathbb{P}(\mathcal{H}_C^{SL}(n,d))$ into a quotient of the Sato Grassmannian. There is also a natural embedding of $\mathcal{H}_C(n, n(g-1))$ into a *relative* Grassmannian of [2, 4, 22, 23]. We show that the linear Jacobian flows on the Hitchin integrable system $(\mathcal{H}_C(n, n(g-1)), \omega, H)$ are exactly the KP equations on the Grassmannian via this second embedding.

Following Quandt [23], we define

Definition 5.1 (Sato Grassmannian)**.** *Let n be a positive integer. The Sato Grassmannian Gr_n is a functor from the category of schemes to the small category of sets. It assigns to every scheme S a set $Gr_n(S)$ consisting of quasi-coherent $\mathcal{O}_S$-submodules W of $\mathcal{O}_S((z))^{\oplus n}$ such that both the kernel and the cokernel of the natural homomorphism*

$$0 \longrightarrow \mathrm{Ker}(\gamma_W) \longrightarrow W \xrightarrow{\gamma_W} \mathcal{O}_S((z))^{\oplus n}/\mathcal{O}_S[[z]]^{\oplus n} \longrightarrow \mathrm{Coker}(\gamma_W) \longrightarrow 0 \tag{5.1}$$

are coherent $\mathcal{O}_S$ modules. We refer to this condition simply the Fredholm condition*. Here we denote by $\mathcal{O}_S[[z]]$ the ring of formal power series in z with coefficients in $\mathcal{O}_S$, and $\mathcal{O}_S((z)) = \mathcal{O}_S[[z]] + \mathcal{O}_S[z^{-1}]$.*

Remark 5.1. A more general and powerful theory of relative Sato Grassmannians has been recently established by Plaza Martín [22]. It would be an interesting project to study possible relations between [22] and the geometric Langlands correspondence.

Remark 5.2. The Grassmannian $Gr_n(\mathbb{C})$ defined over the point scheme $\mathrm{Spec}(\mathbb{C})$ has the structure of an infinite-dimensional pro-ind scheme over $\mathbb{C}$. If S is irreducible, then $Gr_n(S)$ is a disjoint union of an infinite number of components *indexed* by the Grothendieck group $K(S)$:

$$\mathrm{index} : Gr_n(S) \ni W \longmapsto \mathrm{index}(\gamma_W) \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\gamma_W) - \mathrm{Coker}(\gamma_W) \in K(S). \tag{5.2}$$

Remark 5.3. The *big cell* is the open subscheme of the index 0 piece of the Grassmannian $Gr_n^0(S)$ consisting of W's such that γ_W is an isomorphism. This is the stage where the Lax and Zakharov-Shabat formalisms of integrable nonlinear partial differential equations, such as KP, KdV, and many other equations, are interpreted as infinite-dimensional dynamical systems [24, 25]. The fundamental result is the identification (6.17) of the big cell with the group of 0-th order monic pseudo-differential operators [16, 23, 24, 25].

Remark 5.4. The above W is a closed subset of $\mathcal{O}_S((z))^{\oplus n}$ with respect to the topology defined by the filtration

$$\cdots \supset z^{\nu-1} \cdot \mathcal{O}_S[[z]]^{\oplus n} \supset z^{\nu} \cdot \mathcal{O}_S[[z]]^{\oplus n} \supset z^{\nu+1} \cdot \mathcal{O}_S[[z]]^{\oplus n} \supset \cdots .$$

One of the consequences of the Fredholm condition is that

$$W \cap z^{\nu} \cdot \mathcal{O}_S[[z]]^{\oplus n} = 0 \qquad \text{for } \nu >> 0. \tag{5.3}$$

Consider the group

$$\Gamma_n(S) = \mathcal{O}_S^\times \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + z \cdot \begin{bmatrix} \mathcal{O}_S[[z]] & & & \\ & \mathcal{O}_S[[z]] & & \\ & & \ddots & \\ & & & \mathcal{O}_S[[z]] \end{bmatrix} \tag{5.4}$$

consisting of $n \times n$ invertible diagonal matrices with entries in the ring of formal power series whose constant term is scalar diagonal. It acts on $Gr_n(S)$ by left-multiplication without fixed points. Indeed, let $W \in Gr_n(S)$ and $c+\gamma \in \mathcal{O}_S^\times \cdot I_n + z \cdot \mathcal{O}_S[[z]]^{\oplus n}$ satisfy that $W = (c+\gamma)\cdot W$. This means that $\gamma \cdot w \in W$ for every $w \in W$ since W is an $\mathcal{O}_S$-module. Since $\gamma \in z \cdot \mathcal{O}_S[[z]]^{\oplus n}$, it then contradicts to (5.3) unless $\gamma = 0$. The *quotient Grassmannian*

$$Z_n \overset{\text{def}}{=} Gr_n/\Gamma_n \tag{5.5}$$

is a smooth infinite-dimensional scheme. We denote by $\overline{W}$ the point of Z_n corresponding to $W \in Gr_n$.

The tangent space $T_W Gr_n$ of the Sato Grassmannian at W is given by the space of continuous $\mathcal{O}_S$-homomorphisms

$$T_W Gr_n = \mathrm{Hom}_{\mathcal{O}_S}\big(W, \mathcal{O}_S((z))^{\oplus n}/W\big).$$

The tangent space to Z_n is then given by

$$T_{\overline{W}} Z_n \cong \mathrm{Hom}_{\mathcal{O}_S}\big(W, \mathcal{O}_S((z))^{\oplus n}\big/(\Gamma_n \cdot W)\big). \tag{5.6}$$

This expression does not depend on the choice of the lift $W \in Gr_n$ of $\overline{W} \in Z_n$.

Every element $a \in \mathcal{O}_S((z))^{\oplus n}$ defines a homomorphism

$$KP(a)_W : W \ni w \longmapsto \overline{a \cdot w} \in \mathcal{O}_S((z))^{\oplus n}\big/(\Gamma_n \cdot W)$$

through the left multiplication as a diagonal matrix, which in turn determines a global vector field

$$\mathcal{O}_S((z))^{\oplus n} \ni a \longmapsto KP(a) \in H^0(Z_n, TZ_n).$$

We call $KP(a)$ the n-component *KP flow* associated with a. As explained in [16, 17, 19, 23], the quotient of the index zero Grassmannian Z_n^0 is naturally identified with the set of *Lax operators* (6.18), and the action of $\mathcal{O}_S((z))^{\oplus n}$ on a Lax operator is written as an infinite system of nonlinear partial differential equations called *Lax equations*. This system for the case of $S = \mathrm{Spec}(\mathbb{C})$ is the n-component *Kadomtsev-Petviashvili hierarchy* [16].

Associated to the Hitchin fibration $H : \mathcal{H}_C(n,d) \to V_{GL}^*$ we have a family of spectral curves:

$$\begin{tikzcd} \mathfrak{C}_{V_{GL}^*}(n,d) \arrow[rr, "\text{inclusion}"] \arrow[dr, "F"'] & & T^*C \times V_{GL}^* \arrow[dl, "p_2"] \\ & V_{GL}^* & \end{tikzcd} \tag{5.7}$$

Here the fiber $F^{-1}(s)$ of $s \in V_{GL}^*$ is the spectral curve $C_s \subset T^*C \times \{s\}$, and p_2 is the projection to the second factor. We note that the ramification points of the covering $\pi : C_s \to C$ are determined by the resultant of the defining equation

$$x^n + s_1 x^{n-1} + \cdots + s_n = 0 \tag{5.8}$$

of C_s and its derivative

$$nx^{n-1} + (n-1)s_1x^{n-1} + \cdots + s_{n-1} = 0. \tag{5.9}$$

For every $s \in V_{GL}^*$, we denote by $\mathrm{Res}(s)$ this resultant. The Sylvester matrix of these polynomials (5.8) and (5.9) show that

$$\mathrm{Res}(s) \in H^0(C, K_C^{\otimes n(n-1)}).$$

Since the linear system $|K_C|$ is base-point-free, for every choice of $p \in C$, the subset

$$U_p = \{s \in V_{GL}^* \mid C_s \text{ is non-singular and } \pi : C_s \to C \text{ is unramified at } p\} \tag{5.10}$$

is Zariski open in V_{GL}^*.

Theorem 5.5. *There is a rational map*

$$\mu : \mathbb{P}(\mathcal{H}_C^{SL}(n,d)) \longrightarrow Z_n^{d-n(g-1)}(\mathbb{C})$$

of the partial projective moduli space of Higgs bundles into the quotient Grassmannian of index $d - n(g-1)$. This map is generically injective. At a general point of the image of the embedding the n-component KP flows defined on $Z_n(\mathbb{C})$ are tangent to the Hitchin fibration $PH : \mathbb{P}(\mathcal{H}_C^{SL}(n,d)) \longrightarrow \mathbb{P}_w(V_{SL}^*)$.

Proof. A point of the partial projective Higgs moduli space represents an isomorphism class of spectral data $(\pi : C_s \to C, \mathcal{L})$, where $\mathcal{L}$ is a line bundle on C_s of degree $d + (n^2 - n)(g-1)$. Choose a point $p \in C$, a coordinate z of the formal completion $\hat{C}_p$ of C at p, and $s \in U_p \cap V_{SL}^*$ so that C_s is non-singular and π is unramified over p. The formal coordinate z defines an identification $\mathcal{O}_{\hat{C}_p} = \mathbb{C}[[z]]$. We also choose a local trivialization of $\mathcal{L}$ around $\pi^{-1}(p)$, i.e., an isomorphism

$$\beta : \mathcal{L}|_{\hat{C}_{s,\pi^{-1}(p)}} \xrightarrow{\sim} \mathcal{O}_{\hat{C}_{s,\pi^{-1}(p)}} = \mathbb{C}[[z]]^{\oplus n}. \tag{5.11}$$

Since the formal completion $\hat{C}_{s,\pi^{-1}(p)}$ is the disjoint union of n copies of $\hat{C}_p$, the formal coordinate z also defines an identification $\mathcal{O}_{\hat{C}_{s,\pi^{-1}(p)}} = \mathbb{C}[[z]]^{\oplus n}$.

Now define

$$W = \beta(H^0(C_s \setminus \pi^{-1}(p), \mathcal{L})) \subset \mathbb{C}((z))^{\oplus n},$$

which is the set of meromorphic sections of $\mathcal{L}$ that are holomorphic on $C_s \setminus \pi^{-1}(p)$ and have finite poles at $\pi^{-1}(p)$. Since we have

$$\begin{cases} \mathrm{Ker}(\gamma_W) \cong H^0(C_s, \mathcal{L}) \\ \mathrm{Coker}(\gamma_W) \cong H^1(C_s, \mathcal{L}), \end{cases}$$

W is a point of the Grassmannian $Gr_n^{d-n(g-1)}$ of index $d - n(g-1)$. The different choice of the local trivialization β' in (5.11) leads to an element $\beta' \circ \beta^{-1} \in \Gamma_n$. Therefore, the point $\overline{W} \in Z_n^{d-n(g-1)}(\mathbb{C})$ is uniquely determined by $(\pi : C_s \to C, p, z, \mathcal{L})$. Conversely this set of geometric data is uniquely determined by $\overline{W}$ (see Section 5 of [16]). Thus the rational map μ is generically one-to-one. Notice that the Γ_n action on the Grassmannian is inessential from the geometric point of view because it simply changes the local trivialization of (5.11).

The tangent space at a general point of $\mathbb{P}(\mathcal{H}_C^{SL}(n,d))$ is

$$H^1(C_s, \mathcal{O}_{C_s}) \oplus V_{SL}^*/\mathbb{C}.$$

Since the Kodaira-Spencer map (3.14) is injective, it suffices to show injectivity of the natural map

$$\begin{aligned} d\mu : H^1(C_s, \mathcal{O}_{C_s}) \oplus H^1(C_s, K_{C_s}^{-1}) \longrightarrow & \operatorname{Hom}(W, \mathbb{C}((z))^{\oplus n}/W) \\ & = T_W Gr_n(\mathbb{C}), \end{aligned}$$

which is induced by the local trivialization of $\mathcal{O}_{C_s}$ and K_C coming from the choice of the local coordinate z. Using z, we define

$$A_W = H^0(C_s \setminus \pi^{-1}(p), \mathcal{O}_{C_s}) \subset \mathbb{C}((z))^{\oplus n},$$

which is a point of the Grassmannian satisfying

$$\begin{cases} \operatorname{Ker}(\gamma_{A_W}) \cong H^0(C_s, \mathcal{O}_{C_s}) \\ \operatorname{Coker}(\gamma_{A_W}) \cong H^1(C_s, \mathcal{O}_{C_s}). \end{cases}$$

We note that A_W is a ring and W is an A_W-module. Since

$$\operatorname{Coker}(\gamma_{A_W}) = \frac{\mathbb{C}((z))^{\oplus n}}{A_W + \mathbb{C}[[z]]^{\oplus n}},$$

$H^1(C_s, \mathcal{O}_{C_s})$ is injectively mapped to $\operatorname{Hom}(W, \mathbb{C}((z))^{\oplus n}/W)$.

The local coordinate z determines a local trivialization of K_C

$$K_C|_{\hat{C}_p} \xrightarrow{\sim} \mathcal{O}_{\hat{C}_p} \cdot dz \cong \mathcal{O}_{\hat{C}_p},$$

and hence that of $K_{C_s}^{-1}$

$$K_{C_s}^{-1}|_{\hat{C}_{s,\pi^{-1}(p)}} \xrightarrow{\sim} \left(\mathcal{O}_{\hat{C}_p}\right)^{\oplus n} \cdot \frac{\partial}{\partial z} \cong \mathbb{C}[[z]]^{\oplus n} \cdot \frac{\partial}{\partial z}.$$

This trivialization gives

$$H^1(C_s, K_{C_s}^{-1}) \cong \frac{\mathbb{C}((z))^{\oplus n} \cdot \partial/\partial z}{D_W + \mathbb{C}[[z]]^{\oplus n} \cdot \partial/\partial z},$$

where

$$D_W = H^0(C_s \setminus \pi^{-1}(p), K_{C_s}^{-1}) \subset \mathbb{C}((z))^{\oplus n} \cdot \partial/\partial z.$$

Note that $s \in V_{GL}^*$ determines an element of $H^0(C_s, K_{C_s})$ because

$$s \in V_{GL}^* = \bigoplus_{i=1}^{n} H^0(C, K_C^{\otimes i}) \cong H^0(C, \pi_* \pi^* K_C^{\otimes n}) \cong H^0(C_s, K_{C_s}).$$

This holomorphic 1-form on C_s induces a homomorphism

$$\mathcal{L} \to \mathcal{L} \otimes K_{C_s},$$

or equivalently,

$$K_{C_s}^{-1} \otimes \mathcal{L} \longrightarrow \mathcal{L}.$$

In terms of the local coordinate z, this homomorphism gives an action of D_W on W. Since $D_W \cdot W \subset W$, we conclude that $H^1(C_s, K_{C_s}^{-1})$ is injectively mapped to $T_W Gr_n(\mathbb{C})$. In this construction $H^1(C_s, \mathcal{O}_{C_s})$ and $H^1(C_s, K_{C_s}^{-1})$ have no common element in $T_W Gr_n(\mathbb{C})$ except for 0. This establishes the injectivity of $d\mu$.

In [16], it is proved that the orbit of the n-component KP flows starting from $\overline{W}$ is isomorphic to $\operatorname{Pic}^{d+(n^2-n)(g-1)}(C_s)$ (Theorem 5.8, [16]), which is the fiber of the Hitchin map PH. This completes the proof of the theorem. □

The above theorem does not say anything about the Hitchin integrable system, because the partial projective moduli space is not a symplectic manifold and we do not have any integrable systems on it. To directly compare the Jacobian flows of the Hitchin systems and the KP flows, we use the *relative* Grassmannian of [23] defined on the scheme U_p of (5.10). So let $\pi^{-1}(p) = \{p_1, \dots, p_n\}$, and denote by $z_i = \pi^*(z)$ the formal coordinate of $\hat{C}_{s,p_i}$. Choose a linear coordinate system $(h_1, \dots, h_N)$ for V_{GL}^* as in (2.24). Recall that $\pi_* \mathcal{O}_{C_s} \cong \oplus_{i=0}^{n-1} K_C^{\otimes -i}$, and from (2.21) we have

$$H^1(C_s, \mathcal{O}_{C_s}) \cong \bigoplus_{i=0}^{n-1} H^1(C, K_C^{\otimes -i}) = V_{GL}.$$

Using the Čech cohomology computation based on the covering $C = (C \setminus \{p\}) \cup \hat{C}_p$, we expand each $h_k \in V_{GL}$ as

$$h_k = \sum_{i=1}^{n} \sum_{j \geq 1} t_{ijk} z_i^{-j}, \qquad t_{ijk} \in \mathbb{C}. \tag{5.12}$$

The n-component KP flows on the Grassmannian $Gr_n(U_p)$ is defined by a formal action of

$$\exp\left(\sum_{i=1}^{n} \sum_{j \geq 1} \sum_{k=1}^{N} t_{ijk} z_i^{-j} \right). \tag{5.13}$$

For degree $d = n(g-1)$, we have the following:

Theorem 5.6. *Let us choose a point $p \in C$, a formal coordinate z of $\hat{C}_p$, and a square root of the canonical sheaf $K_C^{1/2}$. Then the fibration*

$$\begin{tikzcd} \mathfrak{C}_{V_{GL}^*}(n, n(g-1)) \arrow[rr, "\pi"] \arrow[dr, "F"'] & & C \times V_{GL}^* \arrow[dl, "p_2"] \\ & V_{GL}^* & \end{tikzcd}$$

determines a point $W \in Gr_n^0(U_p)$. There is a birational map from $\mathcal{H}_C(n, n(g-1))$ to the orbit of the n-component KP flows on the quotient Grassmannian $Z_n^0(U_p)$ starting from $\overline{W}$. The Hitchin integrable system on the Higgs moduli space $\mathcal{H}_C(n, n(g-1))$ is the pull-back of the n-component KP equations via this birational map.

Proof. We regard C, p, z, and $K_C^{1/2}$ being defined over the trivial family $C \times V_{GL}^*$. There is a natural choice of a local trivialization of $K_C^{1/2}$ on $\hat{C}_p$ determined by the coordinate z. Note that

$$\mathcal{K}^{1/2} \overset{\text{def}}{=} \pi^*(K_C^{n/2}) \tag{5.14}$$

is a square root of the relative dualizing sheaf $\mathcal{K} = \pi^*(K_C^n)$ on

$$\mathfrak{C}_{V_{GL}^*}(n, n(g-1)).$$

Take an arbitrary point $s \in U_p$. Since $p \in C$ is a point of C at which π is unramified, the formal coordinate z determines a local trivialization of $\mathcal{K}^{1/2}$ along $\pi^{-1}(p) \subset \mathfrak{C}_{V_{GL}^*}(n, n(g-1))$. Define

$$W = H^0(\mathfrak{C}_{V_{GL}^*}(n, n(g-1)) \setminus \pi^{-1}(p), \mathcal{K}^{1/2}) \subset \mathcal{O}_{U_p}((z))^{\oplus n}, \tag{5.15}$$

using this local trivialization. Since

$$\begin{cases} \mathrm{Ker}(\gamma_{W_s}) \cong H^0(C_s, K_{C_s}^{1/2}) \\ \mathrm{Coker}(\gamma_{W_s}) \cong H^1(C_s, K_{C_s}^{1/2}) \end{cases}$$

for each $s \in U_p$, we have $W \in Gr_n^0(U_p)$. Let us analyze the action of (5.13) at W. On each C_s, $h_k = \sum t_{ijk} z_i^{-j}$ defines an element of $H^1(C_s, \mathcal{O}_{C_s})$. The exponential of (5.13) is the Lie integration map

$$V_{GL} \cong H^1(C_s, \mathcal{O}_{C_s}) \xrightarrow{\exp} H^1(C_s, \mathcal{O}_{C_s})/H^1(C_s, \mathbb{Z}) = \mathrm{Jac}(C_s). \tag{5.16}$$

Therefore, the action of (5.13) at the point W produces simultaneous deformations of the line bundle

$$\mathcal{K}^{1/2} \longmapsto \mathcal{K}^{1/2} \otimes \exp\left(\sum_{i=1}^{n}\sum_{j\geq 1}\sum_{k=1}^{N} t_{ijk} z_i^{-j}\right) \tag{5.17}$$

on $\mathfrak{C}_{V_{GL}^*}(n, n(g-1))$. Consequently, the orbit of the n-component KP flows on $Z_n^0(U_p)$ starting at $\overline{W}$ is naturally identified with the family of $\mathrm{Pic}^{n^2(g-1)}(C_s)$ on U_p. Since the Hitchin fibration $H : \mathcal{H}_C(n, n(g-1)) \to V_{GL}^*$ is also a family of $\mathrm{Pic}^{n^2(g-1)}(C_s)$ on V_{GL}^*, we have a rational map from $H : \mathcal{H}_C(n, n(g-1))$ to the n-component KP orbit of $\overline{W}$ in $Z_n^0(U_p)$.

Recall that the Hitchin integrable system on $H : \mathcal{H}_C(n, n(g-1))$ is the linear Jacobian flows defined by the coordinate functions $(h_1, \ldots, h_N)$. We note that this is exactly what (5.17) gives. $\square$

Remark 5.7. So far we have assumed that C_s is non-singular. Although it will not be an Abelian variety, we can still define the Jacobian $\mathrm{Jac}(C_s)$ using (5.16) when the spectral curve C_s is singular. As long as we choose $p \in C$ so that $\pi^{-1}(p)$ avoids the singular locus of C_s, the KP flows are well defined. Moreover, the theory of *Heisenberg KP flows* introduced in [1, 16] allows us to consider the covering $\pi : C_s \to C$ right at a ramification point. It is more desirable to define the Grassmmanian over the whole V_{GL}^* and to deal with the entire family $F : \mathfrak{C}_{V_{GL}^*} \to V_{GL}^*$ together with the moduli stack of Higgs bundles instead of the stable moduli we have considered here, since the framework of the Sato Grassmannian allows us to consider all vector bundles on C and degenerated spectral curves. We refer to [4] for a study in this direction.

6. Serre duality and formal adjoint of pseudo-differential operators

In this section we analyze the Serre duality of Higgs bundles in terms of the language of the Sato Grassmannian, and identify the involution on the corresponding pseudo-differential operators.

The Krichever construction ([17, 19, 25]) assigns a point W of the Sato Grassmannian,

$$Gr_n(\mathbb{C}) \ni W = \beta(H^0(C \setminus \{p\}, F)) \subset \mathbb{C}((z))^{\oplus n}, \tag{6.1}$$

to a set of geometric data (C, p, z, F, β), where C is an irreducible algebraic curve, $p \in C$ is a non-singular point, z is a formal parameter of the completion $\hat{C}_p$, F is a torsion-free sheaf of rank n on C, and

$$\beta : F|_{\hat{C}_p} \xrightarrow{\sim} \mathcal{O}_{\hat{C}_p}^{\oplus n} = \mathbb{C}[[z]]^{\oplus n}$$

is a local trivialization of F around p. We continue to assume that C is non-singular, hence F is locally-free on C. As noted in Section 5, the choice of a local coordinate z automatically determines a local trivialization of K_C:

$$\alpha : K_C|_{\hat{C}_p} \xrightarrow{\sim} \mathcal{O}_{\hat{C}_p} \cdot dz \cong \mathcal{O}_{\hat{C}_p}. \tag{6.2}$$

For every homomorphism $\xi : F \to K_C$, we have an element

$$\beta^*(\xi) \overset{\text{def}}{=} \alpha \circ \xi \circ \beta^{-1} \in \left(\mathcal{O}_{\hat{C}_p}^{\oplus n}\right)^* = \mathcal{O}_{\hat{C}_p}^{\oplus n}$$

that is determined by the commutative diagram

$$\begin{array}{ccc} F|_{\hat{C}_p} & \xrightarrow{\ \xi\ } & K_C|_{\hat{C}_p} \\ {\scriptstyle\beta}\Big\downarrow{\scriptstyle\wr} & & {\scriptstyle\wr}\Big\downarrow{\scriptstyle\alpha} \\ \mathcal{O}_{\hat{C}_p}^{\oplus n} & \xrightarrow{\ \beta^*(\xi)\ } & \mathcal{O}_{\hat{C}_p}. \end{array} \tag{6.3}$$

Thus we have the canonical choice of the local trivialization

$$\beta^* : F^* \otimes K_C|_{\hat{C}_p} \xrightarrow{\sim} \mathcal{O}_{\hat{C}_p}^{\oplus n}.$$

Definition 6.1 (Serre Dual)**.** *The* Serre dual *of the geometric date* (C, p, z, F, β) *is the set of data* $(C, p, z, F^* \otimes K_C, \beta^*)$*.*

To identify the counterpart of the Serre duality on the Sato Grassmannian, we introduce a non-degenerate symmetric paring in $\mathbb{C}((z))$ by

$$\langle a(z), b(z)\rangle = \frac{1}{2\pi i}\oint a(z)b(z)dz = \text{ the coefficient of } z^{-1} \text{ in } a(z)b(z), \tag{6.4}$$

and define

$$\langle f(z), g(z)\rangle_n = \sum_{i=1}^{n} \langle f_i(z), g_i(z)\rangle, \qquad f(z), g(z) \in \mathbb{C}((z))^{\oplus n}. \tag{6.5}$$

Lemma 6.1. *Let* $W \in Gr_n(\mathbb{C})$ *be a point of the Sato Grassmannian of index* d*. Then*

$$W^\perp = \{f \in \mathbb{C}((z))^{\oplus n} \mid \langle f, w\rangle_n = 0 \textit{ for all } w \in W\}$$

is a point of $Gr_n(\mathbb{C})$ *of index* $-d$*. Moreover, we have*

$$\begin{cases} \mathrm{Ker}(\gamma_W)^* \cong \mathrm{Coker}(\gamma_{W^\perp}) \\ \mathrm{Coker}(\gamma_W)^* \cong \mathrm{Ker}(\gamma_{W^\perp}), \end{cases} \tag{6.6}$$

where γ_W *is defined by the exact sequence*

$$0 \longrightarrow \mathrm{Ker}(\gamma_W) \longrightarrow W \xrightarrow{\gamma_W} \mathbb{C}((z))^{\oplus n}/\mathbb{C}[[z]]^{\oplus n} \longrightarrow \mathrm{Coker}(\gamma_W) \longrightarrow 0.$$

Proof. Take an element $f \in W^\perp \cap \mathbb{C}[[z]]^{\oplus n}$. Then

$$\langle f, W + \mathbb{C}[[z]]^{\oplus n}\rangle_n = 0.$$

Thus it induces a linear map

$$f : \mathbb{C}((z))\big/(W + \mathbb{C}[[z]]^{\oplus n}) \ni g \longmapsto \langle f, g\rangle_n \in \mathbb{C},$$

defining a natural inclusion

$$\operatorname{Ker}(\gamma_{W^\perp}) \subset \operatorname{Coker}(\gamma_W)^*. \tag{6.7}$$

Now let $h^* \in \operatorname{Coker}(\gamma_W)^*$ be an arbitrary element. Choose a sequence

$$\{g_1^i, g_2^i, g_3^i \dots\}_{i=1,\dots,n}$$

of elements of $\mathbb{C}((z))^{\oplus n}$ such that

(1) $g_j^i = \mathbf{e}^i z^{-j} + \sum_{\ell=1}^n \sum_{k\geq 1} g_{\ell jk} \mathbf{e}^\ell z^{-j+k}$;
(2) if W has an element whose leading term is $\mathbf{e}^i z^{-j}$ for $j > 0$, then $g_j^i \in W$.

Here $\mathbf{e}^i = (0, \dots, 0, 1, 0, \dots, 0)^T$ is the i-th standard basis vector for $\mathbb{C}^n$. We denote by $\bar{g}_j^i$ the projection image of g_j^i in $\mathbb{C}((z))^{\oplus n}/(W + \mathbb{C}[[z]]^{\oplus n})$. Consider the set of equations

$$\langle f, g_j^i \rangle_n = h^*(\bar{g}_j^i), \qquad i = 1, \dots, n; \quad j = 1, 2, 3, \dots \tag{6.8}$$

for $f \in \mathbb{C}[[z]]^{\oplus n}$. If we write $f(z) = \sum_{i=1}^n \sum_{j\geq 0} a_{ij} \mathbf{e}^i z^j$, then (6.8) is equivalent to

$$\begin{aligned}
a_{i0} &= h^*(g_1^i) \\
a_{i1} &= h^*(g_2^i) - \sum_{\ell=1}^n a_{\ell 0} g_{\ell 21} \\
a_{i2} &= h^*(g_3^i) - \left(\sum_{\ell=1}^n a_{\ell 1} g_{\ell 31} + a_{\ell 0} g_{\ell 32} \right) \\
a_{i3} &= h^*(g_4^i) - \left(\sum_{\ell=1}^n a_{\ell 2} g_{\ell 41} + a_{\ell 1} g_{\ell 42} + a_{\ell 0} g_{\ell 43} \right) \\
&\vdots
\end{aligned}$$

It is obvious that (6.8) has a unique solution. By construction we have $\langle f, W \rangle_n = 0$, hence $f \in \operatorname{Ker}(\gamma_{W^\perp})$. Therefore, the inclusion (6.7) is indeed a surjective map.

The application of the same argument to $W \cap \mathbb{C}[[z]]^{\oplus n}$ establishes that

$$\operatorname{Ker}(\gamma_W) \cong \operatorname{Coker}(\gamma_{W^\perp})^*.$$

□

Remark 6.2. We note that $W^{\perp\perp} = W$. It is obvious that $W \subset W^{\perp\perp}$. The relation (6.6) then makes the inclusion relation actually the equality.

Theorem 6.3. *Let $W \in Gr_n(\mathbb{C})$ be the point of the Grassmannian corresponding to a set of geometric data (C, p, z, F, β). Then the point corresponding to its Serre dual $(C, p, z, F^* \otimes K_C, \beta^*)$ is $W^\perp$.*

Proof. Let W' be the point of the Grassmannian corresponding to $(C, p, z, F^* \otimes K_C, \beta^*)$. Note that

$$\operatorname{index}(\gamma_W) = \dim H^0(C, F) - \dim H^1(C, F) = -\operatorname{index}(\gamma_{W'}).$$

Take an arbitrary $f \in H^0(C \setminus \{p\}, F)$ and $g \in H^0(C \setminus \{p\}, F^* \otimes K_C)$. Then

$$g \cdot f \in H^0(C \setminus \{p\}, K_C).$$

From (6.3) we see that $\beta^*(g)\cdot\beta(f)=\alpha(g\cdot f)$. Abel's theorem tells us that the sum of the residues of the meromorphic 1-form $g\cdot f$ on C is 0. Since $g\cdot f$ is holomorphic everywhere on C except for p, we have

$$0=\mathrm{res}_p(g\cdot f)=\langle\beta^*(g),\beta(f)\rangle_n.$$

Therefore, $W'\subset W^\perp$. Since $\gamma_{W'}$ and $\gamma_{W^\perp}$ have the isomorphic kernels and cokernels, we conclude that $W'=W^\perp$. $\square$

The ring of *ordinary differential operators* is defined as

$$\mathcal{D}=\mathbb{C}[[x]]\left[\frac{d}{dx}\right]. \tag{6.9}$$

Extending the powers of the differentiation to negative integers, we define the ring of *pseudo-differential operators* by

$$\mathcal{E}=\mathcal{D}+\mathbb{C}[[x]]\left[\left[\left(\frac{d}{dx}\right)^{-1}\right]\right]. \tag{6.10}$$

Let $\mathcal{E}x$ be the left maximal ideal of $\mathcal{E}$ generated by x. Then $\mathcal{E}/\mathcal{E}x$ is a left $\mathcal{E}$-module. Let us denote $\partial=d/dx$. As a $\mathbb{C}$-vector space we identify

$$\mathcal{E}/\mathcal{E}x=\mathbb{C}((\partial^{-1})). \tag{6.11}$$

Two useful formulas for calculating pseudo-differential operators are

$$\partial^n\cdot a(x)=\sum_{i\ge0}\binom{n}{i}a^{(i)}(x)\partial^{n-i}, \tag{6.12}$$

where $a^{(i)}(x)$ is the i-th derivative of $a(x)$, and

$$a(x)\cdot\partial^n=\sum_{i\ge0}(-1)^i\binom{n}{i}\partial^{n-i}\cdot a^{(i)}(x). \tag{6.13}$$

A pseudo-differential operator has thus two expressions

$$\begin{aligned}P=\sum_{n\in\mathbb{Z}}a_n(x)\partial^n&=\sum_{n\in\mathbb{Z}}\sum_{i\ge0}(-1)^i\binom{n}{i}\partial^{n-i}\cdot a_n^{(i)}(x)\\&=\sum_{m\in\mathbb{Z}}\sum_{i\ge0}(-1)^i\binom{m+i}{i}\partial^m\cdot a_{m+i}^{(i)}(x).\end{aligned}$$

The natural projection $\mathcal{E}\to\mathcal{E}/\mathcal{E}x$ is given by

$$\begin{aligned}\rho:\mathcal{E}\ni P=\sum_n a_n(x)\partial^n&\\ \longmapsto\sum_m\sum_{i\ge0}(-1)^i\binom{m+i}{i}a_{m+i}^{(i)}(0)\partial^m&\in\mathbb{C}((\partial^{-1})).\end{aligned} \tag{6.14}$$

The relation to the Grassmannian comes from the identification

$$\mathbb{C}((z))\ni f(z)\longmapsto f(\partial^{-1})\cdot\partial^{-1}\in\mathbb{C}((\partial^{-1})). \tag{6.15}$$

Then $\mathbb{C}((z))$ becomes an $\mathcal{E}$-module. The action of $P\in\mathcal{E}$ on $f(z)\in\mathbb{C}((z))$ is given by the formula

$$P\cdot f(z)=\rho(P\cdot f(\partial^{-1})\partial^{-1})\in\mathbb{C}((z)).$$

Definition 6.2 (Adjoint)**.** *The* adjoint *of* $P = \sum_\ell a_\ell(x)\partial^\ell$ *is defined by*

$$P^* = \sum_\ell \partial^\ell \cdot a_\ell(-x).$$

Remark 6.4. Our adjoint is slightly different from the *formal adjoint* of [13, 29], which is defined to be $\sum_\ell (-\partial)^\ell \cdot a_\ell(x)$. This is due to the fact that we are considering the action of pseudo-differential operators on the function space $\mathbb{C}((z))$ through *Fourier transform.* Thus ∂ acts as the multiplication of z^{-1}, and x acts as the differentiation.

Let us compute the (m,n)-matrix entry of P. We find

$$\begin{aligned}
&\langle z^m, P \cdot z^n\rangle \\
&= \left\langle z^m, \rho\left(\sum_\ell a_\ell(x)\partial^{\ell-n-1}\right)\right\rangle \\
&= \left\langle z^m, \rho\left(\sum_\ell \sum_i (-1)^i \binom{\ell-n-1}{i} \partial^{\ell-n-1-i} \cdot a_\ell^{(i)}(x)\right)\right\rangle \\
&= \left\langle z^m, \sum_\ell \sum_i (-1)^i \binom{\ell-n-1}{i} a_\ell^{(i)}(0) \cdot z^{-\ell+n+i}\right\rangle \\
&= \sum_i (-1)^i \binom{m+i}{i} a^{(i)}_{m+n+i+1}(0).
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
&\langle z^n, P^* \cdot z^m\rangle \\
&= \left\langle z^n, \rho\left(\sum_\ell \partial^\ell a_\ell(-x) \cdot \partial^{-m-1}\right)\right\rangle \\
&= \left\langle z^n, \rho\left(\sum_\ell \sum_i (-1)^i(-1)^i \binom{-m-1}{i} \partial^{\ell-m-1-i} \cdot a_\ell^{(i)}(-x)\right)\right\rangle \\
&= \left\langle z^n, \sum_\ell \sum_i \binom{-m-1}{i} a_\ell^{(i)}(0) \cdot z^{-\ell+m+i}\right\rangle \\
&= \sum_i \binom{-m-1}{i} a^{(i)}_{m+n+i+1}(0).
\end{aligned}$$

Following [16], we introduce the ring of differential and pseudo-differential operators with matrix coefficients and denote them by $gl_n(\mathcal{D})$ and $gl_n(\mathcal{E})$, respectively. The adjoint of a matrix pseudo-differential operator is defined in an obvious way:

$$P = [P_{ij}]_{i,j=1,\dots,n} \Longrightarrow P^* \overset{\text{def}}{=} [P^*_{ji}]_{i,j=1,\dots,n}. \tag{6.16}$$

Proposition 6.5. *For arbitrary* $f(z), g(z) \in \mathbb{C}((z))^{\oplus n}$ *and* $P \in gl_n(\mathcal{E})$ *we have the adjoint formula*

$$\langle f(z), P \cdot g(z)\rangle_n = \langle P^* \cdot f(z), g(z)\rangle_n.$$

Proof. The statement follows from the above computations, noting the identity

$$(-1)^i \binom{m+i}{i} = \binom{-m-1}{i},$$

and the usual matrix transposition. □

A more direct relation between pseudo-differential operators and the Sato Grassmannian is that the identification of the *big cell* of $Gr_n(\mathbb{C})$ with the group

$$G = I_n + gl_n(\mathbb{C}[[x]])[[\partial^{-1}]] \cdot \partial^{-1} \tag{6.17}$$

of monic pseudo-differential operators of order 0 ([16], Theorem 6.5).

Theorem 6.6. *Let $W \in Gr_n(\mathbb{C})$ be in the big cell, and correspond to a pseudo-differential operator $S \in G$. Then $W^\perp$ corresponds to $(S^*)^{-1} \in G$.*

Proof. The correspondence of $S \in G$ and W in the big cell is the equation

$$W = S^{-1} \cdot \mathbb{C}[z^{-1}]^{\oplus n} \cdot z^{-1}.$$

For $f(z), g(z) \in \mathbb{C}[z^{-1}]^{\oplus n} \cdot z^{-1}$, we have

$$\langle S^* \cdot f(z), S^{-1} \cdot g(z)\rangle_n = \langle f(z), SS^{-1} \cdot g(z)\rangle_n = 0.$$

□

The *Lax operator* is defined by

$$\mathbf{L} = S \cdot I_n \cdot \partial \cdot S^{-1}. \tag{6.18}$$

Lax operators bijectively correspond to points of the big cell of the quotient Grassmannian $Z_n(\mathbb{C})$. If $\overline{W}$ corresponds to $\mathbf{L}$, then $\overline{W}^\perp$ corresponds to $\mathbf{L}^*$. Therefore, the Serre duality indeed corresponds to the adjoint action of the pseudo-differential operators.

7. The Hitchin integrable system for Sp_{2m} and corresponding KP-type equations

In this section we identify the KP-type equations that are equivalent to the Hitchin integrable system for Sp_{2m}. Since the moduli space of the Sp_{2m}-Higgs bundles is identified as the fixed-point-set of the Serre duality involution on $\mathcal{H}_C(2m, 2m(g-1))$ (Proposition 2.4), we will see that the evolution equations are the KP equations restricted to a certain type of self-adjoint Lax operators.

Let $(E, \phi) = (E^* \otimes K_C, -\phi^*)$ be a fixed point of the Serre duality operation on the Higgs moduli space $\mathcal{H}_C(2m, 2m(g-1))$, and $(\mathcal{L}_E, C_s)$ the corresponding spectral data. We choose a point $p \in C$ and a local coordinate z as in Section 5. We assume that the spectral cover $\pi : C_s \to C$ is unramified at p. This time we choose a local trivialization

$$\beta : \mathcal{L}_E|_{\hat{C}_{s,\pi^{-1}(p)}} \xrightarrow{\sim} \mathcal{O}_{\hat{C}_{s,\pi^{-1}(p)}} \cong \mathbb{C}[[z]]^{\oplus 2m}.$$

Then the Serre dual of the data $(\pi_s : C_s \to C, p, z, \mathcal{L}_E, \phi, \beta)$ is

$$(\pi_{s^*} : C_{s^*} \to C, p, z, \epsilon^*(\mathcal{L}_E^* \otimes K_{C_s}), -\phi^*, \epsilon^*\beta^*).$$

In general the dual trivialization is defined by

$$\begin{array}{ccc}
\mathcal{L}_E|_{\hat{C}_{s,\pi_s^{-1}(p)}} & \xrightarrow{\xi} & K_{C_s}|_{\hat{C}_{s,\pi_s^{-1}(p)}} \\
\beta \downarrow \wr & & \wr \downarrow \alpha \\
\mathcal{O}_{\hat{C}_{s,\pi_s^{-1}(p)}} & \xrightarrow{\beta^*(\xi)} & \mathcal{O}_{\hat{C}_{s,\pi_s^{-1}(p)}} \\
\epsilon \downarrow \wr & & \wr \downarrow \epsilon \\
\mathcal{O}_{\hat{C}_{s^*,\pi_{s^*}^{-1}(p)}} & \xrightarrow{\epsilon^*\beta^*(\xi)} & \mathcal{O}_{\hat{C}_{s^*,\pi_{s^*}^{-1}(p)}}.
\end{array}$$

Note that in the current case we have $s = s^*$ and $\epsilon : C_s \to C_s$ is a non-trivial involution. Since ϵ commutes with π, it induces an involution of the set $\pi^{-1}(p) = \{p_1, \dots, p_{2m}\}$. Let us number the $2m$ distinct points so that

$$\epsilon(p_1, \dots, p_{2m}) = (p_1, \dots, p_{2m}) \cdot A,$$

where

$$A = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}. \tag{7.1}$$

By the same argument we used in Theorem 6.3, if we define

$$W = \beta(H^0(C_s \setminus \pi^{-1}(p), \mathcal{L}_E)) \subset \mathbb{C}((z))^{\oplus 2m},$$

then we have

$$AW^\perp = \epsilon^*\beta^*(H^0(C_{s^*} \setminus \pi^{-1}(p), \epsilon^*(\mathcal{L}_E^* \otimes K_{C_s}))) \subset \mathbb{C}((z))^{\oplus 2m}. \tag{7.2}$$

Let us assume that E is not on the theta divisor of [3] so that $H^0(C, E) = H^1(C, E) = 0$. Then $W \in Gr_{2m}(\mathbb{C})$ is on the big cell, and hence corresponds to a monic 0-th order pseudo-differential operator $S \in G$. Since $W = AW^\perp$, we have

$$S = A \cdot (S^*)^{-1} \cdot A. \tag{7.3}$$

The Lax operator

$$\mathbf{L} = S \cdot I_{2m} \cdot \partial \cdot S^{-1} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{L}_3 & \mathbf{L}_4 \end{bmatrix}$$

then satisfies that

$$\mathbf{L}^* = A \cdot \mathbf{L} \cdot A, \tag{7.4}$$

or equivalently

$$\mathbf{L}_1^* = \mathbf{L}_4, \qquad \mathbf{L}_2^* = \mathbf{L}_2, \qquad \mathbf{L}_3^* = \mathbf{L}_3.$$

The time evolution of S and the Lax operator $\mathbf{L}$ is given by the formula established in Section 6 of [16]. Let $S(t)$ be the solution of the n-component KP equations with the initial data $S(0) = S$. Then $S(t)$ is given by the generalized Birkhoff decomposition of [16, 18]

$$\exp\left(\begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix}\right) \cdot S(0)^{-1} = S(t)^{-1} \cdot Y(t), \tag{7.5}$$

where D_1 and D_2 are diagonal matrices of the shape

$$D_i = \begin{bmatrix} \sum_{j\ge 1} t_{ij1}\partial^j & & \\ & \ddots & \\ & & \sum_{j\ge 1} t_{ijm}\partial^j \end{bmatrix}$$

corresponding to (5.13), and $Y(t)$ is an invertible infinite order differential operator introduced in [18]. We impose that the time evolution $S(t)$ satisfies the same twisted self-adjoint condition (7.3). Applying the adjoint-inverse operation and conjugation by A of (7.1) to (7.5), we obtain

$$\exp\left(A\begin{bmatrix}-D_1 & \\ & -D_2\end{bmatrix}A\right)\cdot S(0)^{-1}=S(t)^{-1}\cdot(A(Y(t)^*)^{-1}A).$$

Therefore, the time evolution (7.5) with the condition $D_2=-D_1$ preserves the twisted self-adjointness (7.3). We have thus established

Theorem 7.1. *The KP-type equations that generate the Hitchin integrable systems on the moduli spaces of Sp_{2m}-Higgs bundles are the reduction of the $2m$-component KP equations that preserve the twisted self-adjointness* (7.3) *for the operator S, or* (7.4) *for the Lax operator $\mathbf{L}$. The time evolution $S\longmapsto S(t)$ preserving the condition is given by the generalized Birkhoff decomposition*

$$\exp\left(\begin{bmatrix}D & \\ & -D\end{bmatrix}\right)\cdot S(0)^{-1}=S(t)^{-1}\cdot Y(t), \tag{7.6}$$

where

$$D=\begin{bmatrix}\sum_{j\geq 1}t_{j1}\partial^j & & \\ & \ddots & \\ & & \sum_{j\geq 1}t_{jm}\partial^j\end{bmatrix}.$$

Remark 7.2. Since the time evolution of (7.6) is given by a traceless matrix $\mathrm{diag}(D,-D)$, every finite-dimensional orbit of this system is a Prym variety by the general theory of [16], which is expected from the Sp Hitchin fibration.

References

[1] Malcolm R. Adams and Maarten J. Bergvelt, The Krichever map, vector bundles over algebraic curves, and Heisenberg algebras, Commun. Math. Phys. **154** (1993), 265–305.

[2] A. Álvarez Vázquez, J. M. Muñoz Porras, and F. J. Plaza Martín, The algebraic formalism of soliton equations over arbitrary base fields, In: Workshop on Abelian Varieties and Theta Functions, Aportaciones Mat. Investig. **13** (1998), 3–40.

[3] Arnaud Beauville, M. S. Narasimhan, and S. Ramanan, Spectral curves and the generalized theta divisor, J. reine angew. Math. **398** (1989), 169–179.

[4] Ron Donagi and Eyal Markman, Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles, Lecture Notes in Mathematics **1620**, Springer-Verlag, 1996.

[5] Ron Donagi and Tony Pantev, Langlands duality for Hitchin systems, arXiv:math/0604617v2 (2006).

[6] Edward Frenkel, Lectures on the Langlands program and conformal field theory, arXiv:hep-th/0512172 (2005).

[7] E. Gómez González, J. M. Muñoz Porras, and F. J. Plaza Martín, Prym varieties, curves with automorphisms and the Sato Grassmannian, math.AG/0207202 [math.AG].

[8] Tamás Hausel and Michael Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, Inventiones Math. **153** (2003), 197–229.

[9] D. Hernandez-Serrano, J. M. Muñoz Porras, and F. J. Plaza Martín, Equations of the moduli of Higgs pairs and infinite Grassmannian, Intern. J. Math., to appear.

[10] Nigel J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (Ser. 3) **55** (1987), 59–126.

[11] Nigel J. Hitchin, Stable bundles and integrable systems, Duke Math. J. **54** (1987), 91–114.

[12] Nigel J. Hitchin, Langlands duality and G_2 spectral curves, arXiv:math.AG/0611524 (2006).

[13] Michio Jimbo and Tetsuji Miwa, Solitons and Infinite-Dimensional Lie Algebras, Publ. Res. Inst. Math. Sci. **19** (1983), 943–1001.
[14] Anton Kapustin and Edward Witten, Electric-magnetic duality and the geometric Langlands program, arXiv:hep-th/0604151v3 (2007).
[15] Yingchen Li and Motohico Mulase, Hitchin systems and KP equations, Intern. J. Math. **7** (1996), 277–244.
[16] Yingchen Li and Motohico Mulase, Prym varieties and integrable systems, Commun. Analysis and Geom. **5** (1997), 279–332.
[17] Motohico Mulase, Cohomological structure in soliton equations and Jacobian varieties, J. Diff. Geom. **19** (1984), 403–430.
[18] Motohico Mulase, Solvability of the super KP equation and a generalization of the Birkhoff decomposition, Inventiones Math. **92** (1988), 1–46
[19] Motohico Mulase, Category of vector bundles on algebraic curves and infinite dimensional Grassmannians, Intern. J. Math. **1** (1990), 293–342.
[20] Motohico Mulase, Geometry of character varieties of surface groups, RIMS Kôkyûroku **1605** (2008), 1–21.
[21] David Mumford, John Fogarty, and Frances Kirwan, Geometric invariant theory, Third Edition, Springer-Verlag, 1994.
[22] F. Plaza Martín, Families of infinite Grassmannians and the induced central extensions, in preparation.
[23] Ines Quandt, On a relative version of the Krichever correspondence, Bayreuther Mathematische Schriften **52** (1997), 1–74.
[24] Mikio Sato and Yasuko Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, In: Nonlinear partial differential equations in applied science, North-Holland Math. Stud. **81**, 1983, pp. 259–271.
[25] Graeme Segal and George Wilson, Loop groups and equations of KdV type, Inst. Hautes Études Sci. Publ. Math. **61** (1985), 5–65.
[26] Takahiro Shiota, Characterization of Jacobian varieties in terms of soliton equations, Inventiones Math. **83** (1986), 333–382.
[27] Takahiro Shiota, Prym varieties and soliton equations, In: Infinite-dimensional Lie algebras and groups, Adv. Ser. Math. Phys. **7**, 1989, pp. 407–448.
[28] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow, Mirror symmetry is T-duality, Nuclear Phys. **B 479** (1996), 243–259.
[29] Kanehisa Takasahi, The world of integrable systems–the fellowship of the Toda lattice (in Japanese), Kyoritsu Shuppan Publ. Co., 2001.
[30] Takayuki Tsuchida, New reductions of integrable matrix PDEs–$Sp(2m)$-invariant systems, arXiv:0712.4373 [nlin.SI] (2008).
[31] Pol Vanhaecke, Integrable systems in the realm of algebraic geometry, Lecture Notes in Mathematics **1638**, Springer-Verlag, 1996.

(Address after September 2008) National Security Agency, Fort George G. Meade, MD 20755-6000

Department of Mathematics, University of California, Davis, CA 95616–8633
E-mail address: `mulase@math.ucdavis.edu`