

ON THE CURVATURE OF METRIC TRIPLES

QINGLAN XIA

ABSTRACT. In this article we introduce a notion of curvature, denoted by $k_X(T)$, for a metric triple T inside a (possibly discrete) metric space X . To define the notion, we employ the information consisting of side lengths of the triple as well as the minimum total distance from vertices of the triple to points of the metric space. This information provides us a unique number $k_X(T)$ such that the triple T can be isometrically embedded into the model space M_k^2 up to $k \leq k_X(T)$. The value $k_X(T)$ agrees with the usual curvature when X is a convex subset of a model space. We also show that the curvature $k_X(T)$ of any metric triple T inside a $CAT(k)$ space is bounded above by k .

1. INTRODUCTION

The purpose of this article is to introduce a notion of curvature for a metric triple inside a (possibly discrete) metric space. Our motivation comes from considering the following problem: suppose X is a (possibly finite) subset of an unknown metric space Y , how can we discover geometric information of Y from those of X ? People are interested in finding some intrinsic and numerically computable geometric quantities of X that could be used to indicate properties of the unknown ambient space Y . As development of scientific computation grow rapidly, this problem becomes even more interesting since X may simply be some scientific data collected from experiments or observations. As an example, we investigate curvature information of X in this article.

As one of the fundamental concepts in geometry, curvature has been studied extensively from those of smooth curves to Riemannian manifolds [4], to geodesic metric spaces (see [1],[2] and references therein), and beyond (e.g. [3], [6]). In the literature, most works about curvature assume that the space itself is at least locally path-connected. In the more general setting when the metric space is not necessarily path-connected, or even simply a finite set, one may consider Menger curvature of metric triples (see [5] and references therein). Menger defined the curvature of a triple of points in a metric space as the reciprocal of the radius of the circle in the Euclidean plane which is circumscribed to a comparison configuration associated to that triple. Later Wald [7] considered the curvature of quadruple of points in a metric space as the curvature k of the model surface M_k^2 (i.e., the surface of constant curvature k) into which the metric quadruple can be isometrically embedded.

In the definition of Menger curvature of a triple T , one uses a comparable triangle of the triple in the Euclidean space. Nevertheless, since the triple could also be isometrically embedded into other model spaces M_k^2 of constant curvature k , one may also consider comparison configurations of the triple in other model surfaces

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M_k^2 and get analogous concepts of curvature. In this sense, Menger's curvature is thoroughly Euclidean.

On the other hand, existence of other points in X may prohibit the triple T , together with other points, to be isometrically embedded into some M_k^2 . This motivates us to consider the minimum distance from the triple to points in X , and enable us to find a unique number $k_X(T)$ so that T can only be isometrically embedded into M_k^2 up to $k \leq k_X(T)$.

The triple-wise defined curvature $k_X(T)$ has some nice properties.

- When X is a convex subset of a model space M_k^2 , the value $k_X(T)$ agrees with the pointwise defined curvature k of X , for each non-degenerate triple T in X .
- The value $k_X(T)$ is intrinsically defined. It depends only on the side lengths of the triple as well as the minimum distance of the triple to X .
- Suppose X is a subset of an unknown metric space Y , then according to Proposition 5.7, $k_X(T) \geq k_Y(T)$. Adding more data of Y to X will decrease the gap between $k_X(T)$ and $k_Y(T)$, and provide a better approximation.
- Suppose X is a CAT(k) space. Then, according to Theorem 6.1, every metric triple (with a bounded perimeter) in X will have the curvature $k_X(T)$ bounded above by k . This result indicates that one could study properties of a more general CAT(k)-type space X by assuming that every triple-wise curvature $k_X(T)$ in X is bounded above by k . Note that in this general setting, the space X is not necessarily locally path connected.

The article is organized as follows. In section §2, we define the function $S(a, b, c, k)$ by studying the Steiner problem on the model space M_k^2 . For a metric triple T inside M_k^2 of side lengths a, b, c , the value $S(a, b, c, k)$ gives the minimum distance from the vertices of the triple to points in M_k^2 . In section §3, we calculate the values of $S(a, b, c, k)$ numerically, and showed its continuity and monotonicity in §4. In section §5, we define the curvature $k_X(T)$ of a metric triple T in any metric space X using the function $S(a, b, c, k)$. We also investigate properties of $k_X(T)$ afterwards. In section §6, we show that in a CAT(k) space X , any metric triple T (with a bounded perimeter) in X will also have the curvature $k_X(T)$ bounded above by k .

2. THE STEINER PROBLEM ON TRIPLES IN THE MODEL SURFACES

For a real number k , the model space M_k^2 with distance $|\cdot|_k$ is the simply connected surface with constant curvature k . That is, if $k = 0$, then M_k^2 is the Euclidean plane. If $k > 0$, then M_k^2 is obtained from the sphere \mathbb{S}^2 by multiplying the distance function by the constant $\frac{1}{\sqrt{k}}$. If $k < 0$, then M_k^2 is obtained from the hyperbolic space \mathbb{H}^2 by multiplying the distance function by the constant $\frac{1}{\sqrt{-k}}$. The diameter of M_k^2 is denoted by $D_k := \pi/\sqrt{k}$ for $k > 0$ and $D_k := \infty$ for $k \leq 0$.

Suppose $a, b, c \in (0, \infty)$ with $a \leq b \leq c \leq a + b$. For any $k \in (-\infty, \left(\frac{2\pi}{a+b+c}\right)^2]$, i.e. $a + b + c \leq 2D_k$, there exists a triangle ΔABC in M_k^2 with side length $|BC|_k = a$, $|AC|_k = b$ and $|AB|_k = c$. Then, we consider the Steiner problem of minimizing

$$(2.1) \quad S(a, b, c, k) := \min_{P \in \Delta ABC} \{ |PA|_k + |PB|_k + |PC|_k \}$$

in the model space M_k^2 . The minimum value is denoted by $S(a, b, c, k)$, which is independent of the choice of the triangle ΔABC in M_k^2 .

In general, when the side lengths a, b, c , are not necessarily in increasing order, one can simply extend the definition of S by requiring $S(a, b, c, k)$ to be a symmetric function of variables a, b, c . Nevertheless, in the following context, we will simply assume that $a \leq b \leq c$.

In the space M_k^2 , if the angle $\angle ACB \geq \frac{2\pi}{3}$, then the minimum value of S is achieved at the vertex C . When the angle $\angle ACB < \frac{2\pi}{3}$, the minimum value of S is achieved at an interior point O of the triangle ΔABC . In this case, the point O is called the Fermat's point of the triangle. A useful fact about the Fermat's point O is that the angles $\angle AOB = \angle BOC = \angle COA = \frac{2\pi}{3}$ in M_k^2 .

Lemma 2.1. *For any $a, b, c \in (0, \infty)$ with $a \leq b \leq c \leq a + b$, and any $k \in (-\infty, \left(\frac{2\pi}{a+b+c}\right)^2]$, it holds that*

$$\frac{a+b+c}{2} \leq S(a, b, c, k) \leq a+b.$$

In particular, if $a+b=c$, then for any $k \in (-\infty, \left(\frac{2\pi}{a+b+c}\right)^2]$,

$$S(a, b, c, k) = a+b=c.$$

Proof. Clearly, by the triangle inequality, for each $P \in \Delta ABC$,

$$2\{|PA|_k + |PB|_k + |PC|_k\} \geq |BC|_k + |AC|_k + |AB|_k.$$

Thus, $2S(a, b, c, k) \geq a+b+c$. Also,

$$S(a, b, c, k) \leq |CA|_k + |CB|_k = a+b.$$

□

The function S has the following properties:

Proposition 2.2. Let S be the function defined by (2.1). Then,

(1) For any $\lambda > 0$, we have

$$(2.2) \quad S\left(\lambda a, \lambda b, \lambda c, \frac{k}{\lambda^2}\right) = \lambda S(a, b, c, k).$$

In particular,

$$(a+b+c)S\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}, (a+b+c)^2 k\right) = S(a, b, c, k).$$

(2) If $k > 0$, then

$$(2.3) \quad S(a, b, c, k) = \frac{1}{\sqrt{k}} S\left(a\sqrt{k}, b\sqrt{k}, c\sqrt{k}, 1\right),$$

and

$$S\left(\frac{a}{\sqrt{k}}, \frac{b}{\sqrt{k}}, \frac{c}{\sqrt{k}}, k\right) = \frac{1}{\sqrt{k}} S(a, b, c, 1).$$

(3) If $k < 0$, then

$$(2.4) \quad S(a, b, c, k) = \frac{1}{\sqrt{-k}} S\left(a\sqrt{-k}, b\sqrt{-k}, c\sqrt{-k}, -1\right),$$

and

$$S\left(\frac{a}{\sqrt{-k}}, \frac{b}{\sqrt{-k}}, \frac{c}{\sqrt{-k}}, k\right) = \frac{1}{\sqrt{-k}} S(a, b, c, 1).$$

Proof. These results follow from direct calculations using the definition of S as well as the relationships between distance functions of M_k^2 and those of \mathbb{S}^2 or \mathbb{H}^2 . \square

Definition 2.3. For any $a, b, c \in (0, \infty)$ with $a \leq b \leq c \leq a + b$, we consider the number k^* defined as follows:

- If $c^2 = a^2 + b^2 + ab = a^2 + b^2 - 2ab \cos\left(\frac{2\pi}{3}\right)$, then $k^* = 0$;
- If $c^2 < a^2 + b^2 + ab$, then let $k^* > 0$ be the number such that

$$\cos(c\sqrt{k^*}) = \cos(a\sqrt{k^*}) \cos(b\sqrt{k^*}) - \frac{1}{2} \sin(a\sqrt{k^*}) \sin(b\sqrt{k^*});$$

- If $c^2 > a^2 + b^2 + ab$, then let $k^* < 0$ be the number such that

$$\cosh(c\sqrt{-k^*}) = \cosh(a\sqrt{-k^*}) \cosh(b\sqrt{-k^*}) + \frac{1}{2} \sinh(a\sqrt{-k^*}) \sinh(b\sqrt{-k^*}).$$

We denote the number k^* by $\Lambda(a, b, c)$.

Geometrically, the number $k^* = \Lambda(a, b, c)$ is the critical parameter so that in any space M_k^2 with $k < k^*$, there exists a Fermat's point O in the triangle ΔABC with side lengths (a, b, c) . For any side lengths (a, b, c) not necessarily in an increasing order, we may simply reorder it into an increasing order $(\tilde{a}, \tilde{b}, \tilde{c})$ and extend the definition $\Lambda(a, b, c) := \Lambda(\tilde{a}, \tilde{b}, \tilde{c})$.

From the definition of $\Lambda(a, b, c)$, it clearly holds that

$$\Lambda(ta, tb, tc) = \frac{1}{t^2} \Lambda(a, b, c)$$

for any $t > 0$. As an example, in Figure 1, we plot the graph of $\Lambda(a, b, c)$ with $a = 1, b = 1.2$ and c varies from $(0.2, 2.2)$.

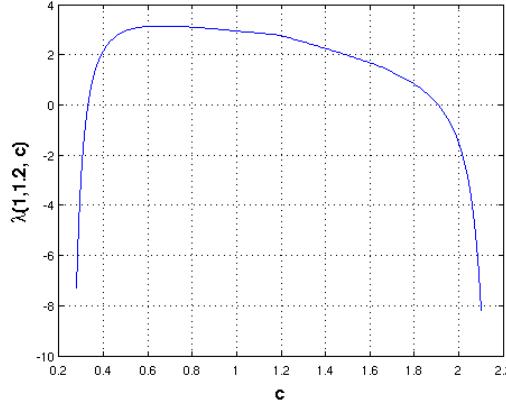


FIGURE 1. Graph of the curve $\Lambda(a, b, c)$ with $a = 1, b = 1.2$ and c varies from $(0.2, 2.2)$.

3. CALCULATION OF THE FUNCTION $S(a, b, c, k)$

Let a, b, c be fixed real values with $0 < a \leq b \leq c \leq a + b$. For any $k \in (-\infty, \left(\frac{2\pi}{a+b+c}\right)^2]$, there exists a triangle ΔABC in the model space M_k^2 with a, b, c as its side lengths. We now consider the properties of $S(a, b, c, k)$ as a function of the variable $k \in (-\infty, \left(\frac{2\pi}{a+b+c}\right)^2]$.

For any $k \in [\Lambda(a, b, c), \left(\frac{2\pi}{a+b+c}\right)^2]$, by the definition of $\Lambda(a, b, c)$, it follows that $\angle ACB \geq \frac{2\pi}{3}$ in M_k^2 . In this case, the minimum value of S in (2.1) is achieved at the point C and thus

$$S(a, b, c, k) = a + b.$$

For any $k \in (-\infty, \Lambda(a, b, c))$, the angle $\angle ACB < \frac{2\pi}{3}$ in M_k^2 , and the minimum value of S in (2.1) is achieved at the Fermat point O in the interior of the triangle ΔABC . In this case,

$$(3.1) \quad S(a, b, c, k) = x + y + z,$$

where $x = |OA|_k$, $y = |OB|_k$ and $z = |OC|_k$. Note that the angles $\angle AOB = \angle BOC = \angle COA = \frac{2\pi}{3}$ in M_k^2 .

We now calculate the value of $S(a, b, c, k)$ in (3.1) based on the signs of k .

3.1. Calculation of $S(a, b, c, 0)$. When $\Lambda(a, b, c) > 0$, i.e., when $a^2 + b^2 + ab > c^2$, one can explicitly calculate the value of $S(a, b, c, 0)$ as follows.

In this case, ΔABC is a triangle in the Euclidean plane. At the Fermat's point O , by the law of cosines, it holds that

$$(3.2) \quad a^2 = y^2 + z^2 + yz,$$

$$(3.3) \quad b^2 = x^2 + z^2 + xz,$$

$$(3.4) \quad c^2 = x^2 + y^2 + xy.$$

On the other hand, when calculating the area Δ of the triangle, we have

$$\frac{1}{2}xy \sin(2\pi/3) + \frac{1}{2}yz \sin(2\pi/3) + \frac{1}{2}xz \sin(2\pi/3) = \Delta$$

Thus, $xy + yz + zx = \frac{4\Delta}{\sqrt{3}}$. As a result, $2(x + y + z)^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\Delta$. Therefore,

$$(3.5) \quad S(a, b, c, 0) = x + y + z = \sqrt{\frac{a^2 + b^2 + c^2 + 4\sqrt{3}\Delta}{2}},$$

where the area Δ can be calculated by Heron's formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

with $s = \frac{a+b+c}{2}$.

3.2. Calculation of $S(a, b, c, k)$ with $k > 0$. To calculate $S(a, b, c, k)$ with $k > 0$, by (2.3) and (2.4), it is sufficient to calculate $S(a, b, c, 1)$.

In this case, ΔABC is a triangle in the unit sphere \mathbb{S}^2 . When $\Lambda(a, b, c) > 1$, i.e., when $\cos c > \cos a \cos b - \frac{1}{2} \sin a \sin b$, S achieves its minimum at the Fermat's point

O . Since the angles at O are all $\frac{2\pi}{3}$, according to the spherical law of cosines, the values x, y, z are giving by solving the following system of trigonometric equations:

$$(3.6) \quad \cos a = \cos y \cos z - \frac{1}{2} \sin y \sin z,$$

$$(3.7) \quad \cos b = \cos x \cos z - \frac{1}{2} \sin x \sin z,$$

$$(3.8) \quad \cos c = \cos x \cos y - \frac{1}{2} \sin x \sin y.$$

Proposition 3.1. For the x, y, z , and a, b, c given as above, let $X = \sin x, Y = \sin y, Z = \sin z$ and $u = \cos a, v = \cos b, w = \cos c$. Then, X, Y, Z satisfy the following system of multivariate polynomial equations:

$$(3.9) \quad (3u^2 + 1)X^2 + (3v^2 + 1)Y^2 + (6uv - 2w)XY - 3X^2Y^2 + 4w^2 = D,$$

$$(3.10) \quad (3u^2 + 1)X^2 + (3w^2 + 1)Z^2 + (6uw - 2v)XZ - 3X^2Z^2 + 4v^2 = D,$$

$$(3.11) \quad (3v^2 + 1)Y^2 + (3w^2 + 1)Z^2 + (6vw - 2u)YZ - 3Y^2Z^2 + 4u^2 = D,$$

where $D = 4u^2 + 4v^2 + 4w^2 - 8uvw$.

Proof. By symmetry, it is sufficient to prove the equation (3.9). Indeed, from (3.6) and (3.7), we have

$$\cos a \cos x - \cos b \cos y = \frac{\sin z}{2} \sin(x - y)$$

and

$$\cos a \sin x - \cos b \sin y = \sin(x - y) \cos z.$$

These two equations as well as (3.8) give

$$\begin{aligned} & \sin^2(x - y) = \sin^2(x - y)(\sin^2 z + \cos^2 z) \\ &= 4(\cos a \cos x - \cos b \cos y)^2 + (\cos a \sin x - \cos b \sin y)^2 \\ &= 4\cos^2 a \cos^2 x + 4\cos^2 b \cos^2 y - 8\cos a \cos b \cos x \cos y \\ & \quad + \cos^2 a \sin^2 x + \cos^2 b \sin^2 y - 2\cos a \cos b \sin x \sin y \\ &= 4\cos^2 a - 3\cos^2 a \sin^2 x + 4\cos^2 b - 3\cos^2 b \sin^2 y \\ & \quad - 8\cos a \cos b(\cos c + \frac{1}{2} \sin x \sin y) - 2\cos a \cos b \sin x \sin y \\ &= 4\cos^2 a + 4\cos^2 b - 3\cos^2 a \sin^2 x - 3\cos^2 b \sin^2 y \\ & \quad - 8\cos a \cos b \cos c - 6\cos a \cos b \sin x \sin y. \end{aligned}$$

On the other hand, by using (3.8) again, we have

$$\begin{aligned} \sin^2(x - y) &= (\sin x \cos y - \cos x \sin y)^2 \\ &= \sin^2 x \cos^2 y + \cos^2 x \sin^2 y - 2\sin x \sin y \cos x \cos y \\ &= \sin^2 x - 2\sin^2 x \sin^2 y + \sin^2 y - 2\sin x \sin y \left(\cos c + \frac{1}{2} \sin x \sin y \right) \\ &= \sin^2 x - 3\sin^2 x \sin^2 y + \sin^2 y - 2\sin x \sin y \cos c. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} & \sin^2 x - 3\sin^2 x \sin^2 y + \sin^2 y - 2\sin x \sin y \cos c \\ &= 4\cos^2 a + 4\cos^2 b - 3\cos^2 a \sin^2 x - 3\cos^2 b \sin^2 y \\ & \quad - 8\cos a \cos b \cos c - 6\cos a \cos b \sin x \sin y. \end{aligned}$$

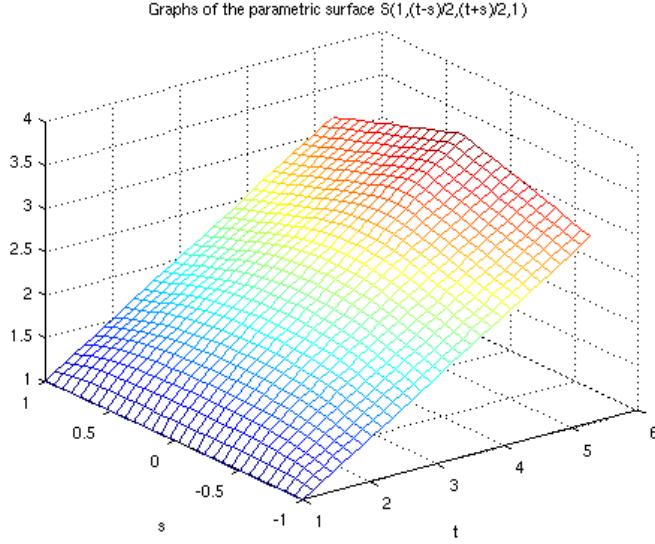


FIGURE 2. Graph of the parametric surface $S(a, b, c, 1)$ with $a = 1, b = (t - s)/2, c = (t + s)/2$ for $t \in (a, 2\pi - a)$ and $s \in (-a, a)$.

That is,

$$\begin{aligned}
 & (1 + 3 \cos^2 a) \sin^2 x - 3 \sin^2 x \sin^2 y + (1 + 3 \cos^2 b) \sin^2 y \\
 & + (6 \cos a \cos b - 2 \cos c) \sin x \sin y \\
 & = 4 \cos^2 a + 4 \cos^2 b - 8 \cos a \cos b \cos c,
 \end{aligned}$$

which is the equation (3.9). \square

From either equations (3.6), (3.7), (3.8) or equations (3.9), (3.10), (3.11), one can calculate the numerical values of $S(a, b, c, 1)$ via numerical methods e.g. Newton's method. In Figure 2, we plot the graph of the parametric surface $S(a, b, c, 1)$ with $a = 1, b = (t - s)/2, c = (t + s)/2$ for $t \in (a, 2\pi - a)$ and $s \in (-a, a)$.

3.3. Calculation of $S(a, b, c, -1)$. Similarly, by the hyperbolic law of cosines, it follows that

$$(3.12) \quad \cosh a = \cosh y \cosh z + \frac{1}{2} \sinh y \sinh z$$

$$(3.13) \quad \cosh b = \cosh x \cosh z + \frac{1}{2} \sinh x \sinh z$$

$$(3.14) \quad \cosh c = \cosh x \cosh y + \frac{1}{2} \sinh x \sinh y$$

for x, y, z and then $S(a, b, c, -1) = x + y + z$.

Proposition 3.2. For the x, y, z , and a, b, c given as above, let $X = \sinh x, Y = \sinh y, Z = \sinh z$ and $u = \cosh a, v = \cosh b, w = \cosh c$. Then, X, Y, Z satisfy the

following system of multivariate polynomial equations:

$$(3.15) \quad (3u^2 + 1)X^2 + (3v^2 + 1)Y^2 + (6uv - 2w)XY + 3X^2Y^2 - 4w^2 + D = 0,$$

$$(3.16) \quad (3u^2 + 1)X^2 + (3w^2 + 1)Z^2 + (6uw - 2v)XZ + 3X^2Z^2 - 4v^2 + D = 0,$$

$$(3.17) \quad (3v^2 + 1)Y^2 + (3w^2 + 1)Z^2 + (6vw - 2u)YZ + 3Y^2Z^2 - 4u^2 + D = 0,$$

where $D = 4u^2 + 4v^2 + 4w^2 - 8uvw$.

Proof. Follows from an analogous proof of Proposition 3.1. \square

Again, from either equations (3.12), (3.13), (3.14) or equations (3.15), (3.16), (3.17), one can calculate the numerical values of $S(a, b, c, -1)$ via Newton's method.

As a result, one can (numerically) calculate the values of $S(a, b, c, k)$ for all $k \in (-\infty, \Lambda(a, b, c)]$. In Figure 3, we plot of the graph of $S(a, b, c, k)$ with $a = 1, b = 1.2$ and $c = 1.3$. Note that $S(a, b, c, k)$ is a continuous strictly increasing function of $k \in (-\infty, \Lambda(a, b, c)]$.

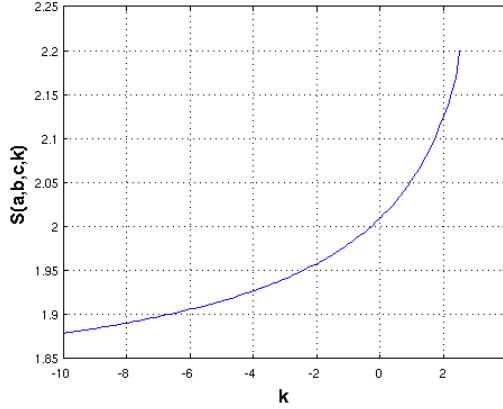


FIGURE 3. Graph of the strictly increasing curve $S(a, b, c, k)$ with $a = 1, b = 1.2, c = 1.3$ and the variable k varies from $(-\infty, \Lambda(a, b, c)]$ with $\Lambda(a, b, c) = 2.5081$ here.

4. CONTINUITY OF THE FUNCTION $S(a, b, c, k)$

As motivated by Figure 3, in this section, we consider the continuity and the monotonicity of the function $S(a, b, c, k)$.

Lemma 4.1. *For each $k \in \mathbb{R}$, $S(a, b, c, k)$ is a continuous function of variables a, b, c in the region given by the inequalities $0 < a \leq b \leq c \leq a+b$ and $k \leq \left(\frac{2\pi}{a+b+c}\right)^2$.*

Proof. When $k = 0$, by (3.5), $S(a, b, c, 0)$ is continuous in a, b, c .

When $k > 0$, by (2.3), it is sufficient to show the continuity of $S(a, b, c, 1)$. In the case that $\Lambda(a, b, c) > 1$, the function $S(a, b, c, 1)$ is defined by using the equations (3.6)–(3.8). The continuity of $S(a, b, c, 1)$ also follows from the continuity of these equations. Indeed, let (x^*, y^*, z^*) be the solution of the equations (3.6)–(3.8) corresponding to (a, b, c) . For any sequence $\{(a_n, b_n, c_n)\}$ converges to (a, b, c) , let (x_n, y_n, z_n) be its corresponding solution of the equations. So $S(a_n, b_n, c_n, 1) =$

$x_n + y_n + z_n$. We want to prove that the sequence $\{(x_n, y_n, z_n)\}$ converges to (x^*, y^*, z^*) . Let $(\bar{x}, \bar{y}, \bar{z})$ be any limit point of the sequence $\{(x_n, y_n, z_n)\}$. By the continuity of the implicit equations (3.6)–(3.8), $(\bar{x}, \bar{y}, \bar{z})$ is also the solution to the equations corresponding to (a, b, c) . As a result, $(\bar{x}, \bar{y}, \bar{z})$ equals to (x^*, y^*, z^*) due to the fact that the system of equations has a unique solution. This shows that the bounded sequence $\{(x_n, y_n, z_n)\}$ converges to (x^*, y^*, z^*) . Therefore, $S(a, b, c, 1) = x^* + y^* + z^* = \lim_{n \rightarrow \infty} x_n + y_n + z_n = \lim_{n \rightarrow \infty} S(a_n, b_n, c_n, 1)$, and hence $S(a, b, c, 1)$ is continuous.

In other cases (e.g., $\Lambda(a, b, c) \leq 1$ when $k > 0$, or the case that $k < 0$), the same continuity result follows from analogous statements derived from the continuity of the corresponding equations. \square

Lemma 4.2. *Given (a, b, c) , $S(a, b, c, k)$ is a continuous function of k .*

Proof. When $k \neq 0$, the continuity of $S(a, b, c, k)$ in k follows from Lemma 4.1 and equations (2.3), (2.4). Thus, we only need to show the continuity of $S(a, b, c, k)$ at $k = 0$.

We first show that $\lim_{k \rightarrow 0^+} S(a, b, c, k) = S(a, b, c, 0)$. When $\Delta(a, b, c) \leq 0$, then $\lim_{k \rightarrow 0^+} S(a, b, c, k) = \lim_{k \rightarrow 0^+} (a + b) = a + b = S(a, b, c, 0)$. When $\Delta(a, b, c) > 0$, by definition,

$$S(a, b, c, 0) = x^* + y^* + z^*$$

where (x^*, y^*, z^*) is the solution to the equations (3.2)–(3.4). Similarly,

$$S(a, b, c, k) = x(k) + y(k) + z(k)$$

where $(x(k)\sqrt{k}, y(k)\sqrt{k}, z(k)\sqrt{k})$ is the solution to the equations (3.6)–(3.8) corresponding to $(a\sqrt{k}, b\sqrt{k}, c\sqrt{k})$. To show that $(x(k), y(k), z(k))$ converges to (x^*, y^*, z^*) as $k \rightarrow 0^+$, it is sufficient to show that any limit point of $(x(k), y(k), z(k))$ equals to (x^*, y^*, z^*) . Indeed, let $(\bar{x}, \bar{y}, \bar{z})$ be the limit of a sequence $(x(k_n), y(k_n), z(k_n))$ for some $k_n > 0$ with $\lim_{n \rightarrow \infty} k_n = 0$. Then by (3.6),

$$\cos(a\sqrt{k_n}) = \cos(y(k_n)\sqrt{k_n}) \cos(z(k_n)\sqrt{k_n}) - \frac{1}{2} \sin(y(k_n)\sqrt{k_n}) \sin(z(k_n)\sqrt{k_n}).$$

Using the little-oh notation, we can rewrite it as follows:

$$\begin{aligned} & 1 - \frac{a^2}{2} k_n + o(k_n) \\ &= \left(1 - \frac{y(k_n)^2}{2} k_n + o(k_n)\right) \left(1 - \frac{z(k_n)^2}{2} k_n + o(k_n)\right) \\ &\quad - \frac{1}{2} \left(y(k_n)\sqrt{k_n} + o(\sqrt{k_n})\right) \left(z(k_n)\sqrt{k_n} + o(\sqrt{k_n})\right) \\ &= \left(1 - \frac{\bar{y}^2}{2} k_n + o(k_n)\right) \left(1 - \frac{\bar{z}^2}{2} k_n + o(k_n)\right) - \frac{1}{2} \left(\bar{y}\sqrt{k_n} + o(\sqrt{k_n})\right) \left(\bar{z}\sqrt{k_n} + o(\sqrt{k_n})\right) \\ &= \left(1 - \frac{\bar{y}^2}{2} k_n - \frac{\bar{z}^2}{2} k_n + o(k_n)\right) - \frac{1}{2} (\bar{y}\bar{z}k_n + o(k_n)) \\ &= 1 - \frac{\bar{y}^2 + \bar{z}^2 + \bar{y}\bar{z}}{2} k_n + o(k_n). \end{aligned}$$

As a result, $a^2 = \bar{y}^2 + \bar{z}^2 + \bar{y}\bar{z}$. i.e., $(\bar{x}, \bar{y}, \bar{z})$ satisfies the equation (3.2). Similarly, it follows that $(\bar{x}, \bar{y}, \bar{z})$ also satisfies the equations (3.3) and (3.4). Therefore,

$(\bar{x}, \bar{y}, \bar{z})$ equals to (x^*, y^*, z^*) , the solution to the equations (3.2)–(3.4). Therefore, $\lim_{k \rightarrow 0^+} S(a, b, c, k) = S(a, b, c, 0)$. Analogous approach also shows that

$$\lim_{k \rightarrow 0^-} S(a, b, c, k) = S(a, b, c, 0),$$

and thus $S(a, b, c, k)$ is continuous at $k = 0$ as desired. \square

By the previous two lemmas, we have the following result.

Proposition 4.3. $S(a, b, c, k)$ is a continuous function.

We now study the monotonicity of the function $S(a, b, c, k)$ in $k \in (-\infty, \Lambda(a, b, c)]$ for each fixed (a, b, c) . To avoid the lengthy analytic proof, we give geometric reasoning here. Consider a collection of spheres representing M_k^2 in the upper half space of \mathbb{R}^3 . Each of them is tangent to the xy -plane at the origin O . Now, let ΔABC be a triangle in the xy -plane with O being its Fermat's point, and we wrap it onto each sphere M_k^2 to get a triangle $\Delta A_k B_k C_k$ while keeping O being its Fermat's point. Note that when one fixes the side lengths $|OA_k|_k = x, |OB_k|_k = y, |OC_k|_k = z$, the corresponding side lengths $|A_k B_k|_k = c(k), |A_k C_k|_k = b(k)$ and $|B_k C_k|_k = a(k)$ are strictly decreasing. Reversely, if one fixes the side lengths $|A_k B_k|_k = c, |A_k C_k|_k = b$ and $|B_k C_k|_k = a$, the corresponding lengths $|OA_k|_k = x(k), |OB_k|_k = y(k), |OC_k|_k = z(k)$ will be strictly increasing. As a result, the function $S(a, b, c, k) = x(k) + y(k) + z(k)$ is strictly increasing in k when $k \geq 0$. Similarly, it is also strictly increasing when $k \leq 0$.

In summary, $S(a, b, c, k)$ is a continuous strictly increasing function of k .

5. CURVATURE OF A METRIC TRIPLE

Definition 5.1. Let (X, d) be a metric space. A metric triple is a set $T = \{p_1, p_2, p_3\} \subseteq X$, together with a set of mutual distance $d_{ij} = d(p_i, p_j); 1 \leq i \leq j \leq 3$.

Without losing of generality, we may assume that $d_{12} \leq d_{23} \leq d_{13} \leq d_{12} + d_{23}$.

Definition 5.2. For any metric triple T in (X, d) , define

$$g_X(T) := \inf_{x \in X} \sum_{i=1}^3 d(p_i, x).$$

Note that $g_X(T) \geq \frac{d_{12} + d_{23} + d_{13}}{2}$.

Definition 5.3. Let T be a metric triple in a metric space (X, d) , and S be the function as defined in (2.1). The curvature $k_X(T)$ of the metric triple T with respect to X is the number $k \in [-\infty, \Lambda(d_{12}, d_{23}, d_{13})]$ such that

$$g_X(T) = S(d_{12}, d_{23}, d_{13}, k).$$

Example 5.4. Suppose (X, d) is a convex subset in the model space M_k^2 for some real number k , and T is a metric triple in X . Then $g_X(T) = S(d_{12}, d_{23}, d_{31}, k)$. Thus, $k_X(T) \leq k$. Moreover, if $k \leq \Lambda(d_{12}, d_{23}, d_{31})$, i.e. the angle at each vertex of the corresponding triangle is no more than $2\pi/3$, then $k_X(T) = k$.

Example 5.5. Let T be a metric triple in a discrete metric space (X, d) with $d(x, y) = 1$ whenever $x \neq y$. Then, $g_X(T) = 2$. In this case, the curvature $k_X(T) = \Lambda(1, 1, 1) = 3.6505 = (\pi - \arccos \frac{1}{3})^2$. The comparable triangle in the model space $M_{k_X(T)}^2$ has an angle $2\pi/3$ at each of its vertices.

According to (2.2), we immediately have the following proposition:

Proposition 5.6. Let $T = \{p_1, p_2, p_3\}$ be a metric triple in the metric space (X, d_X) , and $\tilde{T} = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ be a metric triple in the metric space (Y, d_Y) . If for some $\lambda > 0$, $d_Y(\tilde{p}_i, \tilde{p}_j) = \lambda d_X(p_i, p_j)$ for each $i, j = 1, 2, 3$ and $g_Y(\tilde{T}) = \lambda g_X(T)$, then $k_Y(\tilde{T}) = \frac{1}{\lambda^2} k_X(T)$.

Proposition 5.7. Suppose (X, d_X) can be isometrically embedded in (Y, d_Y) . Then, for any metric triple T in (X, d_X) , we have $k_Y(T) \leq k_X(T)$.

Proof. This is because $g_Y(T) \leq g_X(T)$ and $S(d_{12}, d_{23}, d_{31}, k)$ is an strictly increasing function of k . \square

Lemma 5.8. Let $T = \{p_1, p_2, p_3\}$ and $\tilde{T} = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ be two metric triples in a metric space (X, d) . Then,

$$(5.1) \quad |g_X(T) - g_X(\tilde{T})| \leq \sum_{i=1}^3 d(p_i, \tilde{p}_i).$$

Proof. For all $x \in X$, by the triangle inequality,

$$\sum_{i=1}^3 d(x, p_i) \leq \sum_{i=1}^3 d(x, \tilde{p}_i) + \sum_{i=1}^3 d(p_i, \tilde{p}_i).$$

Taking the infimum over $x \in X$ on both sides of the inequality, we have $g_X(T) \leq g_X(\tilde{T}) + \sum_{i=1}^3 d(p_i, \tilde{p}_i)$. Similarly, $g_X(\tilde{T}) \leq g_X(T) + \sum_{i=1}^3 d(p_i, \tilde{p}_i)$. \square

Proposition 5.9. Let $T^{(n)} = \{p_1^{(n)}, p_2^{(n)}, p_3^{(n)}\}$ be a sequence of metric triples in a metric space (X, d) . If $T^{(n)}$ converges to a metric triple $T = \{p_1, p_2, p_3\}$ in (X, d) in the sense that for each $i = 1, 2, 3$, $d(p_i^{(n)}, p_i) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} k_X(T^{(n)}) = k_X(T).$$

Proof. By (5.1), $\lim_{n \rightarrow \infty} g_X(T^{(n)}) = g_X(T)$. Since S is also a continuous function, we have $\lim_{n \rightarrow \infty} k_X(T^{(n)}) = k_X(T)$. \square

6. CURVATURE OF METRIC TRIPLES IN CAT(k) SPACES

For a real number k , a geodesic metric space (X, d) is said to be CAT(k) if every geodesic triangle ΔABC in X with perimeter less than $2D_k$ satisfies the CAT(k) inequality. That is, let $\Delta A'B'C'$ be a comparison triangle in the model space $(M_k^2, |\cdot|_k)$, with sides of the same length as the sides of ΔABC , such that for any D in edge BC , there is a corresponding point D' in the comparison edge $B'C'$ with $d(B, D) = |B'D'|_k$, $d(C, D) = |C'D'|_k$ and satisfies the inequality $d(A, D) \leq |A'D'|_k$.

Define

$$H_k = \begin{cases} \frac{\pi}{2\sqrt{k}}, & \text{if } k > 0 \\ \infty, & \text{if } k = 0 \\ \frac{1.3877}{\sqrt{-k}}, & \text{if } k < 0. \end{cases}$$

Theorem 6.1. Suppose (X, d) is a CAT(k) space for some real number k . Then, for any metric triple $T = \{A, B, C\}$ in X with $\max\{d(A, B), d(A, C), d(B, C)\} \leq H_k$, it holds that $k_X(T) \leq k$.

Proof. It is sufficient to show that $g_X(T) \leq g_{M_k^2}(T^*)$ where $T^* = \{A^*, B^*, C^*\}$ is the comparison configuration of T in the model space $(M_k^2, |\cdot|_k)$ of equal side lengths. Without losing of generality, we may assume that the Fermat's point O^* of the triangle $A^*B^*C^*$ exists in M_k^2 . Then, $g_{M_k^2}(T^*) = |A^*O^*|_k + |B^*O^*|_k + |C^*O^*|_k$. Let D^* be the point on the edge B^*C^* such that O^* is located on the geodesic A^*D^* . Suppose $|A^*O^*|_k = s|A^*D^*|_k$ and $|B^*D^*|_k = t|B^*C^*|_k$ for some $s, t \in (0, 1)$. Then, let D be the point on the edge BC in the metric space X corresponding to D^* , i.e. $d(B, D) = td(B, C)$. Also, let O be the point on the geodesic AD corresponding to O^* , i.e. $d(A, O) = sd(A, D)$. Note that $g_X(T) \leq d(A, O) + d(B, O) + d(C, O)$. By the CAT(k) inequality, it holds that $|A^*D^*|_k \geq d(A, D) = \sigma|A^*D^*|_k$ for some $\sigma \leq 1$. Thus, it follows that $d(A, O) = \sigma|A^*O^*|_k$ and $d(O, D) = \sigma|O^*D^*|_k$.

Now, we consider a comparison triangle $A'B'D'$ (and $A'C'D'$) in M_k^2 of ABD (and ACD , respectively) in (X, d) . Also, let O' on $A'D'$ be the corresponding point of O on AD in X . So, $|A'O'|_k = d(A, O) = \sigma|A^*O^*|_k$. Again, by the CAT(k) inequality, we have $d(B, O) \leq |B'O'|_k$ and $d(C, O) \leq |C'O'|_k$. Thus,

$$g_X(T) \leq d(A, O) + d(B, O) + d(C, O) \leq |A'O'|_k + |B'O'|_k + |C'O'|_k.$$

Note that when $\sigma = 1$, then one can simply take $A' = A^*, B' = B^*, C' = C^*, D' = D^*, O' = O^*$. In this case, $|A'O'|_k + |B'O'|_k + |C'O'|_k = |A^*O^*|_k + |B^*O^*|_k + |C^*O^*|_k = g_{M_k^2}(T^*)$. Denote

$$(6.1) \quad f(\sigma) = |A'O'|_k + |B'O'|_k + |C'O'|_k.$$

for all $\sigma \in [0, 1]$. By Lemma 6.2, Lemma 6.5 and Lemma 6.8 given below, we have $f(\sigma) \leq f(1)$ whenever $\sigma \in [0, 1]$. Thus, $g_X(T) \leq g_{M_k^2}(T^*)$. \square

Lemma 6.2. *Let f be the function as given in (6.1). When $k = 0$, then $f(\sigma) \leq f(1)$ whenever $\sigma \in [0, 1]$.*

Proof. In the case that $k = 0$, we have $O' = (1-s)A' + sD'$ with $s = ||A^*O^*||/||A^*D^*||$. Here, $||\cdot|| = |\cdot|_0$ denotes the Euclidian distance on $M_0^2 = \mathbb{R}^2$. Thus, by means of the law of cosines, we have

$$\begin{aligned} & ||B'O'||^2 = ||(1-s)B'A' + sB'D'||^2 \\ &= (1-s)^2||B'A'||^2 + s^2||B'D'||^2 + s(1-s)(||B'A'||^2 + ||B'D'||^2 - ||A'D'||^2) \\ &= (1-s)||B'A'||^2 + s||B'D'||^2 - s(1-s)||A'D'||^2 \\ &= (1-s)||B^*A^*||^2 + s||B^*D^*||^2 - s(1-s)\sigma^2||A^*D^*||^2 \\ &= \frac{1}{||A^*D^*||^2} (||D^*O^*|| \times ||B^*A^*||^2 + ||A^*O^*|| \times ||B^*D^*||^2) - ||A^*O^*|| \times ||A^*D^*||\sigma^2. \end{aligned}$$

Similarly, we have

$$||C'O'||^2 = \frac{1}{||A^*D^*||^2} (||D^*O^*|| \times ||C^*A^*||^2 + ||A^*O^*|| \times ||C^*D^*||^2) - ||A^*O^*|| \times ||A^*D^*||\sigma^2.$$

As a result,

$$\begin{aligned} f(\sigma) &= ||A'O'|| + ||B'O'|| + ||C'O'|| \\ &= \sigma||A^*O^*|| + \sqrt{\frac{||A^*O^*|| \times ||B^*D^*||^2 + ||D^*O^*|| \times ||A^*B^*||^2}{||A^*D^*||} - \sigma^2||A^*O^*|| \times ||O^*D^*||} \\ &\quad + \sqrt{\frac{||A^*O^*|| \times ||C^*D^*||^2 + ||D^*O^*|| \times ||A^*C^*||^2}{||A^*D^*||} - \sigma^2||A^*O^*|| \times ||O^*D^*||}. \end{aligned}$$

Direct calculation yields that $f''(\sigma) \leq 0$ and

$$f'(\sigma) = \|A^*O^*\| - \frac{\|A^*O^*\| \times \|O^*D^*\|}{\|B^*O'\|} \sigma - \frac{\|A^*O^*\| \times \|O^*D^*\|}{\|C^*O'\|} \sigma.$$

Thus,

$$f'(1) = \|A^*O^*\| - \frac{\|A^*O^*\| \times \|O^*D^*\|}{\|B^*O^*\|} - \frac{\|A^*O^*\| \times \|O^*D^*\|}{\|C^*O^*\|}.$$

Since O^* is the Fermat point, the angles at O^* are all $\frac{2\pi}{3}$. Denote the angle $\angle O^*B^*D^* = \alpha$, then by the law of sine, we have

$$\begin{aligned} f'(1) &= \|A^*O^*\|(1 - \frac{\|O^*D^*\|}{\|B^*O^*\|} - \frac{\|O^*D^*\|}{\|C^*O^*\|}) \\ &= \|A^*O^*\|(1 - \frac{\sin \alpha}{\sin(\frac{2\pi}{3} - \alpha)} - \frac{\sin(\frac{\pi}{3} - \alpha)}{\sin(\frac{\pi}{3} + \alpha)}) \\ &= \|A^*O^*\|(1 - 1) = 0. \end{aligned}$$

Now, since $f''(\sigma) \leq 0$ and $f'(1) = 0$, it holds that $f'(\sigma) \geq f'(1) = 0$ for $\sigma \in [0, 1]$. Thus, f is increasing on $[0, 1]$, and $f(\sigma) \leq f(1)$ whenever $\sigma \in [0, 1]$. \square

To consider the case when $k > 0$, we need the following two lemmas first. For simplicity, we let $|\cdot|$ denote the standard distance $|\cdot|_1$ on the unit sphere \mathbb{S}^2 .

Lemma 6.3. *Let $\Delta B'D'O'$ be a triangle in the unit sphere $(\mathbb{S}^2, |\cdot|)$. Then,*

$$(6.2) \quad \cot(|O'B'|) = \cot(|O'D'|) \cos(\alpha) + \cot(\gamma) \frac{\sin(\alpha)}{\sin(|O'D'|)}$$

where $\alpha = \angle B'O'D'$ and $\gamma = \angle B'D'O'$.

Proof. By the spherical laws of cosines and sines, it follows that

$$\begin{aligned} \cot(|O'B'|) &= \frac{\cos(|O'B'|)}{\sin(|O'B'|)} \\ &= \frac{\cos(|B'D'|) \cos(|O'D'|) + \sin(|B'D'|) \sin(|O'D'|) \cos(\gamma)}{\sin(|O'B'|)} \\ &= \frac{[\cos(|O'B'|) \cos(|O'D'|) + \sin(|O'B'|) \sin(|O'D'|) \cos(\alpha)] \cos(|O'D'|)}{\sin(|O'B'|)} \\ &\quad + \frac{\sin(|B'D'|) \sin(|O'D'|) \cos(\gamma)}{\sin(|O'B'|)} \\ &= \cot(|O'B'|) \cos^2(|O'D'|) + \sin(|O'D'|) \cos(|O'D'|) \cos(\alpha) + \frac{\sin(\alpha) \sin(|O'D'|)}{\sin(\gamma)} \cos(\gamma). \end{aligned}$$

Simplifying it leads to (6.2). \square

Lemma 6.4. *Let $\Delta A^*B^*C^*$ be a triangle on the unit sphere \mathbb{S}^2 with a Fermat's point O^* inside the triangle. Let D^* be the point on the arc B^*C^* such that O^* is on the arc A^*D^* . Also, let $\Delta A'B'D'$ be a triangle on \mathbb{S}^2 such that $|A'B'| = |A^*B^*|$, $|B'D'| = |B^*D^*|$ and $|A'D'| = \sigma|A^*D^*|$ for some $\sigma \in (0, 1]$. Let O' be the point on the arc $A'D'$ such that $|A'O'| = \sigma|A^*O^*|$ and $|O'D'| = \sigma|O^*D^*|$. If $\max\{|A^*B^*|, |A^*C^*|, |B^*C^*|\} \leq \frac{\pi}{2}$, then for all $\sigma \in [0, 1]$,*

$$(6.3) \quad \frac{d(|O'B'|)}{d\sigma} \geq -\frac{|O^*A^*|}{2} - \frac{|A^*D^*| \sin(|A^*O^*|)}{\sin(|A^*D^*|)} \sin\left(\frac{\pi}{3}\right) \cot(\gamma),$$

where $\gamma = \angle B'D'O'$.

Proof. By the spherical law of cosines, we have

$$\begin{aligned} \cos(|O'B'|) &= \cos(|A'B'|) \cos(|A'O'|) + \sin(|A'B'|) \sin(|A'O'|) \cos(\angle B'A'O') \\ &= \cos(|A'B'|) \cos(|A'O'|) + \sin(|A'B'|) \sin(|A'O'|) \frac{\cos(|B'D'|) - \cos(|A'B'|) \cos(|A'D'|)}{\sin(|A'B'|) \sin(|A'D'|)} \\ &= \cos(|A'B'|) \frac{\sin(|O'D'|)}{\sin(|A'D'|)} + \cos(|B'D'|) \frac{\sin(|A'O'|)}{\sin(|A'D'|)}. \end{aligned}$$

Thus,

$$(6.4) \quad \cos(|O'B'|) = \cos(|A^*B^*|) \frac{\sin(|O^*D^*|\sigma)}{\sin(|A^*D^*|\sigma)} + \cos(|B^*D^*|) \frac{\sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)}.$$

By the law of cosines, it follows that

$$(6.5) \quad \cos(|A^*B^*|) = \cos(|B'O'|) \cos(|A^*O^*|\sigma) - \sin(|B'O'|) \sin(|A^*O^*|\sigma) \cos(\alpha)$$

and

$$(6.6) \quad \cos(|B^*D^*|) = \cos(|B'O'|) \cos(|O^*D^*|\sigma) + \sin(|B'O'|) \sin(|O^*D^*|\sigma) \cos(\alpha).$$

Now, by taking derivatives on (6.4) with respect to σ , and a simplification using (6.5) and (6.6), we have

$$\begin{aligned} & -\sin(|O'B'|) \frac{d(|O'B'|)}{d\sigma} \\ &= \frac{|O^*D^*| \cos(|O^*D^*|\sigma) \sin(|A^*D^*|\sigma) - |A^*D^*| \sin(|O^*D^*|\sigma) \cos(|A^*D^*|\sigma)}{\sin^2(|A^*D^*|\sigma)} \cos(|A^*B^*|) \\ & \quad + \frac{|O^*A^*| \cos(|O^*A^*|\sigma) \sin(|A^*D^*|\sigma) - |A^*D^*| \sin(|O^*A^*|\sigma) \cos(|A^*D^*|\sigma)}{\sin^2(|A^*D^*|\sigma)} \cos(|B^*D^*|) \\ &= \frac{\cos(|B'O'|)}{\sin(|A^*D^*|\sigma)} [|A^*D^*| \sin(|O^*D^*|\sigma) \sin(|A^*O^*|\sigma) + \frac{\sin(|B'O'|)}{\sin(|A^*D^*|\sigma)} \cos(\alpha) \\ & \quad \times [|O^*A^*| \cos(|O^*A^*|\sigma) \sin(|O^*D^*|\sigma) - |O^*D^*| \cos(|O^*D^*|\sigma) \sin(|A^*O^*|\sigma)]]. \end{aligned}$$

Thus, by means of (6.2), it follows that

$$\begin{aligned} & -\frac{d(|O'B'|)}{d\sigma} \\ &= \frac{|A^*D^*| \sin(|O^*D^*|\sigma) \sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)} \cot(|B'O'|) \\ & \quad + \frac{|O^*A^*| \cos(|O^*A^*|\sigma) \sin(|O^*D^*|\sigma) - |O^*D^*| \cos(|O^*D^*|\sigma) \sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)} \cos(\alpha) \\ &= \frac{|A^*D^*| \sin(|O^*D^*|\sigma) \sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)} (\cot(|O^*D^*|\sigma) \cos(\alpha) + \cot(\gamma) \frac{\sin(\alpha)}{\sin(|O^*D^*|\sigma)}) \\ & \quad + \frac{|O^*A^*| \cos(|O^*A^*|\sigma) \sin(|O^*D^*|\sigma) - |O^*D^*| \cos(|O^*D^*|\sigma) \sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)} \cos(\alpha) \\ &= |O^*A^*| \cos(\alpha) + \frac{|A^*D^*| \sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)} \sin(\alpha) \cot(\gamma). \end{aligned}$$

As a result, we have for all $\sigma \in [0, 1]$,

$$(6.7) \quad \frac{d(|O'B'|)}{d\sigma} = -|O^*A^*| \cos(\alpha) - \frac{|A^*D^*| \sin(|A^*O^*|\sigma)}{\sin(|A^*D^*|\sigma)} \sin(\alpha) \cot(\gamma).$$

Now, we want to show that $\frac{d^2(|O'B'|)}{d\sigma^2} \leq 0$ for all $\sigma \in [0, 1]$. Let $p(\sigma) = \cos(|O'B'|)$. Then,

$$(6.8) \quad \frac{d^2(|O'B'|)}{d\sigma^2} = \frac{-p''(\sigma)}{\sqrt{1-p(\sigma)^2}} - \frac{(p'(\sigma))^2 p(\sigma)}{(\sqrt{1-p(\sigma)^2})^3}.$$

Now, we investigate $p(\sigma)$ via the equation (6.4). Clearly, when

$$\max\{|A^*B^*|, |A^*C^*|, |B^*C^*|\} \leq \frac{\pi}{2},$$

both $\cos(|A^*B^*|) \geq 0$ and $\cos(|B^*D^*|) \geq 0$. On the other hand, direct calculations give that when $b \geq a > 0$,

$$\frac{d^2}{dx^2} \left(\frac{\sin(ax)}{\sin(bx)} \right) = \frac{\sin(ax)}{\sin(bx)} [(b \cot(bx) - a \cot(ax))^2 + (\frac{b^2}{\sin^2(bx)} - \frac{a^2}{\sin^2(ax)})] \geq 0.$$

Thus, by equation (6.4), it follows that $p(\sigma) \geq 0$ and $p''(\sigma) \geq 0$ for all $\sigma \leq 1$. This shows that $\frac{d^2(|O'B'|)}{d\sigma^2} \leq 0$. As a result, when $0 \leq \sigma \leq 1$,

$$\frac{d(|O'B'|)}{d\sigma} \geq \frac{d(|O'B'|)}{d\sigma} \Big|_{\{\sigma=1\}} = -\frac{|O^*A^*|}{2} - \frac{|A^*D^*| \sin(|A^*O^*|)}{\sin(|A^*D^*|)} \sin\left(\frac{\pi}{3}\right) \cot(\gamma).$$

Here, in the last equality, we used (6.7) and the fact that O^* is the Fermat's point. \square

Lemma 6.5. *Let f be the function as given in (6.1). When $k > 0$, then $f(\sigma) \leq f(1)$ whenever $\sigma \in [0, 1]$.*

Proof. Without losing generality, we may assume that $k = 1$. So for any $\sigma \leq 1$, by (6.3),

$$\begin{aligned} f'(\sigma) &= \frac{d(|A'O'| + |B'O'| + |C'O'|)}{d\sigma} \\ &\geq |A^*O^*| - \frac{|A^*O^*|}{2} - \frac{|A^*D^*| \sin(|A^*O^*|)}{\sin(|A^*D^*|)} \sin\left(\frac{\pi}{3}\right) \cot(\gamma) \\ &\quad - \frac{|A^*O^*|}{2} - \frac{|A^*D^*| \sin(|A^*O^*|)}{\sin(|A^*D^*|)} \sin\left(\frac{\pi}{3}\right) \cot(\pi - \gamma) = 0. \end{aligned}$$

Thus, $f(\sigma) \leq f(1)$. \square

Now, we consider the case of $k < 0$.

Lemma 6.6. *Using the same notations as in the Lemma 6.4 except that the unit sphere \mathbb{S}^2 is replaced by the hyperbolic plane $(\mathbb{H}^2, |\cdot|_h) = (M_{-1}^2, |\cdot|_{-1})$. If*

$$(6.9) \quad \max\{|A^*B^*|_h, |A^*C^*|_h, |B^*C^*|_h\} \leq 1.3877,$$

then for all $\sigma \in [0, 1]$,

$$(6.10) \quad \frac{d(|O'B'|_h)}{d\sigma} \geq -\frac{|O^*A^*|_h}{2} - \frac{|A^*D^*|_h \sinh(|A^*O^*|_h)}{\sinh(|A^*D^*|_h)} \sin\left(\frac{\pi}{3}\right) \cot(\gamma),$$

where $\gamma = \angle B'D'O'$.

Proof. In the hyperbolic case, analogous statements as in the proof of Lemma 6.4 still give

(6.11)

$$\cosh(|O'B'|_h) = \cosh(|A^*B^*|_h) \frac{\sinh(|O^*D^*|_h\sigma)}{\sinh(|A^*D^*|_h\sigma)} + \cosh(|B^*D^*|_h) \frac{\sinh(|A^*O^*|_h\sigma)}{\sinh(|A^*D^*|_h\sigma)},$$

and for all $\sigma \in [0, 1]$,

$$(6.12) \quad \frac{d(|O'B'|_h)}{d\sigma} = -|O^*A^*|_h \cos(\alpha) - \frac{|A^*D^*|_h \sinh(|A^*O^*|_h\sigma)}{\sinh(|A^*D^*|_h\sigma)} \sin(\alpha) \cot(\gamma).$$

To prove (6.10), it is sufficient to show that for all $\sigma \in [0, 1]$, $\frac{d^2(|O'B'|_h)}{d\sigma^2} \leq 0$. The proof of this fact requires some further works.

Let $p(\sigma) = \cosh(|O'B'|_h)$. Then,

$$(6.13) \quad \frac{d^2(|O'B'|_h)}{d\sigma^2} = \frac{p''(\sigma)}{\sqrt{p(\sigma)^2 - 1}} - \frac{(p'(\sigma))^2 p(\sigma)}{(\sqrt{p(\sigma)^2 - 1})^3}.$$

Together with equation (6.11), Lemma 6.7 below implies that $p''(\sigma) \leq 0$ under the condition (6.9). Since $p(\sigma) > 1$, it follows that $\frac{d^2(|O'B'|_h)}{d\sigma^2} \leq 0$. \square

Lemma 6.7. *Suppose $0 < a \leq b \leq 1.3877$, then for all $x \in [0, 1]$, it holds that*

$$\frac{d^2}{dx^2} \left(\frac{\sinh(ax)}{\sinh(bx)} \right) \leq 0.$$

Proof. Direct calculations give that

$$\frac{d^2}{dx^2} \left(\frac{\sinh(ax)}{\sinh(bx)} \right) = \frac{\sinh(ax)}{\sinh(bx)} g(x),$$

where

$$\begin{aligned} g(x) &= (b \coth(bx) - a \coth(ax))^2 + \frac{b^2}{\sinh^2(bx)} - \frac{a^2}{\sinh^2(ax)} \\ &= 2b \coth(bx)(b \coth(bx) - a \coth(ax)) + a^2 - b^2. \end{aligned}$$

Note that on the interval $(-\pi, \pi)$, the function $\coth(x)$ can be expressed as its Taylor series

$$\coth(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots,$$

where B_{2n} is the $2n$ -th Bernoulli number given by

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} (1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \dots).$$

Thus, when $0 \leq ax \leq bx < \pi$,

$$\begin{aligned} 0 &\leq b \coth(bx) - a \coth(ax) \\ &= b \left(\frac{1}{bx} + \frac{bx}{3} - \frac{b^3 x^3}{45} + \frac{2b^5 x^5}{945} - \dots \right) - a \left(\frac{1}{ax} + \frac{ax}{3} - \frac{a^3 x^3}{45} + \frac{2a^5 x^5}{945} - \dots \right) \\ &= \frac{x}{3} (b^2 - a^2) - \frac{x^3}{45} (b^4 - a^4) + \frac{2x^5}{945} (b^6 - a^6) - \dots \\ &\leq \frac{x}{3} (b^2 - a^2) - \frac{x^3}{45} (b^4 - a^4) + \frac{2x^5}{945} (b^6 - a^6). \end{aligned}$$

Also,

$$0 \leq b \coth(bx) = b\left(\frac{1}{bx} + \frac{bx}{3} - \frac{b^3x^3}{45} + \frac{2b^5x^5}{945} - \dots\right) \leq b\left(\frac{1}{bx} + \frac{bx}{3}\right) = \frac{1}{x} + \frac{b^2x}{3}.$$

As a result,

$$\begin{aligned} g(x) &= 2b \coth(bx)(b \coth(bx) - a \coth(ax)) + a^2 - b^2 \\ &\leq 2\left(\frac{1}{x} + \frac{b^2x}{3}\right)\left[\frac{x}{3}(b^2 - a^2) - \frac{x^3}{45}(b^4 - a^4) + \frac{2x^5}{945}(b^6 - a^6)\right] + a^2 - b^2 \\ &= \frac{a^2 - b^2}{3}\left[1 + \frac{2a^2 - 8b^2}{15}x^2 + \frac{12b^4 + 12a^2b^2 - 2a^4}{315}x^4 - \frac{4b^2(a^4 + b^4 + a^2b^2)}{945}x^6\right] \\ &\leq \frac{a^2 - b^2}{3}\left[1 + \frac{0 - 8b^2}{15}x^2 + \frac{12b^4 + 0 - 2b^4}{315}x^4 - \frac{4b^2(b^4 + b^4 + b^4)}{945}x^6\right] \\ &= \frac{a^2 - b^2}{945}[315 - 168(bx)^2 + 10(bx)^4 - 4(bx)^6] \leq 0, \end{aligned}$$

whenever $0 < a \leq b$ and $(bx)^2 \leq 1.9257$. Thus, when $0 < b \leq \sqrt{1.9257} = 1.3877$, it holds that $g(x) \leq 0$ for all $x \in [0, 1]$ and for all $0 < a \leq b$. \square

Lemma 6.8. *Let f be the function given in (6.1). When $k < 0$, then $f(\sigma) \leq f(1)$ whenever $\sigma \in [0, 1]$.*

Proof. Follows from an analogous proof of Lemma 6.5, via equation (6.10). \square

REFERENCES

1. M. R. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, Berlin Heidelberg, 1999.
2. D. Burago, Y. Burago, and S. Ivanov, A Course in Metric Geometry, in: Graduate Studies in Math., vol. 33, Amer. Math. Soc., Providence, RI, 2001.
3. R. Forman, Bochner's method for cell complexes and combinatorial Ricci curvature, Discrete Comput. Geom. 29:3 (2003), 323–374.
4. J. M. Lee, Riemannian Manifolds: An Introduction to Curvature. Graduate Texts in Mathematics, Vol. 176. (1997)
5. E. Saucan, Curvature – Smooth, Piecewise-Linear and Metric, book chapter in What is Geometry?, Advanced Studies in Mathematics and Logic, 237-268, Polimetrica, Milano, 2006.
6. J. M. Sullivan, Curvatures of Smooth and Discrete Surfaces. Discrete Differential Geometry, Oberwolfach Seminars 38, Birkhauser, 2008.
7. A. Wald, Begründung einer koordinatenlosen Differentialgeometrie der Flächen, Ergebnisse eines mathematischen Kolloquiums, 7, 24–46 (1935).

UNIVERSITY OF CALIFORNIA AT DAVIS, DEPARTMENT OF MATHEMATICS, DAVIS, CA, 95616

Email address: qlxia@math.ucdavis.edu

URL: <http://math.ucdavis.edu/~qlxia>