

ON THE TRANSPORT DIMENSION OF MEASURES*

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Abstract. In this article, we define the transport dimension of probability measures on \mathbb{R}^m using ramified optimal transportation theory. We show that the transport dimension of a probability measure is bounded above by the Minkowski dimension and below by the Hausdorff dimension of the measure. Moreover, we introduce a metric, called “the dimensional distance,” on the space of probability measures on \mathbb{R}^m . This metric gives a geometric meaning to the transport dimension: with respect to this metric, we show that the transport dimension of a probability measure equals the distance from it to any finite atomic probability measure.

Key words. optimal transport path, ramified optimal transportation, fractal dimension of measures, transport dimension, irrigation dimension

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1. Introduction. The theory of ramified optimal transportation aims at finding an optimal transport path between two given probability measures. One of the measures is representing the source, while the other is representing the target. A transport path is typically in the form of a tree-shaped branching structure. A natural question is, Given a probability measure, is it possible to transport it to a Dirac mass via a finite cost transport path? The answer to this question crucially depends on the dimensional information of the given measure.

In [8], Devillanova and Solimini studied the irrigability dimension of measures using optimal transportation theory. For any given probability measure μ on \mathbb{R}^m , the irrigability dimension of μ is defined by

$$\dim_I(\mu) := \inf_{0 \leq \alpha < 1} \left\{ \frac{1}{1 - \alpha} : \mu \text{ is } \alpha\text{-irrigable} \right\}.$$

The main theorem in [8] shows that

$$(1.1) \quad \max \{ \dim_H(\mu), 1 \} \leq \dim_I(\mu) \leq \max \{ 1, \dim_M(\mu) \},$$

where $\dim_H(\mu)$ ($\dim_M(\mu)$, respectively) denotes the infimum of the Hausdorff dimension (Minkowski dimension, respectively) of sets that μ is concentrated on. By definition, the irrigability dimension $\dim_I(\mu)$ of a measure μ must be larger or equal to 1 as the parameter α is in the range of $[0, 1)$.

In this article, we aim at removing the maximum constraint from (1.1) by using a different approach of ramified optimal transportation (i.e., using optimal transport paths) and also allowing the parameter α to be negative. This generalization allows us to consider measures which have fractal dimensions (e.g., the Cantor measure) less than 1. We introduce an analogous concept called “the transport dimension of μ ,” and show in Theorem 3.5.2 that

$$\dim_H(\mu) \leq \dim_T(\mu) \leq \dim_M(\mu)$$

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with a slight modification of the definition of $\dim_M(\mu)$. The major difference between $\dim_I(\mu)$ and $\dim_T(\mu)$ is that $\dim_T(\mu)$ is allowed to be less than 1. For instance, for the Cantor measure, we show that $\dim_T(\mu)$ is $\frac{\ln 2}{\ln 3}$ which is exactly the dimension of the Cantor set.

Moreover, we find a geometric meaning to the transport dimension by introducing a new metric, called *the dimensional distance*, on the space of equivalent classes of probability measures. With respect to this metric, the distance of any probability measure to a Dirac mass (or any atomic probability measure) equals the distance to the transport dimension of the measure. In other words, the transport dimension of a measure quantitatively describes how far the measure is from being an atomic measure.

We refer to [3] as well as references therein for a modern account of optimal transportation with branching structures. In particular, a partial list of most relevant works is listed in the references: [1], [2], [4], [6], [7], [9], [10], [11], [12], [14], [15], [16], [18], [19], and of course [8].

Organization of the paper. We first recall some basic concepts about optimal transport paths in section 2 with some necessary modifications. We show that for any $\alpha \in (-\infty, 1)$, there exists a metric d_α on the space $\mathcal{A}(\mathbb{R}^m)$ of atomic probability measures. Then, we consider the metric completion $\mathcal{P}_\alpha(\mathbb{R}^m)$ of $\mathcal{A}(\mathbb{R}^m)$ with respect to each d_α . We show that when a probability measure is concentrated on a set whose box dimension is less than $\frac{1}{1-\alpha}$, then the measure belongs to \mathcal{P}_α . After that, we begin to consider the transport dimension of measures in section 3 with some comparison with other dimensions of measures. In particular, we show that the transport dimension of a measure is bounded above by its Minkowski dimension and bounded below by its Hausdorff dimension. We also show the transport dimension of the Cantor measure is $\frac{\ln 2}{\ln 3}$. In section 4, we consider the dimensional distance between probability measures. The main result there says that the transport dimension of a probability measure is given by the distance of the measure to any atomic probability measure with respect to the dimensional distance.

2. The d_α metric on probability measures for $\alpha \in (-\infty, 1)$.

2.1. Transport paths between atomic measures. We first recall some basic concepts about transport paths between measures of equal total mass as studied in [11], with some necessary modifications.

Recall that a (finite) atomic measure on \mathbb{R}^m is in the form of

$$(2.1) \quad \mathbf{a} = \sum_{i=1}^k m_i \delta_{x_i}$$

with distinct points $x_i \in \mathbb{R}^m$ and positive real numbers m_i , where δ_x denotes the Dirac mass located at the point x . The measure \mathbf{a} is a probability measure if its mass $\|\mathbf{a}\| := \sum_{i=1}^k m_i = 1$. Let $\mathcal{A}(\mathbb{R}^m)$ be the space of all atomic probability measures on \mathbb{R}^m .

DEFINITION 2.1.1. *Given two atomic measures*

$$(2.2) \quad \mathbf{a} = \sum_{i=1}^k m_i \delta_{x_i} \quad \text{and} \quad \mathbf{b} = \sum_{j=1}^{\ell} n_j \delta_{y_j}$$

in \mathbb{R}^m of equal mass, a transport path from \mathbf{a} to \mathbf{b} is a weighted directed graph G consisting of a vertex set $V(G)$, a directed edge set $E(G)$, and a weight function

$w : E(G) \rightarrow (0, +\infty)$ such that $\{x_1, x_2, \dots, x_k\} \cup \{y_1, y_2, \dots, y_\ell\} \subset V(G)$ and for any vertex $v \in V(G)$,

(2.3)

$$\sum_{e \in E(G), e^- = v} w(e) = \sum_{e \in E(G), e^+ = v} w(e) + \begin{cases} m_i & \text{if } v = x_i \text{ for some } i = 1, \dots, k, \\ -n_j & \text{if } v = y_j \text{ for some } j = 1, \dots, \ell, \\ 0, & \text{otherwise,} \end{cases}$$

where e^- and e^+ denote the starting and ending endpoints of each edge $e \in E(G)$.

Note that the balanced equation (2.3) simply means the conservation of mass at each vertex. In terms of polyhedral chains, we simply have $\partial G = b - a$.

For any two atomic measures \mathbf{a} and \mathbf{b} on \mathbb{R}^m of equal total mass, let

$$Path(\mathbf{a}, \mathbf{b})$$

be the space of all transport paths from \mathbf{a} to \mathbf{b} . It is easy to see that $Path(\mathbf{a}, \mathbf{b})$ is always nonempty.

DEFINITION 2.1.2. For any $\alpha \leq 1$ and any transport path $G \in Path(\mathbf{a}, \mathbf{b})$, we define

$$M_\alpha(G) := \sum_{e \in E(G)} w(e)^\alpha \text{ length}(e).$$

We now consider the following ramified optimal transport problem.

Problem 1. Given two atomic measures \mathbf{a} and \mathbf{b} of equal mass on \mathbb{R}^m and $-\infty < \alpha < 1$, find a minimizer of

$$M_\alpha(G)$$

among all transport paths $G \in Path(\mathbf{a}, \mathbf{b})$.

An M_α minimizer in $Path(\mathbf{a}, \mathbf{b})$ is called an α -optimal transport path from \mathbf{a} to \mathbf{b} .

Note that in [11, Definition 2.2], we only allow that $0 \leq \alpha \leq 1$. But now, for the purpose of studying fractal dimension of measures in this article, we also allow negative α . Negative α also corresponds to many transport problems in reality. For instance, the “cost” (i.e., risk here) for one person to pass through a dangerous region may be higher than the cost for two or many persons. Negative α is used for such phenomenon. Another application of ramified optimal transportation with negative α may be found in [17].

DEFINITION 2.1.3. For any $\alpha \leq 1$, we define

$$d_\alpha(\mathbf{a}, \mathbf{b}) = \inf \{M_\alpha(G) : G \in Path(\mathbf{a}, \mathbf{b})\}$$

for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}(\mathbb{R}^m)$.

Remark 2.1.4. If $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are two atomic measures of equal total mass Λ , then we set

$$d_\alpha(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) = \Lambda^\alpha d_\alpha(\mathbf{a}, \mathbf{b}),$$

where $\mathbf{a} = \frac{1}{\Lambda} \tilde{\mathbf{a}}$ and $\mathbf{b} = \frac{1}{\Lambda} \tilde{\mathbf{b}}$ are the normalization of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$.

2.2. Behavior of transport paths for a negative α . The behavior of an α -optimal transport path is quite different for a negative α and a nonnegative α . For instance, in [11, Proposition 2.1], we showed that for nonnegative α , an α -optimal transport path contains no cycles. On the other hand, from the following Example 2.2.1, an α -optimal transport path may prefer to have a cycle when α is negative.

Example 2.2.1. Let $\mathbf{a} = \delta_0$, $\mathbf{b} = 0.8\delta_0 + 0.2\delta_1 \in \mathcal{A}(\mathbb{R})$; then we may construct two transport paths from \mathbf{a} to \mathbf{b} as in Figure 1. One contains a cycle, while the other does not.

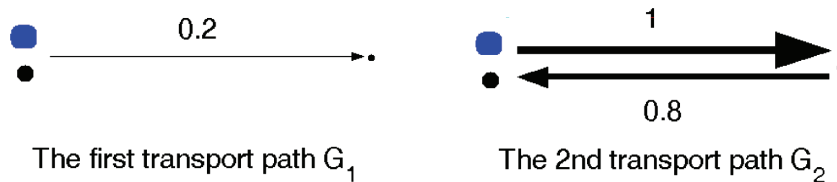


FIG. 1. For $\alpha < 0$, an optimal transport path may contain a cycle.

The first transport path G_1 consists of only one directed edge from 0 to 1 with a weight 0.2. The second transport path G_2 consists of two directed edges: one from 0 to 1 with a weight 1 and the other from 1 to 0 with a weight 0.8. Then, when $\alpha < -\frac{1}{2}$, we have

$$\frac{M_\alpha(G_2)}{M_\alpha(G_1)} = \frac{1^\alpha + 0.8^\alpha}{0.2^\alpha} \leq \frac{0.8^\alpha + 0.8^\alpha}{0.2^\alpha} = 2^{1+2\alpha} < 1.$$

Thus, $M_\alpha(G_2) < M_\alpha(G_1)$; i.e., a path containing a cycle may have less M_α cost.

A transport path may also contain another type of cycles, which are unpleasant for our study.

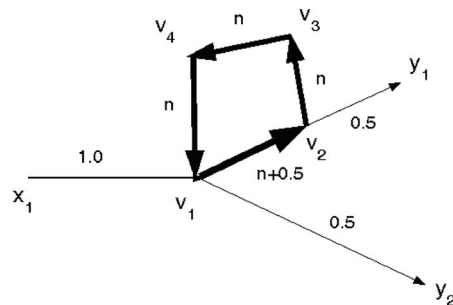


FIG. 2. An unpleasant cycle appeared in a transport path.

Example 2.2.2. From Definition 2.1.1, it is possible for a transport path to contain another type of directed cycles which are similar to the one appearing in Figure 2. Using this directed cycle, the original weight “0.5” of the edge in the example may either be repeatedly counted or be combined with some mass from unknown sources (which may have nothing to do with \mathbf{a} or \mathbf{b}) and lead to a new very high weight “ $n + 0.5$ ” in the end. When α is negative and n is large, $(n+0.5)^\alpha$ may become a very small positive number, much less than 0.5^α . Thus, for negative α , one may significantly decrease the M_α cost of a transport path by repeatedly adding a directed cycle of this type with a very high weighing constant n . Since the value n^α approaches zero as

n approaches infinity when $\alpha < 0$, allowing such kind of directed cycles in the family $\text{Path}(\mathbf{a}, \mathbf{b})$ may eventually lead to an unpleasant result: $d_\alpha(\mathbf{a}, \mathbf{b}) = 0$ for $\mathbf{a} \neq \mathbf{b}$.

The cycle in Example 2.2.1 is desirable, while the cycle in Example 2.2.2 is not what we want. To overcome this conflict of interest, we adopt the following convention on the definition of transport paths when α is negative.

CONVENTION 2.2.3. *For any vertex $v \in V(G)$ of a transport path G , if there exists a list of vertices $\{v_1, v_2, \dots, v_n\}$ such that $v_1 = v_n = v$ and $[v_i, v_{i+1}]$ is a directed edge in $E(G)$ with a positive edge length for each $i = 1, 2, \dots, n - 1$, then we will view v_n as a different copy of the point v . In other words, v_n and v are two different vertices in $V(G)$, and thus, the balance equation (2.3) must be separately held at each of them.*

Using this convention, one cannot “combine” the weights of edges from the vertex v_n with weights of edges from the different vertex v_1 . So the weight of an edge cannot be repeatedly counted, leading to a very large number, and must be bounded above by the total mass of \mathbf{a} . As a result, the convention implies two consequences: A transport path will no longer contain a directed cycle in the type of Example 2.2.2. Moreover, we have a universal upper bound (i.e., the total mass of the source) on the weight of each edge

$$(2.4) \quad w(e) \leq \|\mathbf{a}\|$$

for each edge e of a transport path G from \mathbf{a} to \mathbf{b} . In particular, if \mathbf{a} is a probability measure, then $w(e) \leq 1$.

For nonnegative α , since an α -optimal transport path contains no cycles (as shown in [11, Proposition 2.1]), there is no need to adopt this convention.

Many properties about transport paths have been studied in [11], [12], [13], [14], [16], and [3] when $0 \leq \alpha < 1$. For instance, we showed in [11, Theorem 5.1] that $(\mathcal{A}(\mathbb{R}^m), d_\alpha)$ is a geodesic space, and it is indeed a metric induced by a quasi metric (see [16]). As the main purpose of this article is studying dimensions of measures, we will leave the study of properties of optimal transport paths in the situation of $\alpha < 0$ to a later article. Currently, we only need to show that d_α is still a metric on $\mathcal{A}(\mathbb{R}^m)$ when $\alpha < 0$.

2.3. The d_α metric when $\alpha < 0$. We denote $S(p, r)$ ($\bar{B}(p, r)$, respectively) to be the sphere (the closed ball, respectively) centered at $p \in \mathbb{R}^m$ of radius $r > 0$. Note that for any transport path G , the restriction of G on any closed ball $\bar{B}(p, r_0)$ gives another transport path $G|_{\bar{B}(p, r_0)}$.

LEMMA 2.3.1. *Suppose \mathbf{a} and \mathbf{b} are two atomic measures on \mathbb{R}^m of equal total mass, and G is a transport path from \mathbf{a} to \mathbf{b} . If the intersection of $G \cap S(p, r)$ as sets is nonempty for almost all $r \in [0, r_0]$ for some $r_0 > 0$, then for any $\alpha < 0$, we have*

$$\mathbf{M}_\alpha(G|_{\bar{B}(p, r_0)}) \geq \Lambda^\alpha r_0,$$

where Λ is any upper bound of the weights of edges in $G|_{\bar{B}(p, r_0)}$.

Proof. For any $r > 0$, let

$$\mathbf{E}_r := \{e \in E(G) : e \cap S(p, r) \neq \emptyset\}$$

be the family of all edges of G that intersects with the sphere $S(p, r)$. By assumption, \mathbf{E}_r is nonempty for almost all $r \in [0, r_0]$.

Let L be any fixed ray with endpoint p , and let $P : \mathbb{R}^m \rightarrow L$ be the projection that maps any point $x \in \mathbb{R}^m$ to the point $P(x) \in L$ with $|P(x) - p| = |x - p|$. Note

that for any edge $e \in E(G)$, under the projection P , the length of the segment $P(e)$ is no more than the length of e . Therefore,

$$\begin{aligned} \mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) &= \sum_{e \in E(G|_{B(p,r_0)})} [w(e)]^\alpha \text{length}(e) \\ &\geq \sum_{e \in E(G|_{B(p,r_0)})} [w(e)]^\alpha \text{length}(P(e)) \\ &= \int_0^{r_0} \sum_{e \in E(G|_{B(p,r_0)})} [w(e)]^\alpha \chi_{P(e)}(r) dr \\ &= \int_0^{r_0} \sum_{e \in \mathbf{E}_r} [w(e)]^\alpha dr, \end{aligned}$$

where in the second equality χ_A denotes the characteristic function on a set A . So, we have shown that

$$(2.5) \quad \mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) \geq \int_0^{r_0} \sum_{e \in \mathbf{E}_r} [w(e)]^\alpha dr.$$

Now, since $w(e) \leq \Lambda$ for any $e \in \mathbf{E}_r \neq \emptyset$ and $\alpha < 0$, (2.5) yields

$$\mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) \geq \int_0^{r_0} \max_{e \in \mathbf{E}_r} [w(e)]^\alpha dr \geq \int_0^{r_0} \Lambda^\alpha dr = \Lambda^\alpha r_0. \quad \square$$

COROLLARY 2.3.2. *Suppose $\alpha \leq 0$ and $p \in \mathbb{R}^m$. Then, for any atomic measure*

$$\mathbf{a} = \sum_{i=1}^k m_i \delta_{x_i}$$

on \mathbb{R}^m with mass $\|\mathbf{a}\| := \sum_{i=1}^k m_i > 0$ and any $G \in \text{Path}(\mathbf{a}, \|\mathbf{a}\| \delta_p)$, we have

$$\mathbf{M}_\alpha(G) \geq \|\mathbf{a}\|^\alpha |p - x_i|$$

for each $i = 1, 2, \dots, k$.

Proof. By (2.4), $w(e) \leq \|\mathbf{a}\|$ for any edge $e \in E(G)$. Then, the result follows from Lemma 2.3.1 by setting $r_0 = \max_{1 \leq i \leq k} \{|p - x_i|\}$ and $\Lambda = \|\mathbf{a}\|$. \square

The inequality (2.5) also gives a lower bound estimate for positive α .

COROLLARY 2.3.3. *Suppose $0 \leq \alpha < 1$. For any $\mathbf{a} \in \mathcal{A}(\mathbb{R}^m)$ in the form of (2.1), $p \in \mathbb{R}^m$, and $r_0 > 0$, we have*

$$(2.6) \quad \left[\sum_{d(p,x_i) > r_0} m_i \right]^\alpha \leq \frac{d_\alpha(\mathbf{a}, \delta_p)}{r_0}.$$

Proof. Let $\lambda = \sum_{d(p,x_i) > r_0} m_i$. Let G be any transport path from \mathbf{a} to δ_p . Then, for any $0 < r \leq r_0$, we have

$$\sum_{e \in \mathbf{E}_r} w(e) \geq \lambda.$$

By the inequality (2.5), since the function $f(x) = x^\alpha$ is concave on $[0, 1]$ when $0 \leq \alpha < 1$, we have

$$\begin{aligned} \mathbf{M}_\alpha(G|_{\bar{B}(p,r_0)}) &\geq \int_0^{r_0} \sum_{e \in \mathbf{E}_r} [w(e)]^\alpha dr \\ &\geq \int_0^{r_0} \left[\sum_{e \in \mathbf{E}_r} w(e) \right]^\alpha dr \geq \int_0^{r_0} \lambda^\alpha dr = \lambda^\alpha r_0. \end{aligned}$$

Therefore, we have (2.6). \square

PROPOSITION 2.3.4. d_α is a metric on $\mathcal{A}(\mathbb{R}^m)$.

Proof. Obviously, we only need to consider the case $\alpha < 0$. In this case, it is clear that d_α is nonnegative and symmetric. Now, for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{A}(\mathbb{R}^m)$ and any path $G_1 \in \text{Path}(\mathbf{a}, \mathbf{b})$ and $G_2 \in \text{Path}(\mathbf{b}, \mathbf{c})$, let G_3 be the disjoint union of the directed weighted graphs G_1 and G_2 . That is,

$$V(G_3) = (V(G_1) \setminus V(\mathbf{b})) \amalg (V(G_2) \setminus V(\mathbf{b})) \amalg V(\mathbf{b})$$

and $E(G_3) = E(G_1) \amalg E(G_2)$, where the symbol \amalg denotes the disjoint union of sets and $V(\mathbf{b})$ denotes the vertex set (i.e., the support) of the measure \mathbf{b} . Then, by using Convention 2.2.3, G_3 is a transport path from \mathbf{a} to \mathbf{c} with

$$\mathbf{M}_\alpha(G_3) = \mathbf{M}_\alpha(G_1) + \mathbf{M}_\alpha(G_2).$$

Thus, by taking infimum, we have the triangle inequality

$$d_\alpha(\mathbf{a}, \mathbf{c}) \leq d_\alpha(\mathbf{a}, \mathbf{b}) + d_\alpha(\mathbf{b}, \mathbf{c}).$$

Now, we need only to check that if $\mathbf{a} \neq \mathbf{b}$, then $d_\alpha(\mathbf{a}, \mathbf{b}) > 0$. We may assume that \mathbf{a} and \mathbf{b} are in the forms of (2.2). If the supports of \mathbf{a} and \mathbf{b} are different, i.e., $\{y_1, y_2, \dots, y_l\} \neq \{x_1, x_2, \dots, x_k\}$ as sets, we may assume that $y_1 \notin \{x_1, x_2, \dots, x_k\}$. In this case, we set $p = y_1$ and

$$r_0 := \min\{|p - x_i| : i = 1, 2, \dots, k\} > 0.$$

If $\{x_1, x_2, \dots, x_k\} = \{y_1, y_2, \dots, y_l\}$ as sets, then we may assume

$$\mathbf{a} = \sum_{i=1}^k m_i \delta_{x_i} \text{ and } \mathbf{b} = \sum_{i=1}^k n_i \delta_{x_i},$$

and $m_1 \neq n_1$. In this case, we set $p = x_1$ and let

$$r_0 := \min\{|p - x_i| : m_i \neq n_i, i = 2, \dots, k\} > 0.$$

In either of these two cases, for any transport path G from \mathbf{a} to \mathbf{b} , the intersection of G with the sphere $S(p, r)$ is nonempty for any $0 < r < r_0$. By Lemma 2.3.1,

$$\mathbf{M}_\alpha(G) \geq r_0$$

since $w(e) \leq 1$ for any edge $e \in E(G)$. Therefore,

$$d_\alpha(\mathbf{a}, \mathbf{b}) \geq r_0 > 0. \quad \square$$

2.3.1. The completion of $\mathcal{A}(\mathbb{R}^m)$ with respect to d_α .

DEFINITION 2.3.5. For any $\alpha \in (-\infty, 1]$, let $\mathcal{P}_\alpha(\mathbb{R}^m)$ be the completion of the metric space $\mathcal{A}(\mathbb{R}^m)$ with respect to the metric d_α .

We will simply write $\mathcal{P}_\alpha(\mathbb{R}^m)$ as \mathcal{P}_α .

Note that when $\alpha = 1$, the metric d_1 is the usual Monge’s distance on $\mathcal{A}(\mathbb{R}^m)$ and \mathcal{P}_1 is the space \mathcal{P} of all probability measures on \mathbb{R}^m . Now, the following lemma implies that each element in \mathcal{P}_α can be viewed as a probability measure on \mathbb{R}^m .

LEMMA 2.3.6. If $\beta < \alpha$, then $\mathcal{P}_\beta(\mathbb{R}^m) \subseteq \mathcal{P}_\alpha(\mathbb{R}^m)$, and for all μ, ν in $\mathcal{P}_\beta(\mathbb{R}^m)$, we have $d_\beta(\mu, \nu) \geq d_\alpha(\mu, \nu)$.

Proof. Note that for any path G between any two probability measures \mathbf{a} and \mathbf{b} in $\mathcal{A}(\mathbb{R}^m)$, we have

$$\mathbf{M}_\alpha(G) = \sum_{e \in E(G)} w(e)^\alpha \text{length}(e) \leq \sum_{e \in E(G)} w(e)^\beta \text{length}(e) = \mathbf{M}_\beta(G).$$

This is because $w(e) \leq 1$ for any $e \in E(G)$. Hence,

$$\begin{aligned} d_\alpha(\mathbf{a}, \mathbf{b}) &= \inf\{\mathbf{M}_\alpha(G) : G \in \text{Path}(\mathbf{a}, \mathbf{b})\} \\ &\leq \inf\{\mathbf{M}_\beta(G) : G \in \text{Path}(\mathbf{a}, \mathbf{b})\} = d_\beta(\mathbf{a}, \mathbf{b}). \end{aligned}$$

Therefore, any Cauchy sequence $\{\mathbf{a}_n\}$ in $\mathcal{A}(\mathbb{R}^m)$ with respect to the metric d_β is also a Cauchy sequence with respect to d_α . Thus, $\mathcal{P}_\beta(\mathbb{R}^m) \subseteq \mathcal{P}_\alpha(\mathbb{R}^m)$ with

$$d_\alpha(\mu, \nu) \leq d_\beta(\mu, \nu)$$

for any μ, ν in $\mathcal{P}_\beta(\mathbb{R}^m)$. □

As discussed in [8], the Minkowski–Bouligand dimension, also known as box-counting dimension, of a set A is given by

$$(2.7) \quad \dim_{\text{box}}(A) = \inf\{\beta > 0 : A \text{ can be covered by } C_\beta \delta^{-\beta} \text{ balls of radius } \delta \text{ for all } \delta \text{ less than } 1 \text{ and some constant } C_\beta\}.$$

A positive Borel measure μ on \mathbb{R}^m is said to be *concentrated* on a Borel set A if $\mu(\mathbb{R}^m \setminus A) = 0$.

THEOREM 2.3.7. Suppose μ is a probability measure concentrated on a subset A of \mathbb{R}^m with $\dim_{\text{box}}(A) < \frac{1}{1-\alpha}$ for some $0 \leq \alpha < 1$; then $\mu \in \mathcal{P}_\alpha$.

Proof. Since $\dim_{\text{box}}(A) < \frac{1}{1-\alpha}$, we may fix a constant β such that $\dim_{\text{box}}(A) < \beta < \frac{1}{1-\alpha}$. By (2.7), for every $n \in \mathbb{N}$, the set A can be covered by balls $\{B(x_i^{(n)}, \frac{1}{2^n})\}_{i=1}^{N_n}$ of radius $\frac{1}{2^n}$ centering at $x_i^{(n)}$, where the number N_n of balls of radius $\frac{1}{2^n}$ is bounded above by $C_\beta (\frac{1}{2^n})^{-\beta}$ for some constant C_β . Using Vitali’s covering theorem (see, for instance, [5, section 1.5.1]), we may assume that the balls $\{B(x_i^{(n)}, \frac{1}{5 \cdot 2^n})\}_{i=1}^{N_n}$ are disjoint. For each ball $B(x_i^{(n)}, \frac{1}{2^n})$, let

$$\Phi_i^{(n)} = \left\{ x_j^{(n-1)} : B\left(x_i^{(n)}, \frac{1}{2^n}\right) \cap B\left(x_j^{(n-1)}, \frac{1}{2^{n-1}}\right) \neq \emptyset \right\}$$

be the set of centers of all balls in the preceding cover that intersects with $B(x_i^{(n)}, \frac{1}{2^n})$. Then, one can easily check that the cardinality $k_i^{(n)}$ of $\Phi_i^{(n)}$ is less than $(\frac{17}{2})^m$.

Using this family of covers, we construct a Cauchy sequence $\{\mathbf{a}_n\}$ for μ as follows. For each n , let

$$\mathbf{a}_n = \sum_{i=1}^{N_n} m_i^{(n)} \delta_{x_i^{(n)}},$$

where the positive number $m_i^{(n)} = \mu(B(x_i^{(n)}, \frac{1}{2^n}) \setminus \bigcup_{h=1}^{i-1} B(x_h^{(n)}, \frac{1}{2^n}))$. Then, for each $n > 1$, we may construct a partition of $\mathbf{a}_n = \sum_{j=1}^{N_{n-1}} \mathbf{a}_{n,j}^{(n-1)}$ with respect to \mathbf{a}_{n-1} by setting

$$\mathbf{a}_{n,j}^{(n-1)} = \sum_{i=1}^{N_n} m_{i,j}^{(n)} \delta_{x_i^{(n)}},$$

where the number

$$m_{i,j}^{(n)} =: \mu \left(\left(B \left(x_i^{(n)}, \frac{1}{2^n} \right) \setminus \bigcup_{h=1}^{i-1} B \left(x_h^{(n)}, \frac{1}{2^n} \right) \right) \cap B \left(x_j^{(n-1)}, \frac{1}{2^{n-1}} \right) \right)$$

if $x_j^{(n-1)} \in \Phi_i^{(n)}$, and we set $m_{i,j}^{(n)}$ to be zero otherwise. Also, let $G_{n,i}^{n-1}$ be the path sending a mass $m_{i,j}^{(n)}$ from the center $x_i^{(n)}$ to every point $x_j^{(n-1)} \in \Phi_i^{(n)}$. Then,

$$\begin{aligned} d_\alpha(\mathbf{a}_{n-1}, \mathbf{a}_n) &\leq \sum_{i=1}^{N_n} \mathbf{M}_\alpha(G_{n,i}^{n-1}) \\ &\leq \sum_{i=1}^{N_n} \sum_{j=1}^{N_{n-1}} (m_{i,j}^{(n)})^\alpha |x_i^{(n)} - x_j^{(n-1)}| \\ &\leq \sum_{i=1}^{N_n} k_i^{(n)} (m_i^{(n)})^\alpha \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} \right). \end{aligned}$$

Now, by the Hölder inequality, we have

$$\begin{aligned} &d_\alpha(\mathbf{a}_{n-1}, \mathbf{a}_n) \\ &\leq \left(\frac{17}{2} \right)^m \left(\frac{3}{2^n} \right) \left(\sum_{i=1}^{N_n} m_i^{(n)} \right)^\alpha \left(\sum_{i=1}^{N_n} 1 \right)^{1-\alpha} \\ &\leq \left(\frac{17}{2} \right)^m \left(\frac{3}{2^n} \right) (N_n)^{1-\alpha}, \text{ because } \sum_{i=1}^{N_n} m_i^{(n)} = 1 \\ &\leq \frac{3}{2^n} \cdot \left(\frac{17}{2} \right)^m \left(C_\beta \left(\frac{1}{2^n} \right)^{-\beta} \right)^{1-\alpha} = 3 \cdot \left(\frac{17}{2} \right)^m (C_\beta)^{1-\alpha} b^n, \end{aligned}$$

where $b = (\frac{1}{2})^{1-\beta(1-\alpha)} \in (0, 1)$, because $\beta < \frac{1}{1-\alpha}$. As a result, $\{\mathbf{a}_n\}$ is a Cauchy sequence representing μ with respect to the metric d_α ; therefore, $\mu \in \mathcal{P}_\alpha$. \square

COROLLARY 2.3.8. *Suppose μ is a probability measure on \mathbb{R}^m with a compact support. Then, $\mu \in \mathcal{P}_\alpha$ for any $\alpha \in (1 - \frac{1}{m}, 1)$.*

3. Dimension of measures. In this section, we will study properties of measures that belong to a special subset of \mathcal{P}_α and then define the transport dimension of measures.

3.1. d_α -admissible Cauchy sequence.

DEFINITION 3.1.1. Let $\{\mathbf{a}_k\}_{k=1}^\infty$ be a sequence of atomic measures of equal total mass in the form of

$$\mathbf{a}_k = \sum_{i=1}^{N_k} m_i^{(k)} \delta_{x_i^{(k)}}$$

for each k . We say that this sequence is a d_α -admissible Cauchy sequence if for any $\epsilon > 0$, there exists an N such that, for all $n > k \geq N$, there exists a partition of

$$\mathbf{a}_n = \sum_{i=1}^{N_k} \mathbf{a}_{n,i}^{(k)}$$

with respect to \mathbf{a}_k as sums of disjoint atomic measures and a path (see Figure 3)

$$G_{n,i}^k \in \text{Path}(m_i^{(k)} \delta_{x_i^{(k)}}, \mathbf{a}_{n,i}^{(k)})$$

for each $i = 1, 2, \dots, N_k$ such that

$$\sum_{i=1}^{N_k} \mathbf{M}_\alpha(G_{n,i}^k) \leq \epsilon.$$

Also, we denote $G_n^k = \sum_{i=1}^{N_k} G_{n,i}^k$, which is a path from \mathbf{a}_k to \mathbf{a}_n with $\mathbf{M}_\alpha(G_n^k) \leq \epsilon$. Each d_α -admissible Cauchy sequence corresponds to an element in $\mathcal{P}_\alpha(\mathbb{R}^m)$. Let

$$\mathcal{D}_\alpha(\mathbb{R}^m) \subset \mathcal{P}_\alpha(\mathbb{R}^m)$$

be the set of all probability measures μ which corresponds to a d_α -admissible Cauchy sequence of probability measures. For simplicity, we may write $\mathcal{D}_\alpha(\mathbb{R}^m)$ as \mathcal{D}_α .

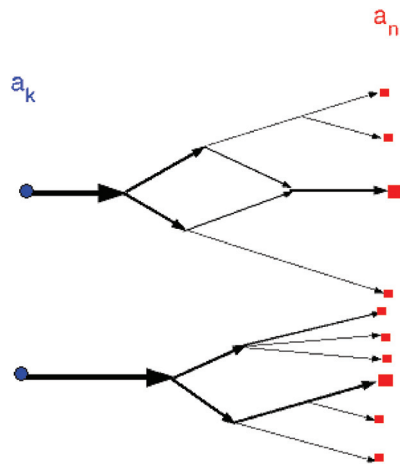


FIG. 3. An example of a transport path between \mathbf{a}_k and \mathbf{a}_n .

It is easy to see that if for each k there is a partition of $\mathbf{a}_{k+1} = \sum_{i=1}^{N_k} \mathbf{a}_{k+1,i}^{(k)}$ with respect to \mathbf{a}_k as sums of disjoint atomic measures and a path $G_{k+1,i}^k \in \text{Path}(m_i^{(k)} \delta_{x_i^{(k)}}, \mathbf{a}_{k+1,i}^{(k)})$ for each $i = 1, 2, \dots, N_k$ such that

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^{N_k} M_{\alpha}(G_{k+1,i}^k) \right) < +\infty,$$

then $\{\mathbf{a}_n\}$ is a d_{α} -admissible Cauchy sequence.

Also, note that if $\mu, \nu \in \mathcal{D}_{\alpha}(\mathbb{R}^m)$, one automatically has $d_{\alpha}(\mu, \nu) < \infty$.

Before discussing properties of \mathcal{D}_{α} , we give some examples of elements of \mathcal{D}_{α} as follows. The first example is a strengthening of Corollary 2.3.8.

Example 3.1.2. Let μ be any probability measure supported on a compact subset of \mathbb{R}^m . Then, $\mu \in \mathcal{D}_{\alpha}$ whenever $\alpha > 1 - \frac{1}{m}$.

Let B be a cube in \mathbb{R}^m of side length l that contains the support of μ . For each n , by using dyadic decomposition of B , we get a family of smaller cubes $\{B_i^{(n)}\}_{i=1}^{N_n}$ of generation n centered at $x_i^{(n)}$ and of side length $\frac{l}{2^n}$, where $N_n = 2^{mn}$. Then, set

$$\mathbf{a}_n = \sum_{i=1}^{N_n} \mu(B_i^{(n)}) \delta_{x_i^{(n)}}.$$

For each n , by setting

$$\mathbf{a}_{n+1,i}^{(n)} = \sum_{x_j^{(n+1)} \in B_i^{(n)}} \mu(B_j^{(n+1)}) \delta_{x_j^{(n+1)}}$$

for each $i = 1, 2, \dots, N_n$, we get a partition of $\mathbf{a}_{n+1} = \sum_{i=1}^{N_{n+1}} \mathbf{a}_{n+1,i}^{(n)}$ with respect to \mathbf{a}_n . Also, by transporting the corresponding mass $\mu(B_j^{(n+1)})$ from $x_j^{(n+1)}$ to each $x_i^{(n)}$, we build an obvious path $G_{n+1,i}^n \in \text{Path}(\mu(B_i^{(n)}) \delta_{x_i^{(n)}}, \mathbf{a}_{n+1,i}^{(n)})$ (see Figure 4).

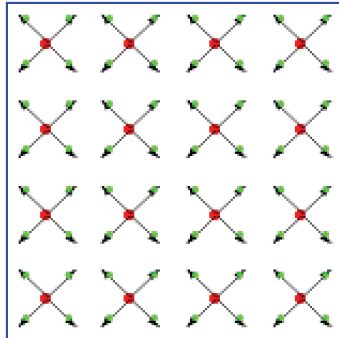


FIG. 4.

Now, it is easy (see [11, Proposition 3.1], for instance) to check that

$$\sum_{k=1}^{\infty} \left(\sum_{i=1}^{N_k} M_{\alpha}(G_{n+1,i}^n) \right) < +\infty$$

whenever $\alpha > 1 - \frac{1}{m}$, and thus, $\mu \in \mathcal{D}_{\alpha}$.

Example 3.1.3 (Cantor measure). Let

$$\mathbf{a}_n = \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{x_{n,i}},$$

where $x_{n,i}$'s are centers of intervals of the i th Cantor interval of length $\frac{1}{3^n}$. By transporting the mass $\frac{1}{2^n}$ to each $x_{n,i}$ from the center of the interval in the previous step, we construct a path G_n^{n-1} from \mathbf{a}_{n-1} to \mathbf{a}_n (see Figure 5).



FIG. 5.

Now,

$$\mathbf{M}_\alpha(G_n^{n-1}) = \sum_{i=1}^{2^n} \left(\frac{1}{2^n}\right)^\alpha \left(\frac{1}{3}\right)^n = \left(\frac{2^{1-\alpha}}{3}\right)^n.$$

Therefore, $\{\mathbf{a}_n\}$ forms a d_α -admissible Cauchy sequence whenever $\frac{2^{1-\alpha}}{3} < 1$, that is, whenever $\frac{1}{1-\alpha} > \frac{\ln 2}{\ln 3}$, which is exactly the fractal dimension of the Cantor set. The measure represented by this Cauchy sequence $\{\mathbf{a}_n\}$ is called the *Cantor measure*. It is the usual $\mathcal{H}^s \llcorner C$, where C is the Cantor set and $s = \frac{\ln 2}{\ln 3}$ is its Hausdorff dimension. This shows that the Cantor measure is in \mathcal{D}_α whenever $\frac{1}{1-\alpha} > \frac{\ln 2}{\ln 3}$. Note that α is allowed to be negative here.

Example 3.1.4 (fat Cantor measure). A fat Cantor set is constructed in the same way as constructing a Cantor set except that an interval of length λ is removed from the middle of $[0, 1]$ for $\lambda \in (0, 1)$. Again, we set

$$\mathbf{a}_n = \frac{1}{2^n} \sum_{i=1}^{2^n} \delta_{x_{n,i}},$$

where $x_{n,i}$'s are centers of intervals of the i th fat Cantor interval of length $\left(\frac{1-\lambda}{2}\right)^n$. By transporting the mass $\frac{1}{2^n}$ to each $x_{n,i}$ from the center of the interval in the previous step, we construct a path G_n^{n-1} from \mathbf{a}_{n-1} to \mathbf{a}_n . Then,

$$\mathbf{M}_\alpha(G_n^{n-1}) = \sum_{i=1}^{2^n} \left(\frac{1}{2^n}\right)^\alpha \frac{1+\lambda}{4} \left(\frac{1-\lambda}{2}\right)^{n-1} = \frac{1+\lambda}{2(1-\lambda)} (2^{1-\alpha} p)^n,$$

where $p = \frac{1-\lambda}{2}$. The sequence $\{\mathbf{a}_n\}$ forms a d_α -admissible Cauchy sequence whenever $2^{1-\alpha} p < 1$, that is, whenever

$$\frac{1}{1-\alpha} > -\frac{\ln 2}{\ln p} = \frac{\ln 2}{\ln 2 - \ln(1-\lambda)},$$

which is the fractal dimension of the fat Cantor set. The measure represented by this Cauchy sequence $\{\mathbf{a}_n\}$ is called the *fat Cantor measure*. This shows that the fat Cantor measure is in \mathcal{D}_α whenever $\frac{1}{1-\alpha} > -\frac{\ln 2}{\ln p}$.

Example 3.1.5 (self-similar measures). Let A be any bounded self-similar set in the sense that A is the finite union of sets A_i for $i = 1, \dots, k$ with each A_i being a σ -rescale of A . Pick any point $x^* \in A$ as the center of A . Then, each copy $A_i^{(n)}$ of A of generation n has a corresponding center $x_i^{(n)}$. Now, we set

$$\mathbf{a}_n = \frac{1}{k^n} \sum_{i=1}^{k^n} \delta_{x_i^{(n)}}.$$

By transporting the mass $\frac{1}{k^n}$ to each $x_i^{(n)}$ from the center $x_j^{(n-1)}$ of the set $A_j^{(n-1)}$ in the previous step, we construct a path G_n^{n-1} from \mathbf{a}_{n-1} to \mathbf{a}_n . Note that $|x_i^{(n)} - x_j^{(n-1)}| \lesssim \sigma^{n-1}L$, where L is the diameter of the set A . Therefore,

$$\mathbf{M}_\alpha(G_n^{n-1}) \approx k^n \left(\frac{1}{k^n}\right)^\alpha \sigma^{n-1}L = \frac{L}{\sigma} (k^{1-\alpha}\sigma)^n.$$

Thus,

$$\begin{aligned} \sum_{k=1}^\infty \mathbf{M}_\alpha(G_n^{n-1}) \text{ is finite} \\ \iff k^{1-\alpha}\sigma < 1 \iff \frac{1}{1-\alpha} > -\frac{\ln k}{\ln \sigma}. \end{aligned}$$

Therefore, the sequence $\{\mathbf{a}_n\}$ forms a d_α -admissible Cauchy sequence whenever

$$\frac{1}{1-\alpha} > -\frac{\ln k}{\ln \sigma},$$

which is the fractal dimension of the self-similar set A . The measure represented by this Cauchy sequence $\{\mathbf{a}_n\}$ is called a *self-similar measure*. This shows that a self-similar measure is in \mathcal{D}_α whenever $\frac{1}{1-\alpha} > -\frac{\ln k}{\ln \sigma}$.

3.2. Hausdorff dimension of measures. Let \mathcal{H}^s denote an s -dimensional Hausdorff measure on \mathbb{R}^m for each $s \geq 0$.

THEOREM 3.2.1. *If $\mu \in \mathcal{D}_\alpha$ for some $\alpha < 1$, then μ is concentrated on a subset A of \mathbb{R}^m with $\mathcal{H}^{\frac{1}{1-\alpha}}(A) = 0$.*

Proof. Since $\mu \in \mathcal{D}_\alpha$, it is represented by a d_α -admissible Cauchy sequence $\{\mathbf{a}_k\}$ in the form of

$$\mathbf{a}_k = \sum_{i=1}^{N_k} m_i^{(k)} \delta_{x_i^{(k)}}.$$

By Definition 3.1.1 and taking a subsequence of $\{\mathbf{a}_k\}$ if necessary, we have that, for any k and for all $n > k$, there exists a partition of

$$\mathbf{a}_n = \sum_{i=1}^{N_k} \mathbf{a}_{n,i}^{(k)}$$

with respect to \mathbf{a}_k and a path $G_{n,i}^k \in \text{Path}(m_i^{(k)} \delta_{x_i^{(k)}}, \mathbf{a}_{n,i}^{(k)})$ for each $i = 1, 2, \dots, N_k$ such that

$$\sum_{i=1}^{N_k} d_\alpha \left(\mathbf{a}_{n,i}^{(k)}, m_i^{(k)} \delta_{x_i^{(k)}} \right) \leq \sum_{i=1}^{N_k} \mathbf{M}_\alpha(G_{n,i}^k) \leq \frac{1}{2^k}.$$

Now, for each fixed k and any $i = 1, 2, \dots, N_k$, the sequence $\{\mathbf{a}_{n,i}^{(k)}\}_{n=k+1}^\infty$ is also a d_α -admissible Cauchy sequence representing a positive Radon measure $\mu_i^{(k)}$ of mass $m_i^{(k)}$. As a result, for each fixed k , the measure μ can be decomposed as

$$\mu = \sum_{i=1}^{N_k} \mu_i^{(k)}$$

such that each $\mu_i^{(k)}$ has mass $m_i^{(k)}$ for each i and

$$\sum_{i=1}^{N_k} d_\alpha \left(\mu_i^{(k)}, m_i^{(k)} \delta_{x_i^{(k)}} \right) \leq \frac{1}{2^k}.$$

Case 1. $0 < \alpha < 1$. Note that from (2.6), we also have the following estimate: for any $\nu \in \mathcal{D}_\alpha$ and $0 \leq \alpha < 1$

$$\nu(\mathbb{R}^m \setminus B(x, r))^\alpha r \leq d_\alpha(\nu, \delta_x) \text{ for any } x \in \mathbb{R}^m \text{ and any } r > 0.$$

So, if $r \geq d_\alpha(\nu, \delta_x)^{1-\alpha}$, then

$$(3.1) \quad \nu(\mathbb{R}^m \setminus B(x, r)) \leq d_\alpha(\nu, \delta_x).$$

Now, let $r_i^{(k)} = d_\alpha(\mu_i^{(k)}, m_i^{(k)} \delta_{x_i^{(k)}})^{1-\alpha}$ and $A_k = \bigcup_{i=1}^{N_k} B(x_i^{(k)}, r_i^{(k)})$. Then,

$$\mathcal{H}^{\frac{1}{1-\alpha}}(A_k) \leq \alpha(k) \sum_{i=1}^{N_k} \left(r_i^{(k)} \right)^{\frac{1}{1-\alpha}} = \alpha(k) \sum_{i=1}^{N_k} d_\alpha \left(\mu_i^{(k)}, m_i^{(k)} \delta_{x_i^{(k)}} \right) \leq \frac{\alpha(k)}{2^k},$$

where $\alpha(k)$ is the constant

$$(3.2) \quad \alpha(k) = \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2} + 1\right)}.$$

Moreover,

$$\begin{aligned} \mu(\mathbb{R}^m \setminus A_k) &\leq \sum_{i=1}^{N_k} \mu_i^{(k)} \left(\mathbb{R}^m \setminus B\left(x_i^{(k)}, r_i^{(k)}\right) \right) \\ &\stackrel{\text{by (3.1)}}{\leq} \sum_{i=1}^{N_k} d_\alpha \left(\mu_i^{(k)}, m_i^{(k)} \delta_{x_i^{(k)}} \right) \leq \frac{1}{2^k}. \end{aligned}$$

We set $A := \bigcup_h \left(\bigcap_{k>h} A_k \right)$. Then, for each h ,

$$\begin{aligned} \mu(\mathbb{R}^m \setminus A) &\leq \mu \left(\mathbb{R}^m \setminus \left(\bigcap_{k>h} A_k \right) \right) = \mu \left(\bigcup_{k>h} (\mathbb{R}^m \setminus A_k) \right) \\ &\leq \sum_{k>h} \mu(\mathbb{R}^m \setminus A_k) \leq \sum_{k>h} \frac{1}{2^k} = \frac{1}{2^h}. \end{aligned}$$

This implies that $\mu(\mathbb{R}^m \setminus A) = 0$. On the other hand, since

$$\mathcal{H}^{\frac{1}{1-\alpha}} \left(\bigcap_{k>h} A_k \right) \leq \mathcal{H}^{\frac{1}{1-\alpha}}(A_k) \leq \frac{\alpha(k)}{2^k} \rightarrow 0,$$

we have $\mathcal{H}^{\frac{1}{1-\alpha}} \left(\bigcap_{k>h} A_k \right) = 0$ for all $h \in \mathbb{N}$ and thus $\mathcal{H}^{\frac{1}{1-\alpha}}(A) = 0$. Thus, μ is concentrated on a $\frac{1}{1-\alpha}$ -negligible set A .

Case 2. $\alpha \leq 0$. We shall denote

$$r_i^{(k)} = \frac{d_\alpha \left(m_i^{(k)} \delta_{x_i^{(k)}}, \mu_i^{(k)} \right)}{\left(m_i^{(k)} \right)^\alpha}$$

and $A_k = \bigcup_{i=1}^{N_k} B(x_i^{(k)}, r_i^{(k)})$.

Suppose the atomic measure $\mathbf{a}_{n,i}^{(k)}$ is expressed as

$$\mathbf{a}_{n,i}^{(k)} = \sum_{j=p_i}^{q_i} m_j^{(n)} \delta_{x_j^{(n)}}.$$

Then, by Corollary 2.3.2, for every $x_j^{(n)}$, $j = p_i, \dots, q_i$, we have

$$d_\alpha \left(m_i^{(k)} \delta_{x_i^{(k)}}, \mathbf{a}_{n,i}^{(k)} \right) \geq \left(m_i^{(k)} \right)^\alpha \left| x_i^{(k)} - x_j^{(n)} \right|.$$

Therefore, $|x_i^{(k)} - x_j^{(n)}| \leq r_i^{(k)}$, so $x_j^{(n)} \in B(x_i^{(k)}, r_i^{(k)})$. Thus, $\mu(\mathbb{R}^m \setminus A_k) = 0$.

Moreover,

$$\begin{aligned} \mathcal{H}^{\frac{1}{1-\alpha}}(A_k) &\leq \sum_{i=1}^{N_k} \left(r_i^{(k)} \right)^{\frac{1}{1-\alpha}} = \sum_{i=1}^{N_k} \left(d_\alpha \left(m_i^{(k)} \delta_{x_i^{(k)}}, \mu_i^{(k)} \right) \right)^{\frac{1}{1-\alpha}} \left(m_i^{(k)} \right)^{\frac{-\alpha}{1-\alpha}} \\ &\leq \left(\sum_{i=1}^{N_k} \left(d_\alpha \left(m_i^{(k)} \delta_{x_i^{(k)}}, \mu_i^{(k)} \right) \right)^{\frac{1}{1-\alpha} \cdot (1-\alpha)} \right)^{\frac{1}{1-\alpha}} \left(\sum_{i=1}^{N_k} \left(m_i^{(k)} \right)^{\frac{-\alpha}{1-\alpha} \cdot \frac{1-\alpha}{1-\alpha}} \right)^{\frac{-\alpha}{1-\alpha}} \\ &= d_\alpha(\mu, \mathbf{a}_k)^{\frac{1}{1-\alpha}}. \end{aligned}$$

Then, for $A = \bigcap A_k$, we have

$$\mu(\mathbb{R}^m \setminus A) = \mu \left(\bigcup_k (\mathbb{R}^m \setminus A_k) \right) \leq \sum_k \mu(\mathbb{R}^m \setminus A_k) = 0.$$

Moreover, we have $\mathcal{H}^{\frac{1}{1-\alpha}}(A) \leq \mathcal{H}^{\frac{1}{1-\alpha}}(A_k) \leq d_\alpha(\mu, \mathbf{a}_k)^{\frac{1}{1-\alpha}} \rightarrow 0$ as $k \rightarrow \infty$. Thus, μ is concentrated on a $\frac{1}{1-\alpha}$ -negligible set A . \square

DEFINITION 3.2.2. For any probability measure μ on \mathbb{R}^m , the Hausdorff dimension of μ is defined to be

$$\dim_H(\mu) = \inf \{ \dim_H(A) : \mu(\mathbb{R}^m \setminus A) = 0 \},$$

where $\dim_H(A)$ is the Hausdorff dimension of a set A .

Thus, by Theorem 3.2.1, we have the following.

COROLLARY 3.2.3. For any $\alpha < 1$ and any $\mu \in \mathcal{D}_\alpha(\mathbb{R}^m)$, we have

$$\dim_H(\mu) \leq \frac{1}{1-\alpha}.$$

LEMMA 3.2.4. Let μ be the Cantor measure as defined in Example 3.1.3. Then, $\dim_H(\mu) = \frac{\ln 2}{\ln 3}$.

Proof. Since μ is clearly concentrated on the Cantor set whose Hausdorff dimension is $\frac{\ln 2}{\ln 3}$, we have $\dim_H(\mu) \leq \frac{\ln 2}{\ln 3}$. Therefore, to show $\dim_H(\mu) = \frac{\ln 2}{\ln 3}$, all we need to show is

$$\dim_H(\mu) \geq \frac{\ln 2}{\ln 3} := s.$$

For this, it is sufficient to show that

$$H^s(A) \geq 1$$

whenever μ is concentrated on a set A . Let $\{C_i\}$ be any collection of sets that covers A . We want to show that

$$(3.3) \quad \sum_i \alpha(s) \left(\frac{\text{diam}(C_i)}{2} \right)^s \geq 1,$$

where $\alpha(s)$ is the constant as given in (3.2). Without loss of generality, we may assume that C_i 's are closed intervals. Also, if $C_i \cap C_j \neq \emptyset$ for some $i \neq j$, then

$$(\text{diam}(C_i \cup C_j))^s \leq (\text{diam}(C_i) + \text{diam}(C_j))^s \leq (\text{diam}(C_i))^s + (\text{diam}(C_j))^s$$

as the function x^s is concave on $[0, 1]$. Therefore, to prove the inequality (3.3), we may replace C_i and C_j by $C_i \cup C_j$. Thus, without loss of generality, we may assume that $\{C_i\}$'s are disjoint closed intervals $C_i = [a_i, b_i]$ with the order

$$0 \leq a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n < \dots \leq 1.$$

Note that for each i ,

$$\mu((b_i, a_{i+1})) = 0$$

as $(b_i, a_{i+1}) \cap A = \emptyset$ and μ is concentrated on A . Therefore, $\mathbf{a}_n((b_i, a_{i+1})) = 0$ when n is large enough. This implies that the Cantor set is disjoint with (b_i, a_{i+1}) . Thus, the Cantor set is also covered by $\{C_i\}$. As a result, $H^s(\cup C_i) \geq H^s(\text{Cantor set}) = 1$. This shows the inequality (3.3), and hence, $\dim_H(\mu) = s$ as desired. \square

3.3. Minkowski dimension of measures. A nested collection

$$\mathcal{F} = \{Q_i^n : i = 1, 2, \dots, N_n \text{ and } n = 1, 2, \dots\}$$

is a collection of Borel subsets of \mathbb{R}^m with the following properties:

1. For each Q_i^n , its diameter

$$(3.4) \quad C_1 \sigma^n \leq \text{diam}(Q_i^n) \leq C_2 \sigma^n$$

for some constants $C_2 \geq C_1 > 0$ and some $\sigma \in (0, 1)$.

2. For any k, l, i, j with $l \leq k$, either $Q_i^k \cap Q_j^l = \emptyset$ or $Q_i^k \subseteq Q_j^l$.
 3. For each Q_j^{n+1} , there exists exactly one Q_i^n (parent of Q_j^{n+1}) such that $Q_j^{n+1} \subseteq Q_i^n$.
 4. For each Q_i^n , there exists at least one Q_j^{n+1} (child of Q_i^n) such that $Q_j^{n+1} \subseteq Q_i^n$.
- Each Q_i^n is called a cube of generation n in \mathcal{F} . If two different cubes Q_i^n and Q_j^n of generation n have the same parent, then they are called *brothers* to each other.

A typical example of a nested collection includes collections of standard cubes in \mathbb{R}^m . That is, let Q be a cube in \mathbb{R}^m of side length L , and k is a fixed natural number. Then, we evenly split Q into k^m cubes of side length $\frac{L}{k}$. Pick some (or all) of these cubes of generation 1 to form a collection \mathcal{Q}_1 . We may then evenly split each of the cubes of generation 1 in \mathcal{Q}_1 into cubes of side length $\frac{L}{k^2}$ to get cubes of generation 2. Pick at least one cube of generation 2 from each cube in \mathcal{Q}_1 ; we get a collection \mathcal{Q}_2 consisting of cubes of generation 2. Then, we may continue this process for each n to get a collection \mathcal{Q}_n consisting of some cubes of side length $\frac{L}{k^n}$. The union of all \mathcal{Q}_n is clearly a nested collection.

DEFINITION 3.3.1. For any nested collection \mathcal{F} , we define its Minkowski dimension

$$(3.5) \quad \dim_M(\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\log(N_n)}{\log\left(\frac{1}{\sigma^n}\right)},$$

provided the limit exists, where N_n is the total number of cubes of generation n .

By (3.4), we have

$$\dim_M(\mathcal{F}) \leq m$$

for any nested collection \mathcal{F} consisting of Borel subsets of \mathbb{R}^m .

DEFINITION 3.3.2. A Radon measure μ is said to be concentrated on a nested collection \mathcal{F} if for each n ,

$$\mu\left(\mathbb{R}^m \setminus \left(\bigcup_{i=1}^{N_n} Q_i^n\right)\right) = 0.$$

Now, we inductively define the centers for cubes in \mathcal{F} as follows. For any cube Q_i^n in \mathcal{F} , if it has more than one child, then one may pick any point p in Q_i^n and call it the center of Q_i^n . If Q_i^n has only one child, then we pick the center of the child as the center of Q_i^n . In case we have an infinite sequence of cubes $\{Q_i^n\}$ such that each cube is the only child of the previous one, then the intersection of the closures of these cubes contains at least one point, and we will call this point the center of each cube in the sequence. It is easy to see that we have defined a center for each cube in \mathcal{F} by this process.

Now, for each Q_i^n in \mathcal{F} , let

$$l(Q_i^n)$$

be the distance from the center of Q_i^n to the center of its parent Q_j^{n-1} . Then, by definition of centers and (3.4), we have

$$(3.6) \quad l(Q_i^n) = \begin{cases} 0 & \text{if } Q_i^n \text{ has no brothers,} \\ \leq C_2 \sigma^{n-1} & \text{otherwise.} \end{cases}$$

We now state our key lemma as follows.

LEMMA 3.3.3. Suppose μ is a probability measure concentrated on a nested collection \mathcal{F} with

$$\dim_M(\mathcal{F}) < \frac{1}{1 - \alpha}$$

for some $\alpha < 1$. If for each n ,

$$(3.7) \quad \sum_{i=1}^{N_n} \mu(Q_i^n)^\alpha l(Q_i^n) \leq C(N_n)^{1-\alpha} \sigma^n$$

for some constant C , then $\mu \in \mathcal{D}_\alpha$.

Proof. Let

$$\beta := \dim_M(\mathcal{F}) < \frac{1}{1-\alpha}.$$

Then, by (3.5), when n is large enough, we have

$$N_n \leq \sigma^{-n\beta}.$$

Now, we construct a d_α -admissible Cauchy sequence $\{\mathbf{a}_n\}$ for μ as follows. For each n , let

$$\mathbf{a}_n = \sum_{i=1}^{N_n} \mu(Q_i^n) \delta_{x_i^{(n)}},$$

where $x_i^{(n)}$ is the center of Q_i^n and $m_i^{(n)} = \mu(Q_i^n)$ denotes the total mass of μ on Q_i^n . Also, we can construct a partition of \mathbf{a}_n with respect to \mathbf{a}_{n-1} by grouping together masses that are located in centers of children of the same cube. That is, for each $j = 1, 2, \dots, N_{n-1}$, let

$$\mathbf{a}_{n-1,j}^{(n)} = \sum \mu(Q_i^n) \delta_{x_i^{(n)}},$$

where the summation is over all i 's such that Q_i^n is a child of Q_j^{n-1} . For each Q_i^n , we may transport the mass $m_i^{(n)}$ from the center of its parent to its center and thus construct a path G_n^{n-1} from \mathbf{a}_{n-1} to \mathbf{a}_n . Then, by (3.7),

$$\begin{aligned} \mathbf{M}_\alpha(G_n^{n-1}) &\leq \sum_{i=1}^{N_n} (\mu(Q_i^n))^\alpha l(Q_i^n) \\ &\leq C(N_n)^{1-\alpha} \sigma^n \\ &\leq C(\sigma^{-n\beta})^{1-\alpha} \sigma^n \text{ when } n \text{ is large enough} \\ &= Cb^n \end{aligned}$$

for $b = \sigma^{1-\beta(1-\alpha)} \in (0, 1)$. As a result, $\{\mathbf{a}_n\}$ is a d_α -admissible Cauchy sequence representing μ , and thus, $\mu \in \mathcal{D}_\alpha$. \square

DEFINITION 3.3.4. For any Radon measure μ , we define the Minkowski dimension of the measure μ to be

$$\dim_M(\mu) := \inf \{ \dim_M(\mathcal{F}) \},$$

where the infimum is over all nested collections \mathcal{F} that μ is concentrated on.

THEOREM 3.3.5. If μ is a probability measure with $\dim_M(\mu) < \frac{1}{1-\alpha}$ for some $0 \leq \alpha < 1$, then $\mu \in \mathcal{D}_\alpha$.

Proof. Since $\dim_M(\mu) < \frac{1}{1-\alpha}$, μ is concentrated on a nested collection \mathcal{F} with

$$\dim_M(\mathcal{F}) < \frac{1}{1-\alpha}.$$

When $\alpha \geq 0$, by the Hölder inequality and (3.6), we have

$$\begin{aligned} & \sum_{i=1}^{N_n} \mu(Q_i^n)^\alpha l(Q_i^n) \\ & \leq \left(\sum_{i=1}^{N_n} \mu(Q_i^n) \right)^\alpha \left(\sum_{i=1}^{N_n} 1 \right)^{1-\alpha} C_2 \sigma^{n-1} \\ & = C_2 N_n^{1-\alpha} \sigma^{n-1}, \text{ because } \sum_{i=1}^{N_n} \mu(Q_i^n) = 1. \end{aligned}$$

Thus, by Lemma 3.3.3, we have $\mu \in \mathcal{D}_\alpha$. \square

3.4. Evenly concentrated measures. Now, we aim at achieving a similar result as in Theorem 3.3.5 for the case $\alpha < 0$. To do this, we introduce the following definition.

DEFINITION 3.4.1. *A Radon measure μ is evenly concentrated on a nested collection \mathcal{F} if, for each cube Q_i^n of generation n in \mathcal{F} , either Q_i^n has no brothers or $\mu(Q_i^n) \geq \frac{\lambda}{N_n}$ for some constant $\lambda > 0$.*

Here, “ Q_i^n has no brothers” means that the parent of Q_i^n has only one child, namely, Q_i^n itself.

Now, we provide some examples of a measure μ that is evenly concentrated on some nested collection.

Example 3.4.2. Suppose k is a fixed natural number and h is a natural number no more than k^m . Let B_0 be a fixed box in \mathbb{R}^m , and we set $\mathcal{B}_1 = \{B_0\}$. Now, for each $n \geq 1$ and for each $B \in \mathcal{B}_n$, we split B evenly into k^m smaller boxes of edge length σ^n and then pick h smaller boxes from them either determinately or randomly (see Figure 6). Let \mathcal{B}_{n+1} be the collections of all smaller boxes as picked above. Thus, for

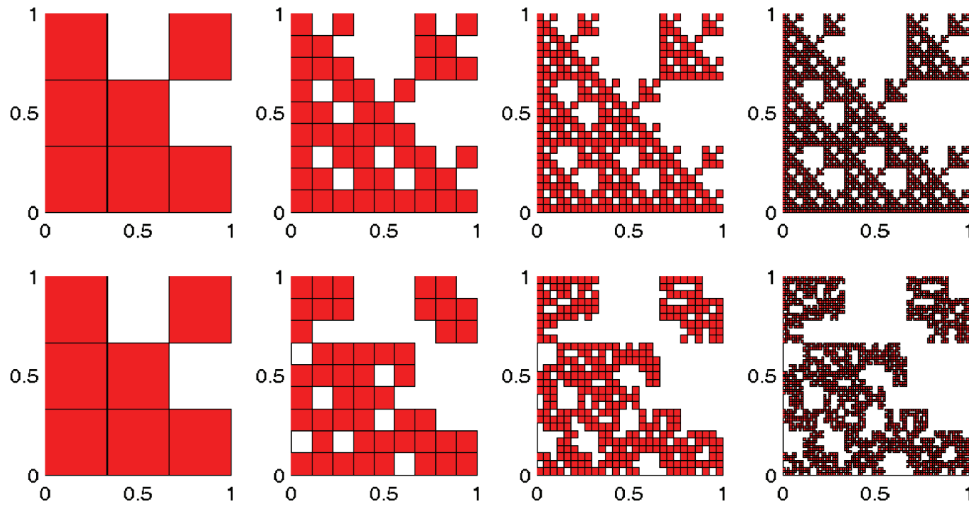


FIG. 6. Here, $k = 3, m = 2$, and $h = 7$. Boxes in the first row are chosen determinately, and boxes in the second row are picked randomly.

each n , the cardinality of \mathcal{B}_{n+1} is $N_{n+1} = hN_n = h^n$ for each n . Let

$$\mathbf{a}_n = \sum_{B \in \mathcal{B}_n} \frac{1}{N_n} \delta_{x_B},$$

where x_B is the center of the box $B \in \mathcal{B}_n$. Then, $\{\mathbf{a}_n\}$ is a sequence of probability measures. For each $B \in \mathcal{B}_n$, we transport the mass $\frac{1}{N_n}$ from the center x_{B^*} of B 's parent $B^* \in \mathcal{B}_{n-1}$ to x_B . In this way, we construct a path G_n^{n-1} from \mathbf{a}_{n-1} to \mathbf{a}_n . For each α ,

$$M_\alpha(G_n^{n-1}) \simeq \sum_{i=1}^{N_n} \left(\frac{1}{N_n}\right)^\alpha \left(\frac{1}{k}\right)^n = \frac{(N_n)^{1-\alpha}}{k^n} = \frac{1}{h^{1-\alpha}} \left(\frac{h^{1-\alpha}}{k}\right)^n.$$

Thus, $\{\mathbf{a}_n\} \in \mathcal{D}_\alpha$ if $\frac{h^{1-\alpha}}{k} < 1$, i.e., $\frac{1}{1-\alpha} > \log_k h$, which is the Minkowski dimension of the nested collection $\mathcal{B} = \bigcup_n \mathcal{B}_n$. Hence, $\{\mathbf{a}_n\}$ is a d_α -admissible Cauchy sequence representing a probability measure μ . Moreover, for each $j > 0$,

$$\mathbf{a}_{n+j}(B) = \sum_{\substack{B^* \in \mathcal{B}_{n+j} \\ B^* \subset B}} \frac{1}{N_{n+j}} \delta_{x_{B^*}} = \frac{h^j}{N_{n+j}} = \frac{1}{N_n} = \mathbf{a}_n(B).$$

Hence, μ is evenly concentrated on the nested collection \mathcal{B} with the property that $\mu(B) = \frac{1}{N_n}$ for each $B \in \mathcal{B}_n$.

Example 3.4.3. Now, we modify the previous example as follows (see Figure 7). For any $B \in \mathcal{B}_n$, suppose

$$\lambda_2 \leq \frac{\text{card}\{B^* \in \mathcal{B}_{n+j} : B^* \subset B\}}{h^j} \leq \lambda_1$$

whenever j is large enough for some constants $0 < \lambda_2 \leq \lambda_1$. Define $\{\mathbf{a}_n\}$, \mathcal{B} , and μ as before. Thus, when j is large enough,

$$\begin{aligned} N_{n+j} &= \sum_{B \in \mathcal{B}_n} \text{card}\{B^* \in \mathcal{B}_{n+j} : B^* \subset B\} \\ &\leq \sum_{B \in \mathcal{B}_n} \lambda_1 h^j = \lambda_1 N_n h^j. \end{aligned}$$

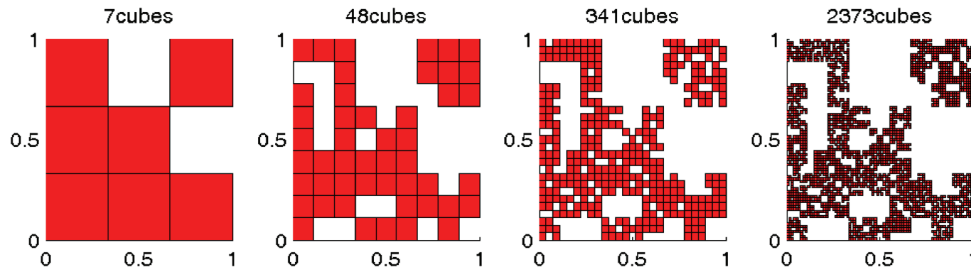


FIG. 7. Here, $k = 3, m = 2$, and $h = 7$. We divide every cube into 9 smaller cubes and then pick about 7 smaller cubes from them with some randomness. For any particular cube, if the number of cubes picked at the current stage is less than (or, respectively, greater than) the desired one (i.e., 7), more (or, respectively, less) smaller cubes inside it will be picked for the next generation.

Then,

$$\begin{aligned} \mu(B) &= \lim_j \mathbf{a}_{n+j}(B) \\ &= \lim_j \frac{\text{card}\{B^* \in \mathcal{B}_{n+j} : B^* \subset B\}}{N_{n+j}} \\ &\geq \lim_j \frac{\lambda_2 h^j}{\lambda_1 N_n h^j} = \frac{\lambda}{N_n} \quad \text{for } \lambda = \frac{\lambda_2}{\lambda_1}. \end{aligned}$$

Therefore, in this case, μ is still evenly concentrated on a nested collection \mathcal{B} .

Recall that a nonnegative Borel measure μ is *Ahlfors regular* of dimension d if there exists a constant $C > 0$ such that

$$C^{-1}r^d \leq \mu(B(x, r)) \leq Cr^d$$

whenever $0 < r \leq \text{diam}(spt(\mu))$ and x lies in the support $spt(\mu)$ of μ .

Example 3.4.4. If μ is an Ahlfors regular measure whose support is a box \tilde{B} in \mathbb{R}^m , then μ is evenly concentrated on a nested collection.

Proof. By splitting the box \tilde{B} into smaller boxes of edge length σ^n , we get a collection \mathcal{B}_n of disjoint boxes. For any $B \in \mathcal{B}_n$, we have

$$a(\sigma^n)^d \leq \mu(B) \leq A(\sigma^n)^d$$

for some d and for some suitable constants $a, A > 0$. Thus,

$$\mu(\tilde{B}) = \sum_{B \in \mathcal{B}_n} \mu(B) \leq \sum_{B \in \mathcal{B}_n} A(\sigma^n)^d \leq AN_n \sigma^{nd},$$

where N_n is the cardinality of the collection \mathcal{B}_n . Therefore, for every $B \in \mathcal{B}_n$,

$$\mu(B) \geq a\sigma^{nd} \geq a \frac{\mu(\tilde{B})}{AN_n} = \frac{\lambda}{N_n},$$

where $\lambda = \frac{a}{A}\mu(\tilde{B}) > 0$. Hence, μ is evenly concentrated on a nested collection. □

DEFINITION 3.4.5. For any Radon measure μ , we define

$$\dim_U(\mu) := \inf \{\dim_M(\mathcal{F})\},$$

where the infimum is over all nested collections \mathcal{F} that μ is evenly concentrated on.

Obviously,

$$\dim_M(\mu) \leq \dim_U(\mu).$$

Now, we have the following theorem for any $\alpha < 1$.

THEOREM 3.4.6. Suppose μ is a probability measure with $\dim_U(\mu) < \frac{1}{1-\alpha}$ for some $\alpha < 1$; then $\mu \in \mathcal{D}_\alpha$.

Proof. Since $\dim_M(\mu) \leq \dim_U(\mu)$, by Theorem 3.3.5, $\mu \in \mathcal{D}_\alpha$ whenever $0 \leq \alpha < 1$. Thus, we only need to consider the case when $\alpha < 0$. Since $\dim_U(\mu) < \frac{1}{1-\alpha}$, μ is evenly concentrated on a nested collection \mathcal{F} with

$$\dim_M(\mathcal{F}) < \frac{1}{1-\alpha}.$$

For each cube Q_i^n in \mathcal{F} , if Q_i^n has no brothers, then $l(Q_i^n) = 0$. If Q_i^n has more than one brother, then

$$\mu(Q_i^n) \geq \frac{\lambda}{N_n} \quad \text{and} \quad l(Q_i^n) \leq C_2 \sigma^{n-1}.$$

Thus, when $\alpha \leq 0$, we always have

$$\mu(Q_i^n)^\alpha l(Q_i^n) \leq \left(\frac{\lambda}{N_n}\right)^\alpha C_2 \sigma^{n-1}.$$

Therefore,

$$\sum_{i=1}^{N_n} \mu(Q_i^n)^\alpha l(Q_i^n) \leq \frac{C_2 \lambda^\alpha}{\sigma} (N_n)^{1-\alpha} \sigma^n.$$

By Lemma 3.3.3, we still have $\mu \in \mathcal{D}_\alpha$ when $\alpha \leq 0$. □

3.5. Comparison of dimensions of measures.

DEFINITION 3.5.1. For any probability measure μ , we define the transport dimension of μ to be

$$\dim_T(\mu) = \inf_{\alpha < 1} \left\{ \frac{1}{1 - \alpha} : \mu \in \mathcal{D}_\alpha(\mathbb{R}^m) \right\}.$$

THEOREM 3.5.2. Let μ be any positive probability measure on \mathbb{R}^m . Then, the following bounds hold:

$$\dim_H(\mu) \leq \dim_T(\mu) \leq \max\{\dim_M(\mu), 1\}.$$

Moreover, we also have

$$\dim_H(\mu) \leq \dim_T(\mu) \leq \dim_U(\mu).$$

Proof. Assume that $\dim_T(\mu) > \max\{\dim_M(\mu), 1\}$ ($\dim_U(\mu)$, respectively). Then, we may pick an $\alpha < 1$ so that

$$\dim_T(\mu) > \frac{1}{1 - \alpha} > \max\{\dim_M(\mu), 1\} \quad (\dim_U(\mu), \text{ respectively}).$$

By Theorem 3.3.5 (or Theorem 3.4.6), $\mu \in \mathcal{D}_\alpha(\mathbb{R}^m)$. This is a contradiction to the definition of $\dim_T(\mu)$.

On the other hand, by Theorem 3.2.1, $\dim_H(\mu) \leq \frac{1}{1-\alpha}$ whenever $\mu \in \mathcal{D}_\alpha(\mathbb{R}^m)$. Thus, $\dim_H(\mu) \leq \dim_T(\mu)$. □

Example 3.5.3. Let μ be the Cantor measure as in Example 3.1.3. Let Q_i^n denote the i th Cantor interval of length $\frac{1}{3^n}$. Then, clearly, μ is evenly concentrated on the nested collection $\mathcal{C} = \{Q_i^n : i = 1, 2, \dots, 2^n\}_{n=1}^\infty$. Note that

$$\dim_M(\mathcal{C}) = \lim_{n \rightarrow \infty} \frac{\log(N_n)}{\log(3^n)} = \lim_{n \rightarrow \infty} \frac{\log(2^n)}{\log(3^n)} = \frac{\ln 2}{\ln 3}.$$

On the other hand, by Lemma 3.2.4, $\dim_H(\mu) = \frac{\ln 2}{\ln 3}$. Therefore, by Theorem 3.5.2,

$$\dim_T(\mu) = \frac{\ln 2}{\ln 3}.$$

4. The dimensional distance between probability measures. For any $\alpha < 1$, let

$$\mathcal{S}_\alpha(\mathbb{R}^m) = \{\Lambda(\mu - \nu) : \Lambda \geq 0, \mu, \nu \in \mathcal{D}_\alpha(\mathbb{R}^m)\}$$

be a collection of signed measures. Clearly, $\mathcal{S}_{\alpha_1}(\mathbb{R}^m) \subseteq \mathcal{S}_{\alpha_2}(\mathbb{R}^m)$ if $\alpha_1 \leq \alpha_2$.

DEFINITION 4.0.1. For any two probability measures μ, ν on \mathbb{R}^m , we define

$$D(\mu, \nu) := \inf_{\alpha < 1} \left\{ \frac{1}{1 - \alpha} : \mu - \nu \in \mathcal{S}_\alpha(\mathbb{R}^m) \right\}.$$

PROPOSITION 4.0.2. $D(\mu, \nu)$ is a pseudometric¹ on the space of probability measures on \mathbb{R}^m .

Proof. It is clear that D is symmetric and nonnegative. Also, for any $\alpha < 1$, $\mu - \mu \in \mathcal{S}_\alpha$, so $D(\mu, \mu) = 0$. Now we only need to check the triangle inequality

$$(4.1) \quad D(\mu_1, \mu_2) + D(\mu_2, \mu_3) \geq D(\mu_1, \mu_3)$$

for any probability measures μ_1, μ_2 , and μ_3 on \mathbb{R}^m .

Note that (4.1) is clearly true if either $D(\mu_1, \mu_2)$ or $D(\mu_2, \mu_3)$ is infinity. Thus, we may assume that both $D(\mu_1, \mu_2)$ and $D(\mu_2, \mu_3)$ are finite. So, $\mu_1 - \mu_2 \in \mathcal{S}_{\alpha_1}$ and $\mu_2 - \mu_3 \in \mathcal{S}_{\alpha_2}$ for some $\alpha_1, \alpha_2 < 1$. Now,

$$\mu_1 - \mu_3 = (\mu_1 - \mu_2) + (\mu_2 - \mu_3) \in \mathcal{S}_{\max\{\alpha_1, \alpha_2\}}(\mathbb{R}^m).$$

Therefore,

$$D(\mu_1, \mu_3) \leq \frac{1}{1 - \max\{\alpha_1, \alpha_2\}} \leq \frac{1}{1 - \alpha_1} + \frac{1}{1 - \alpha_2}.$$

By taking an infimum over α_1 and α_2 , we obtain the triangle inequality (4.1). \square

In general, D is not necessarily a metric. Indeed, for any two atomic probability measures \mathbf{a}, \mathbf{b} , we have $\mathbf{a} - \mathbf{b} \in \mathcal{S}_\alpha(\mathbb{R}^m)$ for any $\alpha < 1$. Thus, $D(\mathbf{a}, \mathbf{b}) = 0$, while \mathbf{a} and \mathbf{b} are not necessarily the same measure. Nevertheless, we may easily extend the pseudometric D to a metric on equivalent classes of measures. To this end we define a notion of the equivalence classes on measures.

DEFINITION 4.0.3. For any two probability measures μ and ν on \mathbb{R}^m , we say

$$\mu \sim \nu \text{ if } D(\mu, \nu) = 0.$$

The equivalent class of μ is denoted by $[\mu]$.

For instance, all atomic probability measures are equivalent to each other.

PROPOSITION 4.0.4. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $D(\mu_1, \nu_1) = D(\mu_2, \nu_2)$.

Proof. By definition of equivalence of measures, we have $D(\mu_1, \mu_2) = 0$ and $D(\nu_1, \nu_2) = 0$. So by the triangle inequality (4.1), we have

$$|D(\mu_1, \nu_1) - D(\mu_2, \nu_2)| \leq D(\mu_1, \mu_2) + D(\nu_1, \nu_2) = 0. \quad \square$$

DEFINITION 4.0.5. For any equivalent class $[\mu]$ and $[\nu]$, define

$$\mathbf{D}([\mu], [\nu]) = D(\mu, \nu).$$

From this definition and Proposition 4.0.2, clearly, we have the following theorem.

THEOREM 4.0.6. \mathbf{D} is a metric on $\mathcal{P}(\mathbb{R}^m) / \sim$.

¹A pseudometric D means that it is nonnegative and symmetric and satisfies the triangle inequality, and $D(\mu, \mu) = 0$. But $D(\mu, \nu) = 0$ does not imply $\mu = \nu$.

DEFINITION 4.0.7. *The metric \mathbf{D} is called the dimensional distance on the space $\mathcal{P}(\mathbb{R}^m)/\sim$ of equivalent classes of probability measures on \mathbb{R}^m .*

We now give a geometric meaning to transport dimension of measures.

THEOREM 4.0.8. *For any positive probability measure μ , we have*

$$\dim_T(\mu) = \mathbf{D}([\mu], [\mathbf{a}]) = D(\mu, \mathbf{a}),$$

where \mathbf{a} is any atomic probability measure.

Proof. Since $\mathbf{a} \in \mathcal{D}_\alpha$ for any $\alpha < 1$, we have

$$\begin{aligned} D(\mu, \mathbf{a}) &= \inf_{\alpha < 1} \left\{ \frac{1}{1-\alpha} : \mu - \mathbf{a} \in S_\alpha \right\} \\ &= \inf_{\alpha < 1} \left\{ \frac{1}{1-\alpha} : \mu \in \mathcal{D}_\alpha \right\} = \dim_T(\mu). \quad \square \end{aligned}$$

This theorem says that the transport dimension of a probability measure μ is the distance from μ to any atomic measure with respect to the dimensional distance. In other words, the dimension information of a measure tells us quantitatively how far the measure is from being an atomic measure.

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