GENERALIZED MORAN SETS GENERATED BY STEP-WISE ADJUSTABLE ITERATED FUNCTION SYSTEMS

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ABSTRACT. In this article we provide a systematic way of creating generalized Moran sets using an analogous iterated function system (IFS) procedure. We use a step-wise adjustable IFS to introduce some variance (such as non-self-similarity) in the fractal limit sets. The process retains the computational simplicity of a standard IFS procedure. In our construction of the generalized Moran sets, we relax the second Moran Structure Condition so that the limit set is not necessarily self-similar. We also weaken the fourth Moran Structure Condition so that there are no limitations on the ratios of the diameters of the sets across a generation. Moreover, we provide upper and lower bounds for the Hausdorff dimension of the fractals created from this generalized process. Specific examples (Cantor-like sets, Sierpinski-like Triangles, etc) with the calculations of their corresponding dimensions are studied.

1. Introduction

The Moran construction is a typical way to generate self-similar fractals, and has been studied extensively in the literature (e.g. [15], [5], [7], [12],[11], [18], [8], and references therein). In this paper, we extend ideas from iterated function systems (IFS) and Moran constructions by describing a new process that allows for the functions to be updated at every iteration while still maintaining the computational simplicity of an IFS. This process provides more variance in the limit sets (such as non-self-similarity) using an analogous approach to an IFS procedure. We also give estimates of the Hausdorff dimension of the limit sets created from such a process, and provide concrete examples.

The classic construction of Moran sets was introduced in [15]. We reproduce the definition here with a more current interpretation to introduce notations.

Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers for $k \geq 1$. Here k will represent the generation, and n_k will be the number of children in generation k that each parent set from generation $k-1$ has. For any $k \in \mathbb{N}$, define

(1.1)
$$D_k = \{(i_1, i_2, \cdots, i_k) : 1 \leq i_j \leq n_k, 1 \leq j \leq k\} \text{ and } D = \bigcup_{k\geq 0} D_k \text{ with } D_0 = \emptyset.$$

Let $\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k$ and $\tau = (\tau_1, \cdots, \tau_m) \in D_m$, then denote

$$\sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m) \in D_{k+m}. \tag{1.2}$$

Using this notation, we may express

$$D_k = \{\sigma * j \mid \sigma \in D_{k-1}, 1 \leq j \leq n_k\} \tag{1.3}$$

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to emphasize the process of moving between generations.

Suppose $\mathcal{J} := \{J_\sigma : \sigma \in D\}$ is a collection of subsets of $\mathbb{R}^N$. Set

$$E_k = \bigcup_{\sigma \in D_k} J_\sigma, \text{ and } F = \bigcap_{k \geq 0} E_k. \tag{1.4}$$

We call F the limit set associated with the collection $\mathcal{J}$.

Definition 1.1 ([18])**.** *Suppose that $J \subset \mathbb{R}^N$ is a compact set with nonempty interior. Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers, and $\{\Phi_k\}_{k\geq 1}$ be a sequence of positive real vectors with*

$$\Phi_k = (c_{k,1}, c_{k,2}, \ldots, c_{k,n_k}), \sum_{1 \leq j \leq n_k} c_{k,j} \leq 1, k \in \mathbb{N}. \tag{1.5}$$

Suppose that $\mathcal{F} := \{J_\sigma : \sigma \in D\}$ is a collection of subsets of $\mathbb{R}^N$, where D is given in (1.1). We say that the collection $\mathcal{F}$ fulfills the Moran Structure provided it satisfies the following Moran Structure Conditions (MSC):

MSC(1) $J_\emptyset = J$.

MSC(2) *For any $\sigma \in D$, J_σ is geometrically similar to J. That is, there exists a similarity $S_\sigma : \mathbb{R}^N \to \mathbb{R}^N$ such that $J_\sigma = S_\sigma(J)$.*

MSC(3) *For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma\star 1}, \ldots, J_{\sigma\star n_k}$ are subsets of J_σ, and $int(J_{\sigma\star i}) \cap int(J_{\sigma\star j}) = \emptyset$ for $i \neq j$.*

MSC(4) *For any $k \geq 1$ and $\sigma \in D_{k-1}, 1 \leq j \leq n_k$,*

$$\frac{diam(J_{\sigma\star j})}{diam(J_\sigma)} = c_{k,j}. \tag{1.6}$$

For the collection $\mathcal{F}$ fulfilling the MSC, the limit set F given in (1.4) is a nonempty compact set. This limit set F is called the Moran set associated with the collection $\mathcal{F}$. This Moran set is self-similar, and has been studied extensively by many authors with various approaches (e.g. [15], [7], [10], [5], [16]).

The fact that there are four conditions to create a Moran set makes the area ripe for generalizations or restrictions. Note that in condition MSC(2), the sets in the new generation are geometrically similar, which is a rather strong condition. In MSC(3) the interiors of the next generation may not overlap, but says nothing else of the spacing between these sets. Condition MSC(4) requires that the sets in the new generation all have the same pattern of ratios for each iteration. There is even a hidden condition in MSC(2) that one may want to do away with in that the locations of the sets $J_{\sigma * j}$ are completely determined by the similarities used on J_σ.

Often times in the literature one may define a self-similar set to be a set satisfying the MSC as well as the fact that the set of similarities $\{S_\sigma\}$ has finite cardinality, the ratios described in MSC(4) do not change over generations (i.e. $c_{k,j} = c_j$), and that the system is deterministic. In this case, the dimensions (Hausdorff, Box, Packing etc.) of the Moran set are known to coincide, see [18]. However, the dimensions may differ when we begin to modify the MSC.

Several approaches have been used to relax MSC in order to create more general limit sets. There are many generalizations for MSC(2). For a self-similar set, one could change MSC(2) to use conformal maps [6] or affine maps [14] instead of similarities. In this setting, however, calculations of the dimension of limit sets can become particularly difficult. One could also study the limit sets generated by infinitely many similarities, as in [13]. In [11], the authors removed MSC(2), but required $\overline{int(J_\sigma)} = J_\sigma$ in their construction, and studied the dimension of the resulting fractals. In [8], Holland and Zhang studied a construction that replaced similarity maps in MSC(2) with a more general class of functions that are not necessarily contractions. In [17], Pesin and Weiss removed the requirement for similarities from MSC(2), but also relaxed MSC(3) from non-intersecting basic sets to non-intersecting balls contained in the basic sets. In particular they pursued sufficient conditions for which the box dimension and Hausdorff dimensions coincide. For more examples of modifications to the Moran set definition, see [18], [19] and the references therein.

A special case of Moran sets can be constructed from an iterated function system (IFS). An iterated function system $\{S_1, S_2, \cdots, S_m\}$ is a finite family of similarities for a fixed natural number $m \geq 2$ (see [10] for more details and applications). In MSC(2), define $n_k = m$ and set $S_\sigma = S_{i_k} \circ S_{i_{k-1}} \circ \cdots \circ S_{i_1}$ for $\sigma = (i_1, i_2, \cdots, i_k) \in D$. Then the resulting Moran set is self-similar and agrees with the attractor of the IFS $\{S_1, S_2, \cdots, S_m\}$. The dimension of the limit set can be quickly calculated from the Moran-Hutchinson formula in [7]. Using iterated function systems is a popular way to construct fractals, and has been used to great effect (e.g. [1], [10], [7], [3]).

A natural question arises: *Can we construct more general fractals (e.g. non-self-similar Moran type sets) using an analogous approach while preserving the computational simplicity of the IFS?* In this paper, we present a method to do so.

We first make the following observations about the general construction of a Moran set. Note that in the construction of a Moran set described in (1.4),

$$J_{\sigma * i} = S_i(J_\sigma), \text{ for all } i = 1, ...m, \text{ and } \sigma \in D. \tag{1.7}$$

Suppose that there is a tuning parameter in the expression of the function S_i (e.g. the coefficients a_i, b_i in a linear function $S_i(x) = a_i x + b_i$). One can tune the values of the parameter to get a comparable function. When J_σ is given, applying the comparable function to J_σ, as in equation (1.7), will not significantly change the computational complexity of constructing $J_{\sigma * i}$. The advantage of doing this at each iteration is that we introduce some variance into the limit set. Another observation is about which space the functions are defined. In classical IFS constructions, the functions are usually defined on all of the ambient space $\mathbb{R}^n$ (as in [8], the functions are $C^{1+\alpha}$ diffeomorphisms on $\mathbb{R}^n$). For our construction, we wish to relax the condition MSC(2) as well. Instead of restricting our attention to functions of higher regularity defined on the whole ambient space $\mathbb{R}^n$, we use maps from a collection of subsets to itself.

This article is organized as follows. In Section 2 we find bounds for the Hausdorff dimension of the limit sets in a general metric space setting of a collection of bounded sets, not necessarily satisfying the MSC conditions. In particular, we introduce the concept *uniform covering condition* in Definition 2.1 for the purpose of studying the lower bound of the Hausdorff dimension of the limit set. Then in Section 3 we formulate the general setup for the construction of Moran-type limit sets using the ideas from a modified IFS procedure, as discussed in the previous paragraph. In our construction we relax MSC(2) so that the limit set is not necessarily self-similar. More importantly, we drop MSC(4) from the construction process so that there are no limitations on the ratios of the diameters of the sets. Specifically, the ratio $\dfrac{diam(J_{\sigma * j})}{diam(J_\sigma)}$ in (1.6) is not limited to depend on just k and j, but varies with σ. This change allows us to produce a mosaic of possible fractals. An important observation is that the computational complexity of generating these fractals is the same as using an analogous, standard IFS. In Section 4 we give estimates of the Hausdorff dimension of the limit sets created from the general construction. In Section 5 we apply the results to specific examples, including modifications of the Cantor set, the Sierpinski triangle, and the Menger sponge. We also give a remark to discuss similarities and differences of this construction with $V-$variable fractals created by Barnsley, Hutchinson, and Stenflo in [2], [3]. In section 6, we explore the sufficient conditions needed for a fractal to satisfy the uniform covering condition, which plays a vital role in computing a lower estimate for the Hausdorff dimension of a fractal.

2. Hausdorff Dimension of the Limit Sets

In this section we investigate the Hausdorff dimension $\dim_H(F)$ of the fractals F defined in (1.4), which do not necessarily satisfy all the MSC conditions. To start, we determine an upper bound for the dimension of the limit set F by considering the step-wise relative ratios between the diameters of sets.

Proposition 2.1. *Suppose $\mathcal{J} := \{J_\sigma : \sigma \in D\}$ is a collection of bounded subsets of a metric space (X, d), and $s > 0$. Let $E_k = \bigcup_{\sigma\in D_k} J_\sigma$, and $F = \bigcap_{k\geq 0} E_k$ be defined as in (1.4). If there exists a sequence of positive numbers $\{c_k\}_{k=1}^\infty$ such that*

$$\liminf_{k\to\infty} \prod_{i=1}^{k} c_i = 0$$

and

$$\sum_{j=1}^{n_k} (\mathrm{diam}(J_{\sigma*j}))^s \leq c_k\, (\mathrm{diam}(J_\sigma))^s\,, \tag{2.1}$$

for all $\sigma \in D_{k-1}$ and all $k = 1, 2, \cdots$, then $dim_H(F) \leq s$.

Proof. We prove by using mathematical induction that for $k = 1, 2, \cdots$,

$$\sum_{\sigma\in D_k} (diam(J_\sigma))^s \leq \left(\prod_{i=1}^{k} c_i\right) (diam(J_\emptyset))^s. \tag{2.2}$$

When $k = 1$, (2.2) follows from (2.1). Now assume (2.2) is true for some $k \geq 1$. Then by (1.3), (2.1), and (2.2),

$$\begin{aligned}\sum_{\sigma\in D_{k+1}} (diam(J_\sigma))^s &= \sum_{\sigma\in D_k}\left(\sum_{j=1}^{n_{k+1}} (diam(J_{\sigma*j}))^s\right)\\ &\leq c_{k+1}\sum_{\sigma\in D_k}(diam(J_\sigma))^s\\ &\leq \left(\prod_{i=1}^{k+1} c_i\right)(diam(J_\emptyset))^s\end{aligned}$$

as desired. By the induction principle, (2.2) holds for all $k = 1, 2, \cdots$. For each k, set

$$\delta_k = \max\{diam(J_\sigma) : \sigma \in D_k\} > 0.$$

Then, by (2.2), $\delta_k \le \left(\prod_{i=1}^k c_i\right)^{1/s} diam(J_\emptyset)$. Moreover, by (2.2)

$$\mathcal{H}^s_{\delta_k}(F) \le \mathcal{H}^s_{\delta_k}(E_k) \le \sum_{\sigma\in D_k} \alpha(s)\left(\frac{\mathrm{diam}(J_\sigma)}{2}\right)^s \le \left(\prod_{i=1}^k c_i\right)\alpha(s)\left(\frac{\mathrm{diam}(J_\emptyset)}{2}\right)^s.$$

Since $\liminf_{k\to\infty}\prod_{i=1}^k c_i = 0$, there exists a sequence $\{k_t\}_{t=1}^\infty$ such that

$$\lim_{t\to\infty}\prod_{i=1}^{k_t} c_i = 0. \tag{2.3}$$

Thus, $\lim_{t\to\infty}\delta_{k_t} = 0$, $\mathcal{H}^s(F) = \lim_{t\to\infty}\mathcal{H}^s_{\delta_{k_t}}(F) = 0$, and hence $\dim_H(F) \le s$. □

Conversely, a lower bound on the Hausdorff dimension of the limit set F can also be obtained as follows.

Definition 2.1 (uniform covering condition)**.** *Let $\mathcal{J} := \{J_\sigma : \sigma\in D\}$ be a collection of compact subsets of a metric space (X,d), and F be the limit set of $\mathcal{J}$ as given in (1.4). $\mathcal{J}$ is said to satisfy the uniform covering condition if there exists a real number $\gamma > 0$ and a natural number N such that for all closed ball B in X, there exists a subset $D_B \subset D$ with cardinality of D_B at most N,*

$$B\cap F \subseteq \bigcup_{\sigma\in D_B} J_\sigma \text{ and } diam(B) \ge \gamma \sum_{\sigma\in D_B} diam(J_\sigma). \tag{2.4}$$

Proposition 2.2. *Let $\mathcal{J} := \{J_\sigma : \sigma\in D\}$ be a collection of compact subsets of a metric space (X,d) with $diam(J_\emptyset) > 0$, and F be the limit set of $\mathcal{J}$ as given in (1.4). If $\mathcal{J}$ satisfies the uniform covering condition, and if for some $s > 0$,*

$$\sum_{j=1}^{n_k} diam(J_{\sigma*j})^s \ge diam(J_\sigma)^s \tag{2.5}$$

for all $\sigma\in D_{k-1}$ and all $k = 1,2,\cdots$, then $dim_H(F) \ge s$.

Proof. We first show that under condition (2.5), there exists a probability measure μ on X concentrated on F such that for each $\sigma \in D$,

$$\mu(J_\sigma) \le \left(\frac{\mathrm{diam}(J_\sigma)}{\mathrm{diam}(J_\emptyset)}\right)^s. \tag{2.6}$$

Let $\mu(J_\emptyset) = 1$, and for each $\sigma\in D_k$ for $k > 0$ and $i = 1,\cdots,n_k$, we inductively set

$$\mu(J_{\sigma*i}) = \frac{\mathrm{diam}(J_{\sigma*i})^s}{\sum_{j=1}^{n_k}\mathrm{diam}(J_{\sigma*j})^s}\mu(J_\sigma).$$

For any Borel set A in X, define

$$\mu(A) = \inf\left\{\sum_{i=1}^\infty \mu(J_{\sigma_i}) : A\cap F \subset \bigcup_{i=1}^\infty J_{\sigma_i} \text{ and } J_{\sigma_i}\in\mathcal{J}\right\}.$$

One can check that μ defines a probability measure on X, concentrated on F.

To prove (2.6) for J_σ, $\forall\sigma\in D$, we proceed by using induction on k when $\sigma\in D_k$. It is clear for $k = 0$. Now assume that (2.6) holds for each $\sigma\in D_k$ for some k.

Then by induction assumption and (2.5), for each $i = 1, \cdots, n_{k+1}$,

$$\begin{aligned}\mu(J_{\sigma*i}) &= \frac{diam(J_{\sigma*i})^s}{\sum_{j=1}^{n_k} \text{diam}(J_{\sigma*j})^s}\mu(J_\sigma) \\ &\leq \frac{diam(J_{\sigma*i})^s}{\sum_{j=1}^{n_k} \text{diam}(J_{\sigma*j})^s}\left(\frac{\text{diam}(J_\sigma)}{\text{diam}(J_\emptyset)}\right)^s \\ &\leq \left(\frac{\text{diam}(J_{\sigma*i})}{\text{diam}(J_\emptyset)}\right)^s.\end{aligned}$$

This proves inequality (2.6).

Now, for any $\delta > 0$, let $\{B_i\}$ be any collection of closed balls with $diam(B_i) \leq \delta$ and $F \subseteq \cup_i B_i$. For each i, let D_{B_i} be the subset of D corresponding to B_i as given in equation (2.4). Note that

$$F \subseteq \bigcup_i B_i \cap F \subseteq \bigcup_i \bigcup_{\sigma\in D_{B_i}} J_\sigma = \bigcup_{\sigma\in\tilde{D}} J_\sigma,$$

where $\tilde{D} := \cup_{i=1}^{\infty} D_{B_i} \subseteq D$.

Let

$$\begin{aligned}C(s) &:= \max\{\sum_{i=1}^{N}(x_i)^s : (x_1, x_2, \cdots, x_N) \in [0,1]^N \text{ with } \sum_{i=1}^{N} x_i = 1\} \\ &= \begin{cases} N^{1-s}, & \text{if } 0 < s < 1 \\ 1, & \text{if } s \geq 1. \end{cases}\end{aligned}$$

and $c(s) = \frac{\alpha(s)}{C(s)}\left(\frac{\gamma diam(J_\emptyset)}{2}\right)^s > 0$. Then, by (2.4) and (2.6),

$$\begin{aligned}& \sum_i \alpha(s)\left(\frac{\text{diam}(B_i)}{2}\right)^s \\ \geq\ & \sum_i \frac{\alpha(s)}{2^s}\left(\gamma \sum_{\sigma\in D_{B_i}} \text{diam}(J_\sigma)\right)^s \\ \geq\ & \sum_i \frac{\alpha(s)}{2^s C(s)}\gamma^s \sum_{\sigma\in D_{B_i}} (\text{diam}(J_\sigma))^s \\ \geq\ & \frac{\alpha(s)}{2^s C(s)}\gamma^s \sum_{\sigma\in\tilde{D}} (\text{diam}(J_\sigma))^s \\ \geq\ & \frac{\alpha(s)}{2^s C(s)}\gamma^s (\text{diam}(J_\emptyset))^s \sum_{\sigma\in\tilde{D}} \mu(J_\sigma) \\ \geq\ & c(s)\mu\left(\sum_{\sigma\in\tilde{D}} J_\sigma\right) \geq c(s)\mu(F) = c(s).\end{aligned}$$

Thus, $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) \geq c(s) > 0$, and hence $\dim_H(F) \geq s$. □

In section 6 we will explore sufficient conditions for $\mathcal{J}$ to satisfy the uniform covering condition.

3. General Setup of $\mathcal{F}$-Limit sets

We now formalize the ideas stated in the Introduction to give a description of the construction of generalized fractals. We concentrate on the maps in order to take advantage of the computational nature of an IFS, but allow for the maps to be updated and changed at each iteration.

Let $\mathcal{X}$ be a collection of nonempty compact subsets of a metric space.

Definition 3.1. *A mapping $f : \mathcal{X} \to \mathcal{X}$ is called a compression on $\mathcal{X}$ if $f(E) \subseteq E$ for each $E \in \mathcal{X}$.*

For each natural number m, let

$$\mathcal{C}_m(\mathcal{X}) = \{(f^{(1)}, f^{(2)}, \ldots, f^{(m)}) : f_i \text{ is a compression on } \mathcal{X}, i = 1, \ldots, m\}.$$

Definition 3.2. *Let $\mathcal{M}$ be a nonempty set. A mapping*

$$\begin{aligned} \mathcal{F} : \quad & \mathcal{M} \to \mathcal{C}_m(\mathcal{X}) \\ & k \to f_k = (f_k^{(1)}, f_k^{(2)}, \cdots, f_k^{(m)}) \end{aligned}$$

is called a marking of $\mathcal{C}_m(\mathcal{X})$ by $\mathcal{M}$. Each element $k \in \mathcal{M}$ is called the marker of f_k.

Given a marking $\mathcal{F}$ and an initial set $E_0 \in \mathcal{X}$, we will construct a generalized Moran set from any sequence of markers in $\mathcal{M}$. Note that any sequence $\{k_\ell\}_{\ell=0}^\infty$ in $\mathcal{M}$ can be represented as a mapping from the ordered set D to $\mathcal{M}$.

Definition 3.3. *Let $\mathcal{F}$ be a marking of $\mathcal{C}_m(\mathcal{X})$ by $\mathcal{M}$, let E_0 be any element in $\mathcal{X}$, and D be as in (1.1). Suppose $\vec{k} : D \to \mathcal{M}$ is a map sending σ to k_σ. For each $\sigma \in D$ and $1 \le j \le m$, we recursively define $J_\emptyset = E_0$ and*

$$J_{\sigma * j} = f_{\mathbf{k}_\sigma}^{(j)}(J_\sigma), \tag{3.1}$$

where f_{k_σ} is given by $\mathcal{F}$ as in (3.1). The limit set

$$F = \bigcap_{k \ge 1} \bigcup_{\sigma \in D_k} J_\sigma \tag{3.2}$$

associated with $\mathcal{J}(\vec{k}) = \{J_\sigma : \sigma \in D\}$ is called the $\mathcal{F}$-limit set generated by $\vec{k}$ with the initial set E_0.

We now make two observations relating the concepts of an $\mathcal{F}$-limit set with the attractor of an IFS.

First we observe that the attractor of an IFS $\{S_1, S_2, \ldots, S_m\}$ on a closed subset Δ of $\mathbb{R}^n$ can be viewed as an $\mathcal{F}$-limit set as follows. Let

$$\mathcal{X} = \{E : E \text{ is a non-empty compact subset of } \Delta, S_i(E) \subseteq E, \text{ for all } i\}.$$

Since each S_i is a contraction on Δ, the set $E_r := \Delta \cap \overline{B(0, r)}$ is a non-empty compact subset of Δ, and $S_i(E_r) \subseteq E_r$ for each i when r is sufficiently large. In other words, $E_r \in \mathcal{X}$ for sufficiently large r. Also, each contraction map S_i acting on Δ naturally determines a map $f^{(i)} : \mathcal{X} \to \mathcal{X}$ given by

$$f^{(i)}(E) = S_i(E) := \{S_i(x) \mid x \in E \subseteq \Delta\} \tag{3.3}$$

for each $E \in \mathcal{X}$. Since $f^{(i)}(E) = S_i(E) \subseteq E$, $f^{(i)}$ is a compression for each i. Set

$$f = (f^{(1)}, f^{(2)}, \ldots, f^{(m)}).$$

For any non-empty set $\mathcal{M}$, define the marking $\mathcal{F}$ of $\mathcal{C}_m(\mathcal{X})$ to be the constant function $\mathcal{F}(k) = f$ for all $k \in \mathcal{M}$. Thus, for each $\sigma \in D_k$ and $i = 1, \ldots, m$, we have that $J_{\sigma * i} = S_i(J_\sigma)$ from (3.1). As a result, for any map $\vec{k} : D \to \mathcal{M}$, the collection $\mathcal{J}(\vec{k}) = \{J_\sigma : \sigma \in D\}$ is independent of the choice of $\vec{k}$. Thus, the associated $\mathcal{F}$-limit set $F = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma$ agrees with the attractor of the given IFS $\{S_1, S_2, \ldots, S_m\}$. Conversely, let $\mathcal{F}$ be a marking of $\mathcal{C}_m(\mathcal{X})$ by $\mathcal{M}$ where $\mathcal{X}$ is a collection of non-empty compact subsets of Δ. Suppose there is a mapping $\vec{k} : D \to \mathcal{M}$ such that the sequence $\{f_{k_\sigma}\}_{\sigma \in D}$ is constant in $\mathcal{C}_m(\mathcal{X})$ (i.e. there exists an $f \in \mathcal{C}_m(\mathcal{X})$ such that $f_{k_\sigma} = f$ for all $\sigma \in D$) and for each $i = 1, 2, \ldots, m$, there exists a contraction S_i on Δ such that equation (3.3) holds for each $E \in \mathcal{X}$. Then the $\mathcal{F}$-limit set F generated by $\vec{k}$ is the attractor of the IFS $\{S_1, S_2, \ldots, S_m\}$. Therefore, choosing $\vec{k} : D \to \mathcal{M}$ to be a constant map will result in a limit set F that is the attractor of an IFS. In the above sense, our approach is a generalization of the standard IFS construction.

An important observation is that replacing $\{k_\sigma\}_{\sigma \in D}$ by another sequence $\{\tilde{k}_\sigma\}_{\sigma \in D}$ in (3.1) will not change the computational complexity of the construction of $\mathcal{J}(\vec{k})$. Thus, generating the limit set F will have a similar computational complexity as generating the attractor of a comparable IFS.

In the following section we will compute the Hausdorff dimension of the constructed $\mathcal{F}$-limit sets. In section 5 we will provide examples along with their dimensions.

4. Hausdorff dimensions of $\mathcal{F}$-Limit sets

As indicated in Propositions 2.1 and 2.2, the relative ratio between the diameters of the sets plays an important role in the calculation of the dimension of the limit set. Therefore, we introduce the following definition.

Definition 4.1. *For any compression $g : \mathcal{X} \to \mathcal{X}$, define*

$$U(g) = \sup_{E \in \mathcal{X}} \frac{\operatorname{diam}(g(E))}{\operatorname{diam}(E)}, \text{ and } L(g) = \inf_{E \in \mathcal{X}} \frac{\operatorname{diam}(g(E))}{\operatorname{diam}(E)}. \tag{4.1}$$

Note that, for each $E \in \mathcal{X}$,

$$L(g) \cdot \operatorname{diam}(E) \leq \operatorname{diam}(g(E)) \leq U(g) \cdot \operatorname{diam}(E). \tag{4.2}$$

For any $\mathbf{k} \in \mathcal{M}$ and $f_{\mathbf{k}} = (f_{\mathbf{k}}^{(1)}, f_{\mathbf{k}}^{(2)}, \cdots, f_{\mathbf{k}}^{(m)}) \in \mathcal{C}_m(\mathcal{X})$, define

$$\mathbf{U_k} = \left(U(f_{\mathbf{k}}^{(1)}), \cdots, U(f_{\mathbf{k}}^{(m)}) \right) \in \mathbb{R}^m,$$

and

$$\mathbf{L_k} = \left(L(f_{\mathbf{k}}^{(1)}), \cdots, L(f_{\mathbf{k}}^{(m)}) \right) \in \mathbb{R}^m.$$

Also, for each $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ and $s > 0$, denote

$$||x||_s = \left(\sum_{i=1}^m |x_i|^s \right)^{\frac{1}{s}}.$$

These notations, Proposition 2.1 and Proposition 2.2 motivate our main theorem.

Theorem 4.1. *Let F be the $\mathcal{F}$-limit set generated by a sequence $\{\mathbf{k}_\sigma\}_{\sigma \in D}$ with initial set $J_\emptyset$, and $s > 0$.*

(a) *If F satisfies the uniform covering condition (2.4) and*

$$\inf_{\sigma\in D}\{||\mathbf{L}_{\mathbf{k}_\sigma}||_s\} \geq 1,$$

then $\dim_H(F) \geq s$.

(b) *If*

$$\sup_{\sigma\in D}\{||\mathbf{U}_{\mathbf{k}_\sigma}||_s\} < 1,$$

then $\dim_H(F) \leq s$.

Proof. (a) By (3.1) and (4.2), for all $\sigma \in D$,

$$\sum_{j=1}^{m} diam(J_{\sigma*j})^s = \sum_{j=1}^{m} diam\left(f_{\mathbf{k}_\sigma}^{(j)}(J_\sigma)\right)^s \geq \sum_{j=1}^{m}\left(L(f_{\mathbf{k}_\sigma}^{(j)})\right)^s diam(J_\sigma)^s \geq diam(J_\sigma)^s.$$

Thus, by Proposition 2.2, $\dim_H(F) \geq s$.

(b) Similarly, for all $\sigma \in D$,

$$\sum_{j=1}^{m} diam(J_{\sigma*j})^s \leq \sum_{j=1}^{m}\left(U(f_{\mathbf{k}_\sigma}^{(j)})\right)^s diam(J_\sigma)^s \leq c\cdot diam(J_\sigma)^s,$$

where

$$c := \sup_{\sigma}\{(||\mathbf{U}_{\mathbf{k}_\sigma}||_s)^s\} < 1.$$

By Proposition 2.1, $\dim_H(F) \leq s$. □

For practical reasons, we find that it is more convenient to represent the mapping $\vec{k} : D \to \mathcal{M}$ by a sequence $\{k_\ell\}_{\ell=0}^{\infty} \subseteq \mathcal{M}$. For each $\sigma = (i_1, i_2, \dots, i_k) \in D_k$, let

$$\ell(\sigma) = \sum_{p=0}^{k-1} m^p i_{k-p} \tag{4.3}$$

be the ordering of σ in the ordered set D. Using this notation, we can rewrite Definition 3.3 as follows.

Definition 4.2 (Revision of Definition 3.3)**.** *Let $\mathcal{F}$ be a marking of $\mathcal{C}_m(\mathcal{X})$ by $\mathcal{M}$, let $\{\mathbf{k}_\ell\}_{\ell=0}^{\infty}$ be a sequence in $\mathcal{M}$, and $E_0 \in \mathcal{X}$ be a starting set. For each $\ell = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, m$, we iteratively denote the set*

$$E_{m\ell+j} = f_{\mathbf{k}_\ell}^{(j)}(E_\ell) \in \mathcal{X},$$

where $f_{\mathbf{k}_\ell}$ is given by $\mathcal{F}$ as in (3.1). Let $\mathcal{G}_m(0) = 0$ and for $n \geq 1$,

$$\mathcal{G}_m(n) = m + m^2 + \cdots + m^n = \frac{m^{n+1} - m}{m-1} \tag{4.4}$$

denote the number of sets in the n^{th} generation, i.e. the cardinality of D_n. The limit set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\ell=\mathcal{G}_m(n-1)+1}^{\mathcal{G}_m(n)} E_\ell \tag{4.5}$$

is called the $\mathcal{F}$-limit set generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$.

In the following, we will use the notation from Definition 4.2 to describe the construction of the $\mathcal{F}$-limit sets. Clearly, using this notation, Theorem 4.1 simply says that if F satisfies the uniform covering condition (2.4) and $\inf_{\ell}\{||\mathbf{L}_{\mathbf{k}_\ell}||_s\} \geq 1$, then $\dim_H(F) \geq s$, and if $\sup_{\ell}\{||\mathbf{U}_{\mathbf{k}_\ell}||_s\} < 1$, then $\dim_H(F) \leq s$.

When both $\{||\mathbf{L}_{\mathbf{k}_\ell}||_s\}_{\ell=0}^{\infty}$ and $\{||\mathbf{U}_{\mathbf{k}_\ell}||_s\}_{\ell=0}^{\infty}$ are convergent sequences, the following corollary enables us to quickly estimate the dimension of F.

Corollary 4.2. *Let F be the limit set generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$.*

(a) *Let $\underline{s_*} := \sup\{s : \liminf_{\ell\to\infty}\{||\mathbf{L}_{\mathbf{k}_\ell}||_s\} > 1\}$. Then*

$$\dim_H(F) \geq \underline{s_*}, \tag{4.6}$$

provided F satisfies the uniform covering condition (2.4).

(b) *Let $\overline{s^*} := \inf\{s : \limsup_{\ell\to\infty}\{||\mathbf{U}_{\mathbf{k}_\ell}||_s\} < 1\}$. Then*

$$\dim_H(F) \leq \overline{s^*}. \tag{4.7}$$

Proof. For any $0 < s < \underline{s_*}$, by the definition of $\underline{s_*}$,

$$\liminf_{\ell\to\infty}\{||\mathbf{L}_{\mathbf{k}_\ell}||_s\} > 1.$$

Thus, when $\ell_* \in \mathbb{N}$ is large enough,

$$\inf_{\ell\geq\ell_*}\{||\mathbf{L}_{\mathbf{k}_\ell}||_s\} \geq 1, \qquad \text{i.e. } \inf_{\ell\geq 0}\{||\mathbf{L}_{\mathbf{k}_{\ell_*+\ell}}||_s\} \geq 1.$$

Since $F \cap E_{\ell_*}$ is the set generated by the triple $(\mathcal{F}, \{\mathbf{k}_{\ell_*+\ell}\}_{\ell=0}^{\infty}, E_{\ell_*})$, by Theorem 4.1, it follows that $\dim_H(F \cap E_{\ell_*}) \geq s$ for any ℓ_* large enough. This implies that $\dim_H(F) \geq s$ for any $s < \underline{s_*}$ and hence $\dim_H(F) \geq \underline{s_*}$. Similarly, we also have $\dim_H(F) \leq \overline{s^*}$. □

In the following corollaries, we will see that bounds of the dimension of F can also be obtained from corresponding bounds on $\mathbf{L}_{\mathbf{k}_\ell}$ and $\mathbf{U}_{\mathbf{k}_\ell}$.

Notation. For any two points $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$ in $\mathbb{R}^m$, we say $x \leq y$ if $x_i \leq y_i$ for each $i = 1, \cdots, m$.

Corollary 4.3. *Let $\mathbf{t} = (t_1, \cdots, t_m)$ and $\mathbf{r} = (r_1, \cdots, r_m)$ be two points in $(0,1)^m \subset \mathbb{R}^m$. Let s_* and s^* be the solutions to $||\mathbf{t}||_{s_*} = 1$, and $||\mathbf{r}||_{s^*} = 1$ respectively, i.e.*

$$t_1^{s_*} + t_2^{s_*} + \cdots + t_m^{s_*} = 1, \text{ and } r_1^{s^*} + r_2^{s^*} + \cdots + r_m^{s^*} = 1.$$

(a) *If $\mathbf{L}_{\mathbf{k}_\ell} \geq \mathbf{t}$ for all ℓ and F satisfies the uniform covering condition (2.4), then $\dim_H(F) \geq s_*$.*
(b) *If $\mathbf{U}_{\mathbf{k}_\ell} \leq \mathbf{r}$ for all ℓ, then $\dim_H(F) \leq s^*$.*
(c) *If $\mathbf{L}_{\mathbf{k}_\ell} = \mathbf{r} = \mathbf{U}_{\mathbf{k}_\ell}$ for all ℓ and F satisfies the uniform covering condition (2.4), then $\dim_H(F) = s^*$.*

Proof. (a) Let $0 < s < s_*$. Then,

$$\inf_{\ell}\{||\mathbf{L}_{\mathbf{k}_\ell}||_s\} \geq ||\mathbf{t}||_s \geq ||\mathbf{t}||_{s_*} = 1.$$

Thus, by Theorem 4.1, $\dim_H(F) \geq s$ for any $s < s_*$, and hence $\dim_H(F) \geq s_*$.

(b) Similarly, let $0 < s^* < s$. Then,

$$\sup_{\ell}\{||\mathbf{U}_{\mathbf{k}_\ell}||_s\} \leq ||\mathbf{r}||_s < ||\mathbf{r}||_{s^*} = 1.$$

Thus, by Theorem 4.1, $\dim_H(F) \leq s$ for any $s > s^*$, and hence $\dim_H(F) \leq s^*$.

(c) follows from (a) and (b). □

A special case of Corollary 4.3 gives the following explicit formulas for the bounds on the dimension of F.

Corollary 4.4. *Let F be the limit set generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$. Let*

$$\mathbf{t} = (t, \cdots, t) \text{ and } \mathbf{r} = (r, \cdots, r),$$

for some $0 < t, r < 1$.

(a) *If $\mathbf{L}_{\mathbf{k}_\ell} \geq \mathbf{t}$ for all ℓ and F satisfies the uniform covering condition (2.4), then $\dim_H(F) \geq \frac{\log m}{-\log t}$.*
(b) *If $\mathbf{U}_{\mathbf{k}_\ell} \leq \mathbf{r}$ for all ℓ, then $\dim_H(F) \leq \frac{\log m}{-\log r}$.*
(c) *If $\mathbf{L}_{\mathbf{k}_\ell} = \mathbf{r} = \mathbf{U}_{\mathbf{k}_\ell}$ for all ℓ and F satisfies the uniform covering condition (2.4), then $\dim_H(F) = \frac{\log m}{-\log r}$.*

Other types of bounds on $\mathbf{L}_{\mathbf{k}_\ell}$ and $\mathbf{U}_{\mathbf{k}_\ell}$ can also be used to provide bounds on $\dim_H(F)$, as indicated by the following result.

Corollary 4.5. *Let F be the limit set generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$.*

(a) *If F satisfies the uniform covering condition (2.4) and*

$$w := \inf_\ell \{||\mathbf{L}_{\mathbf{k}_\ell}||_1\} \geq 1,$$

then $\dim_H(F) \geq \frac{\log(m)}{\log(m)-\log(w)}$.

(b) *If*

$$u := \sup_\ell \{||\mathbf{U}_{\mathbf{k}_\ell}||_1\} < 1,$$

then $\dim_H(F) \leq \frac{\log(m)}{\log(m)-\log(u)}$.

Proof. (a). In this case, for $s = \frac{\log(m)}{\log(m)-\log(w)} \geq 1$, we have

$$\frac{\sum_{j=1}^m \left(L\left(f_{\mathbf{k}_\ell}^{(j)}\right)\right)^s}{m} \geq \left(\frac{\sum_{j=1}^m L\left(f_{\mathbf{k}_\ell}^{(j)}\right)}{m}\right)^s \geq \left(\frac{w}{m}\right)^s$$

for each ℓ. Thus,

$$\inf_\ell \{||\mathbf{L}_{\mathbf{k}_\ell}||_s\} \geq m^{\frac{1}{s}} \frac{w}{m} = 1,$$

then by Theorem 4.1, $\dim_H(F) \geq s$.

(b). In this case, for any $1 \geq s > \frac{\log(m)}{\log(m)-\log(u)}$, we have

$$\frac{\sum_{j=1}^m \left(U\left(f_{\mathbf{k}_\ell}^{(j)}\right)\right)^s}{m} \leq \left(\frac{\sum_{j=1}^m U\left(f_{\mathbf{k}_\ell}^{(j)}\right)}{m}\right)^s \leq \left(\frac{u}{m}\right)^s$$

for each ℓ. Thus,

$$\sup_\ell \{||\mathbf{U}_{\mathbf{k}_\ell}||_s\} \leq m^{\frac{1}{s}} \frac{u}{m} < 1.$$

By Theorem 4.1, $\dim_H(F) \leq s$. Hence, $\dim_H(F) \leq \frac{\log(m)}{\log(m)-\log(u)}$. □

Note that this corollary generally provides better bounds on $\dim_H(F)$ than those obtained from directly applying Theorem 4.1.

5. Examples of $\mathcal{F}$-Limit sets

In this section we describe the construction of both classical fractals and generalized Moran sets in the language of Section 3, and calculate the dimension using the results from Section 4.

5.1. **Cantor-Like Sets.** We first consider Cantor-like sets. Let

$$\mathcal{X} = \{[a,b] : a, b \in \mathbb{R}\} \tag{5.1}$$

be the collection of closed intervals, $m = 2$, and let $\mathcal{M} = [0,1]^2 \subseteq \mathbb{R}$. For each $\mathbf{k} = (k^{(1)}, k^{(2)}) \in \mathcal{M}$, we consider the following two maps,

$$\begin{aligned} f^{(1)}_{\mathbf{k}} : \quad & \mathcal{X} && \to \mathcal{X} \\ & [a,b] && \mapsto [a, k^{(1)}(b-a)+a] \end{aligned}$$

$$\begin{aligned} f^{(2)}_{\mathbf{k}} : \quad & \mathcal{X} && \to \mathcal{X} \\ & [a,b] && \mapsto [k^{(2)}(a-b)+b, b]. \end{aligned}$$

Note that both $f^{(1)}_{\mathbf{k}}$ and $f^{(2)}_{\mathbf{k}}$ are compression maps for any $\mathbf{k} \in \mathcal{M}$. Thus, this defines a marking

$$\begin{aligned} \mathcal{F} : \quad & \mathcal{M} && \to \mathcal{C}_2(\mathcal{X}) \\ & \mathbf{k} && \mapsto f_{\mathbf{k}} = (f^{(1)}_{\mathbf{k}}, f^{(2)}_{\mathbf{k}}). \end{aligned}$$

Here, for each $\mathbf{k} = (k^{(1)}, k^{(2)}) \in \mathcal{M}$, one can clearly see that

$$diam\left(f^{(i)}_{\mathbf{k}}([a,b])\right) = k^{(i)} \cdot diam([a,b]).$$

Thus, $L\left(f^{(i)}_{\mathbf{k}}\right) = k^{(i)} = U\left(f^{(i)}_{\mathbf{k}}\right)$, and hence

$$\mathbf{L}_{\mathbf{k}} = \mathbf{k} = \mathbf{U}_{\mathbf{k}}. \tag{5.2}$$

Let $E_0 = [0,1] \in \mathcal{X}$ be fixed. For any sequence $\{\mathbf{k}_\ell\}_{\ell=0}^\infty \in \mathcal{M}$, we define the following:

$$\begin{aligned}
E^{(0)} &= E_0 \\
E^{(1)} &= f^{(1)}_{\mathbf{k}_0}(E_0) \cup f^{(2)}_{\mathbf{k}_0}(E_0) =: E_1 \cup E_2 \\
E^{(2)} &= f^{(1)}_{\mathbf{k}_1}(E_1) \cup f^{(2)}_{\mathbf{k}_1}(E_1) \cup f^{(1)}_{\mathbf{k}_2}(E_2) \cup f^{(2)}_{\mathbf{k}_2}(E_2) \\
&:= \quad E_3 \quad \cup \quad E_4 \quad \cup \quad E_5 \quad \cup \quad E_6 \\
&\vdots \\
E^{(n)} &= \bigcup_{i=2^{n-1}-1}^{2^n-2} \left(f^{(1)}_{\mathbf{k}_i}(E_i) \cup f^{(2)}_{\mathbf{k}_i}(E_i)\right) := \bigcup_{i=2^{n-1}-1}^{2^n-2} (E_{2i+1} \cup E_{2i+2}) = \bigcup_{\ell=2^n-1}^{2(2^n-1)} E_\ell.
\end{aligned}$$

Note that when $\mathbf{k}_\ell = (\frac{1}{3}, \frac{1}{3})$ for all ℓ, $E^{(n)}$ is the n^{th}-generation of the Cantor set $\mathcal{C}$ and $F = \lim_{n\to\infty} E^{(n)} = \bigcap_n E^{(n)} = \mathcal{C}$.

Observe that the process of constructing the sequence $\{E^{(n)}\}_{n=0}^\infty$ is independent of the values of $\{\mathbf{k}_\ell\}_{\ell=0}^\infty$. To allow for more general outcomes, we can update the linear functions $f^{(1)}_{\mathbf{k}}$ and $f^{(2)}_{\mathbf{k}}$ simply by changing the value of $\mathbf{k}$ at each stage of the construction, which does not change the computational complexity of the process. Using this idea, we now construct some examples of Cantor-like sets by choosing suitable sequences $\{\mathbf{k}_\ell\}_{\ell=0}^{\infty}$.

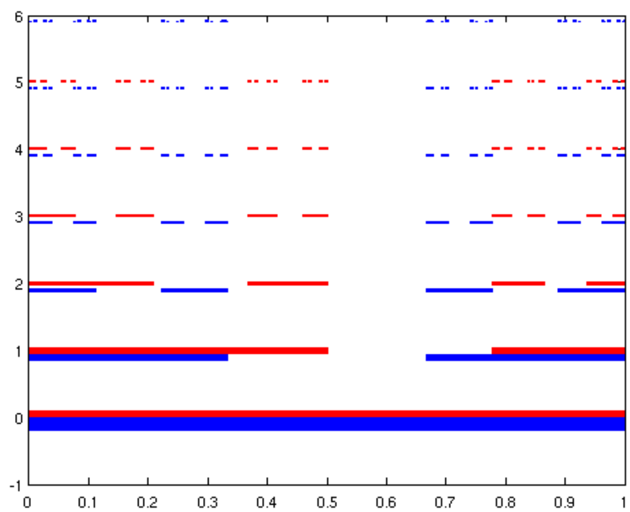


FIGURE 1. Comparison of classical Cantor set (blue) and new Cantor-like set (red)

Example 5.1. *Let* $\mathbf{k}_\ell = \left(\frac{\ell+1}{4\ell+6}, \frac{2\ell+5}{8\ell+16}\right)$ *for* $\ell \geq 0$*, and let* F *be the* $\mathcal{F}$*-limit set generated by the triple* $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$*. In Figure 1 we plot the usual Cantor set* $\mathcal{C}$ *(in blue) below the set* F *(in red) to illustrate the comparison. We can see that the set* F *has the same basic shape as the Cantor set* $\mathcal{C}$*, but is no longer strictly self-similar. In order to compute the Hausdorff dimension of the new Cantor-like set* F*, we apply Corollary 4.2. Note that by equation (5.2),*

$$\lim_{\ell\to\infty} ||\mathbf{L}_{\mathbf{k}_\ell}||_s = \lim_{\ell\to\infty} ||\mathbf{k}_\ell||_s = \frac{2^{\frac{1}{s}}}{4}.$$

So,

$$\underline{s_*} = \sup_s\{\liminf_{\ell\to\infty} ||\mathbf{L}_{\mathbf{k}_\ell}||_s > 1\} = \sup_s \left\{\frac{2^{\frac{1}{s}}}{4} > 1\right\} = \frac{1}{2}.$$

Similarly, we also have $\overline{s^*} = \frac{1}{2}$*. By Corollary 4.2,* $\dim_H(F) = \frac{1}{2}$*. Here,* F *satisfies the uniform covering condition (2.4) since*

$$\sup\left\{k_\ell^{(1)} + k_\ell^{(2)} : \ell = 0, 1, 2, \cdots\right\} = \frac{1}{2} < 1,$$

according to Proposition 6.2.

In the next example, we will construct a random Cantor-like set as follows.

Example 5.2. *For each $\ell \geq 0$, we take $\mathbf{k}_\ell = \left(q_\ell, \frac{1}{2} - q_\ell\right)$ where q_ℓ is a random number between $\frac{1}{8}$ and $\frac{3}{8}$. Let F be the corresponding $\mathcal{F}$-limit set generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^\infty, E_0)$. We plot the first few generations in Figure 2. In this example, the total length of the n^{th} generation $E^{(n)}$ is chosen to be $(\frac{1}{2})^n$, while the scaling factors of the left subintervals at each stage are randomly chosen.*

FIGURE 2. A randomly generated Cantor-like set

We now estimate the dimension of F. By (5.2),

$$\left(\frac{1}{8}, \frac{1}{8}\right) \leq \mathbf{L}_{\mathbf{k}_\ell} = \mathbf{k}_\ell = \mathbf{U}_{\mathbf{k}_\ell} \leq \left(\frac{3}{8}, \frac{3}{8}\right).$$

By Corollary 4.4,

$$\frac{\log(2)}{-\log(1/8)} \leq \dim_H(F) \leq \frac{\log(2)}{-\log(3/8)}.$$

That is,

$$\frac{1}{3} \leq \dim_H(F) \leq \frac{\log(2)}{\log(8/3)} \approx 0.7067.$$

Note that due to Proposition 6.2, F satisfies the uniform covering condition (2.4) since $q_\ell + \frac{1}{2} - q_\ell = \frac{1}{2} < 1$ for each $\ell \geq 0$.

Example 5.3. *In this example, we create a sequence $\{\mathbf{k}_\ell\}_{\ell=0}^\infty$ that results in a limit set with a given measure, e.g. 1/3. Of course, the classic example of such a limiting set is the fat Cantor set. For a different approach, let $\sum_{n=0}^\infty a_n$ be any convergent series of positive terms with limit L. We consider a sequence $\{\mathbf{k}_\ell\}_{\ell=0}^\infty$ defined in the following way.*

Let $n \geq 1$ be the generation of the construction and for each ℓ with $2^{n-1} - 1 \leq \ell \leq 2^n - 2$, define $\mathbf{k}_\ell = (b_n, b_n)$ where

$$b_1 := \frac{\frac{3}{2}L - a_0}{2\left(\frac{3}{2}L\right)} \quad \text{and} \quad b_n := \frac{\frac{3}{2}L - \sum_{i=0}^{n-1} a_i}{2\left(\frac{3}{2}L - \sum_{i=0}^{n-2} a_i\right)} \text{ for } n \geq 2.$$

With this sequence $\{\mathbf{k}_\ell\}_{\ell=0}^{\infty}$, one can find that the length of each interval in the n^{th} generation is

$$b_1 b_2 \cdots b_n = \frac{\frac{3}{2}L - \sum_{i=0}^{n-1} a_i}{2^n \cdot \frac{3}{2}L}.$$

Thus, the total length of the n^{th} generation is

$$\frac{\frac{3}{2}L - \sum_{i=0}^{n-1} a_i}{\frac{3}{2}L} = 1 - \frac{2}{3L}\sum_{i=0}^{n-1} a_i$$

which converges to 1/3 as desired. As an example, we take the convergent series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!} = e$ and use it to create the $\mathcal{F}$-limit set F with measure 1/3. The first few generations are shown in Figure 3.

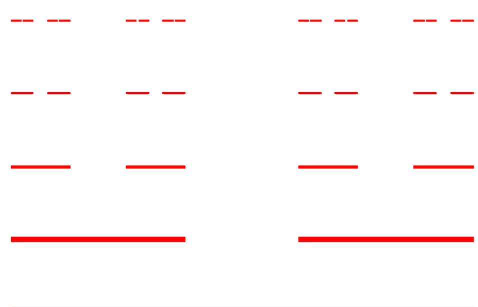


FIGURE 3. Fractal of measure $\frac{1}{3}$ created by using $\sum_{n=0}^{\infty}\frac{1}{n!} = e$

5.2. **Sierpinski Triangle.** The Sierpinski triangle is another well known fractal. Following the general setup in Section 3, we take

$$\mathcal{X} = \{(A, B, C) \mid A, B, C \in \mathbb{R}^2\} \tag{5.3}$$

representing the collection of all triangles ΔABC in $\mathbb{R}^2$, $m = 3$, and $\mathcal{M} = [0,1]^6 \subseteq \mathbb{R}^6$. For each $\mathbf{k} = \left(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}\right) \in \mathcal{M}$ and $i = 1, 2, 3$ we can define affine transformations $f_{\mathbf{k}}^{(i)} : \mathcal{X} \to \mathcal{X}$ as

$$\begin{aligned}
f_{\mathbf{k}}^{(1)}(A,B,C) &= (A, A + k^{(1)}(B - A), A + k^{(2)}(C - A)) \\
f_{\mathbf{k}}^{(2)}(A,B,C) &= (B + k^{(4)}(A - B), B, B + k^{(3)}(C - B)) \\
f_{\mathbf{k}}^{(3)}(A,B,C) &= (C + k^{(5)}(A - C), C + k^{(6)}(B - C), C)
\end{aligned}$$

for every $(A, B, C) \in \mathcal{X}$.

Note that each $f_{\mathbf{k}}^{(i)}$ is a compression map for $i = 1, 2, 3$ and any $\mathbf{k} \in \mathcal{M}$. Thus, this defines a marking

$$\begin{aligned}
\mathcal{F} : \quad \mathcal{M} &\to \mathcal{C}_3(\mathcal{X}) \\
\mathbf{k} &\mapsto f_{\mathbf{k}} = (f_{\mathbf{k}}^{(1)}, f_{\mathbf{k}}^{(2)}, f_{\mathbf{k}}^{(3)}).
\end{aligned}$$

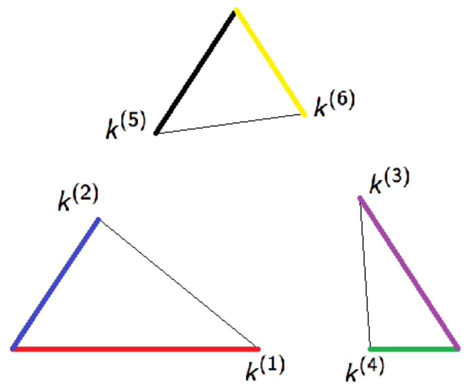
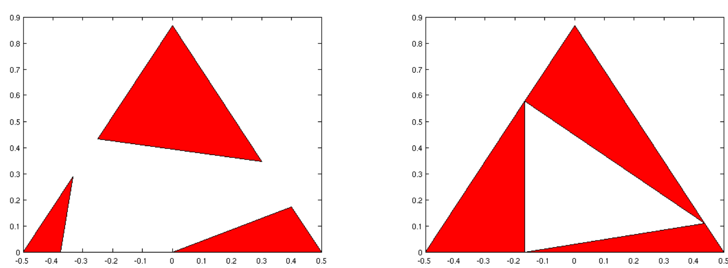


FIGURE 4. Geometric illustration of $\mathbf{k} \in \mathcal{M}$

Of course, to prevent overlaps we can require that $k^{(1)} + k^{(4)} \leq 1, k^{(2)} + k^{(5)} \leq 1, k^{(3)} + k^{(6)} \leq 1$. When each of the inequalities are strict, the images of $f^{(i)}_{\mathbf{k}}$ are three disconnected triangles, as illustrated in Figure 5a. When all equalities hold, the images are connected, as illustrated in Figure 5b.



(A) (B)

FIGURE 5. First generation of disconnected and connected triangles

In the case of the connected sets, the values of $\mathbf{k} = \left(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}\right)$ are determined by $k^{(1)}, k^{(2)}, k^{(3)}$ since $k^{(4)} = 1 - k^{(1)}$, $k^{(5)} = 1 - k^{(2)}$, $k^{(6)} = 1 - k^{(3)}$. In this case, we may also view $\mathbf{k} = \left(k^{(1)}, k^{(2)}, k^{(3)}\right)$ as a vector in $[0,1]^3 \subseteq \mathbb{R}^3$.

To create the normal Sierpinski triangle, we choose

$$E_0 = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 0 & 0 & \sqrt{3}/2 \end{bmatrix}, \tag{5.4}$$

the equilateral triangle of unit side length, and $\mathbf{k}_\ell \in \mathcal{M}$ to be the constant sequence $\mathbf{k}_\ell = \mathbf{k} = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$ so that each iteration maps a triangle to three triangles of half the side length with the desired translation. In this case the $\mathcal{F}$-limit set generated by $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$ corresponds to the standard Sierpinski Triangle as seen in Figure 6.

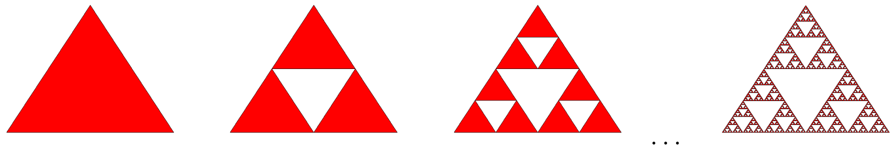


FIGURE 6. Constructing the Sierpinski triangle

To generate Sierpinski-like fractals, we now adjust the values of the marking parameters $\{\mathbf{k}_\ell\}_{\ell=0}^{\infty}$. For each $\mathbf{k} = (k^{(1)}, k^{(2)}, \cdots, k^{(6)}) \in \mathcal{M}$ and $1 \le i \le 3$,

$$U\left(f_{\mathbf{k}}^{(i)}\right) = \sup_{(A,B,C)\in\mathcal{X}} \frac{diam\left(f_{\mathbf{k}}^{(i)}(A,B,C)\right)}{diam\left((A,B,C)\right)} = \max\left\{k^{(2i-1)}, k^{(2i)}\right\},$$

and

$$L\left(f_{\mathbf{k}}^{(i)}\right) = \inf_{(A,B,C)\in\mathcal{X}} \frac{diam\left(f_{\mathbf{k}}^{(i)}(A,B,C)\right)}{diam\left((A,B,C)\right)} = \min\left\{k^{(2i-1)}, k^{(2i)}\right\}.$$

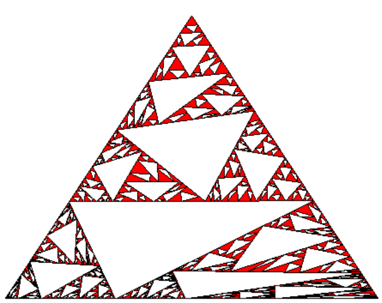
When $\mathbf{k}$ is bounded, i.e. if $\lambda \le k^{(j)} \le \Lambda < 1$ for all $j = 1, \cdots, 6$, then

$$\mathbf{U}_{\mathbf{k}} \le \mathbf{r} := (r, \cdots, r) \text{ and } \mathbf{L}_{\mathbf{k}} \ge \mathbf{s} := (s, \cdots, s),$$

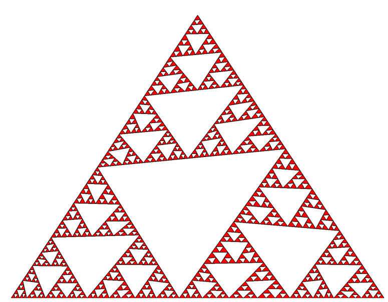
where $r = \max\{1-\lambda, \Lambda\}$ and $s = \min\{1-\lambda, \Lambda\}$.

Following our general process, we construct some random Sierpinski-like sets by introducing randomness into the choice of the sequence $\{\mathbf{k}_\ell\}_{\ell=0}^{\infty}$.

Example 5.4. *Let* $\{\mathbf{k}_\ell\}_{\ell=0}^{\infty} = \left\{\left(k_\ell^{(1)}, k_\ell^{(2)}, k_\ell^{(3)}\right)\right\}_{\ell=0}^{\infty}$ *be a sequence in* $[0,1]^3$ *with each* $k_\ell^{(i)}$ *a random number between given numbers* λ *and* Λ *for each* $i = 1, 2, 3$. *Let* F *be the* $\mathcal{F}$*-limit set generated by* $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$. *Then the* 6^{th} *generation of the construction results in images like Figure 7. Here, in Figure 7a,* $\lambda = \frac{1}{4}$ *and* $\Lambda = \frac{3}{4}$*; while in Figure 7b,* $\lambda = 0.45$ *and* $\Lambda = 0.55$. *Note that the sets are no longer self-similar.*



(A) Each $k_\ell^{(i)}$ is random in $[\frac{1}{4}, \frac{3}{4}]$.



(B) Each $k_\ell^{(i)}$ is random in $[0.45, 0.55]$.

FIGURE 7. Generation 6 of Random Sierpinski triangle

In Figure 7b, we pick $\lambda = 0.45$ and $\Lambda = 0.55$. By Corollary 4.4,

$$\frac{\log(m)}{-\log(s)} \leq \dim_H(F) \leq \frac{\log(m)}{-\log(r)},$$

where $m = 3$, $r = 0.55$ and $s = 0.45$. That is,

$$1.3758 \leq \dim_H(F) \leq 1.8377,$$

provided F satisfies the uniform covering condition (2.4).

Example 5.5. *As in Example 5.4, but replacing E_0 with $\tilde{E}_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the 7^{th} generation of the construction results in an image like Figure 8, when $\lambda = \frac{1}{4}$ and $\Lambda = \frac{3}{4}$.*

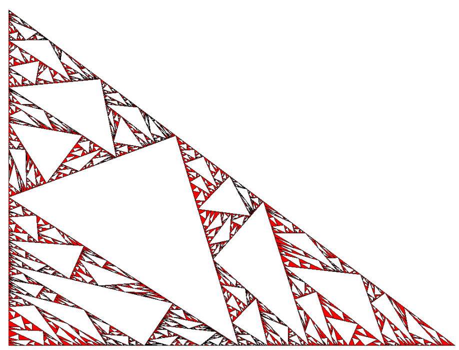


FIGURE 8. Generation 7 of a Random Sierpinski triangle

Example 5.6. *For each $\ell = 0, 1, \cdots,$ let $\mathbf{k}_\ell = \left(k_\ell^{(1)}, k_\ell^{(2)}, \cdots, k_\ell^{(6)}\right)$ where*

$$\begin{aligned} k_\ell^{(1)} &= \frac{1}{2} + \frac{a_\ell}{\sqrt{\ell+1}}, \quad k_\ell^{(2)} = 1 - k_\ell^{(1)}, \\ k_\ell^{(3)} &= \frac{1}{2} + \frac{b_\ell}{\sqrt{\ell+1}}, \quad k_\ell^{(4)} = 1 - k_\ell^{(3)}, \\ k_\ell^{(5)} &= \frac{1}{2} + \frac{c_\ell}{\ell+1}, \quad k_\ell^{(6)} = 1 - k_\ell^{(5)}. \end{aligned}$$

for random numbers $a_\ell, b_\ell, c_\ell \in [-\frac{1}{3}, \frac{1}{3}]$. Let F be the $\mathcal{F}$-limit set F generated by $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$. Then the seventh generation of the construction of F results in an image like Figure 9.

In this case, we can calculate the exact value of the Hausdorff dimension of F. Indeed, by Corollary 4.2,

$$\lim_{\ell\to\infty} (||\mathbf{U}_{\mathbf{k}_\ell}||_s)^s = \frac{3}{2^s} = \lim_{\ell\to\infty} (||\mathbf{L}_{\mathbf{k}_\ell}||_s)^s.$$

Thus, $\dim_H(F) = \frac{\log(3)}{\log(2)}$, provided F satisfies the uniform covering condition (2.4).

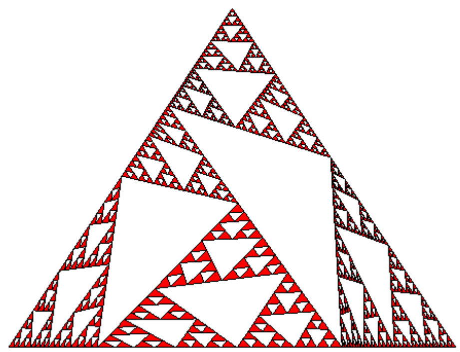


FIGURE 9. Generation 6 of a Sierpinski-type triangle with controlled dimension

5.3. **Menger Sponge.** Let

$$\mathcal{X} = \{(O, A, B, C) \mid O, A, B, C \in \mathbb{R}^3\} \tag{5.5}$$

representing the collection of all rectangular prisms $(OABC)$ in $\mathbb{R}^3$, $m = 20$, and

$$\mathcal{M} = \left\{ \left(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}\right) \in [0,1]^6 : k^{(1)} \leq k^{(2)}, k^{(3)} \leq k^{(4)}, k^{(5)} \leq k^{(6)} \right\}. \tag{5.6}$$

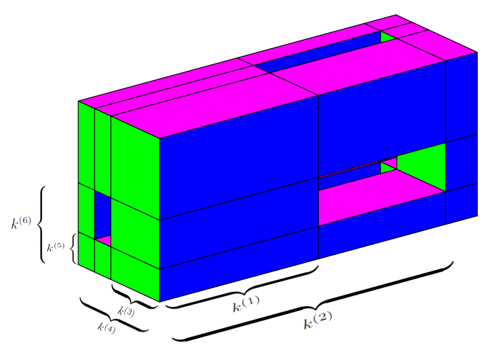


FIGURE 10. Geometric illustration of $\mathbf{k} \in \mathcal{M}$

For each $\mathbf{k} \in \mathcal{M}$ and $i = 1, 2, \ldots, 20$, we can define affine transformations $f_{\mathbf{k}}^{(i)} : \mathcal{X} \to \mathcal{X}$ as follows. For any $\mathbf{k} = (k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}) \in \mathcal{M}$, define

$$T = \begin{bmatrix} 0 & k^{(1)} & k^{(2)} & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & k^{(3)} & k^{(4)} & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & k^{(5)} & k^{(6)} & 1 \end{bmatrix}.$$

Let

$$I \quad = \quad \{(a, b, c) \mid 1 \leq a, b, c \leq 3 \text{ with } a, b, c \in \mathbb{Z}, \text{and no two of } a, b, c \text{ equal to } 2\}.$$

For each $(a,b,c) \in I$ and $\mathbf{k} \in \mathcal{M}$, define

$$M_{\mathbf{k}}(a,b,c) = \begin{bmatrix} 1-(T(a)+R(b)+S(c)) & T(a) & R(b) & S(c) \\ 1-(T(a+1)+R(b)+S(c)) & T(a+1) & R(b) & S(c) \\ 1-(T(a)+R(b+1)+S(c)) & T(a) & R(b+1) & S(c) \\ 1-(T(a)+R(b)+S(c+1)) & T(a) & R(b) & S(c+1) \end{bmatrix}.$$

Note that the set I contains 20 elements, so we can express it as

$$I = \{(a_i, b_i, c_i) \mid 1 \le i \le 20\}.$$

For each $\mathbf{k} \in \mathcal{M}$ and $1 \le i \le 20$, we consider the affine transformation $f_{\mathbf{k}}^{(i)} : \mathcal{X} \to \mathcal{X}$ given by

$$f_{\mathbf{k}}^{(i)}(O,A,B,C) = M_{\mathbf{k}}(a_i,b_i,c_i) \begin{bmatrix} O \\ A \\ B \\ C \end{bmatrix} \tag{5.7}$$

for every $(O,A,B,C) \in \mathcal{X}$. Note that for $i = 1,\dots,20$ and $\mathbf{k} \in \mathcal{M}$, $f_{\mathbf{k}}^{(i)}$ is a compression. Thus, we can define a marking $\mathcal{F} : \mathcal{M} \to \mathcal{C}_{20}(\mathcal{X})$ by sending $\mathbf{k} \mapsto f_{\mathbf{k}} = (f_{\mathbf{k}}^{(1)}, \dots, f_{\mathbf{k}}^{(20)})$. Using this, for any starting rectangular prism $E_0 = (O,A,B,C) \in \mathcal{X}$, we can generate a sequence of sets that follows a similar construction to the Menger Sponge.

Example 5.7. *Let*

$$E_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.8}$$

be the cube of unit side length and choose $\mathbf{k}_\ell \in \mathcal{M}$ to be the constant sequence $\mathbf{k}_\ell = \mathbf{k} = (1/3, 2/3, 1/3, 2/3, 1/3, 2/3)$. Then the $\mathcal{F}$-limit set F generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$ is the classical Menger sponge, as seen in Figure 11.

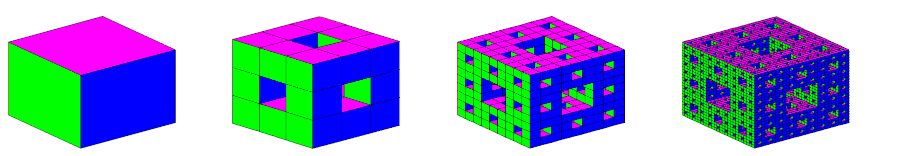


Figure 11. Constructing the Menger cube

Now we consider variations of Menger Sponge. For each $\mathbf{k} = (k^{(1)}, k^{(2)}, \cdots, k^{(6)}) \in \mathcal{M}$ and $1 \le i \le 20$,

$$\begin{aligned} U\left(f_{\mathbf{k}}^{(i)}\right) &= \sup_{(O,A,B,C)\in\mathcal{X}} \frac{diam\left(f_{\mathbf{k}}^{(i)}(O,A,B,C)\right)}{diam\left((O,A,B,C)\right)} \\ &= \sup_{(O,A,B,C)\in\mathcal{X}} \frac{diam\left(M_{\mathbf{k}}(a_i,b_i,c_i)[O,A,B,C]'\right)}{diam\left((O,A,B,C)\right)} \\ &= \max\{T(a_{i+1}) - T(a_i), R(b_{i+1}) - R(b_i), S(c_{i+1}) - S(c_i)\}. \end{aligned}$$

Similarly,

$$L\left(f_{\mathbf{k}}^{(i)}\right) = \min\{T(a_{i+1}) - T(a_i), R(b_{i+1}) - R(b_i), S(c_{i+1}) - S(c_i)\}.$$

When $k^{(2j)} = 1 - k^{(2j-1)}$ for each $j = 1, 2, 3$, it is easy to check that

$$\begin{aligned}\sum_{i=1}^{20} U(f_{\mathbf{k}}^{(i)})^s &= \sum_{i=1}^{20} \max\{T(a_{i+1}) - T(a_i), R(b_{i+1}) - R(b_i), S(c_{i+1}) - S(c_i)\}^s \\ &= 8\max\{k^{(1)}, k^{(3)}, k^{(5)}\}^s + 4\max\{1-2k^{(1)}, k^{(3)}, k^{(5)}\}^s \\ &\quad +4\max\{k^{(1)}, 1-2k^{(3)}, k^{(5)}\}^s + 4\max\{k^{(1)}, k^{(3)}, 1-2k^{(5)}\}^s.\end{aligned}$$

Example 5.8. *Let*

$$\tilde{E_0} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let $\left(k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}, k^{(5)}, k^{(6)}\right) \in \mathcal{M}$ *where each* $k^{(i)}$ *is a random number in* $[0, 1]$*, but still satisfying the condition* $k^{(1)} \le k^{(2)}, k^{(3)} \le k^{(4)}, k^{(5)} \le k^{(6)}$*. Then the first generation* $E^{(1)}$ *of the construction results in a set like Figure 12.*

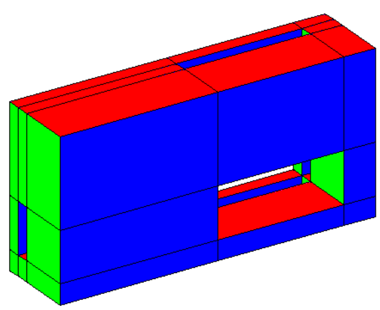
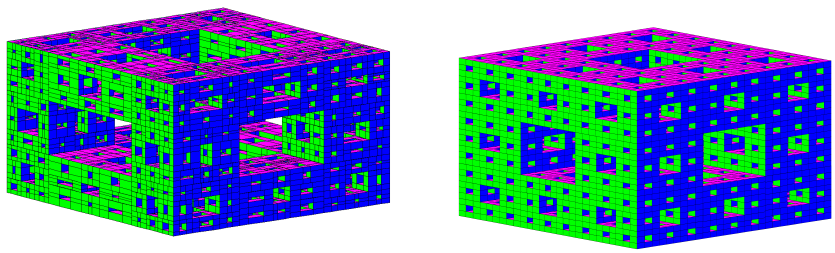


FIGURE 12. First generation of a randomly generated Menger sponge

Example 5.9. *Let* $\mathbf{k}_\ell = \left(k_\ell^{(1)}, k_\ell^{(2)}, k_\ell^{(3)}, k_\ell^{(4)}, k_\ell^{(5)}, k_\ell^{(6)}\right) \in \mathcal{M}$ *with each* $k_\ell^{(2j-1)}$ *a random number between given parameters* λ *and* Λ *and* $k_\ell^{(2j)} = 1 - k_\ell^{(2j-1)}$ *for each* $j = 1, 2, 3$*. Let* F *be the* $\mathcal{F}$*-limit set generated by* $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^{\infty}, E_0)$*. Then the third iteration of the construction of* F *results in images like Figure 13. Here, in Figure 13a the parameters* $\lambda = 0$ *and* $\Lambda = \frac{1}{2}$*, while in Figure 13b the parameters* $\lambda = 0.32$ *and* $\Lambda = 0.35$*.*

We now calculate the dimension of the limit fractal F illustrated by Figure 13b in Example 5.9. Note that in general, when $\lambda \le k^{(2j-1)} \le \Lambda$ for each $j = 1, 2, 3$, it follows that

$$(||\mathbf{U}_{\mathbf{k}}||_s)^s = \sum_{i=1}^{20} U\left(f_{\mathbf{k}}^{(i)}\right)^s \le 8\Lambda^s + 12\max\{1-2\lambda, \Lambda\}^s.$$



(A) $\lambda = 0, \quad \Lambda = \frac{1}{2}$ (B) $\lambda = 0.32, \Lambda = 0.35$

FIGURE 13. Generation 3 of random Menger sponge

Similarly,

$$(||\mathbf{L}_\mathbf{k}||_s)^s \geq 8\lambda^s + 12\min\{1-2\Lambda, \lambda\}^s.$$

In particular, when $\lambda = 0.32$ and $\Lambda = 0.35$, for any $s > 2.901$,

$$\begin{aligned}(||\mathbf{U}_\mathbf{k}||_s)^s &\leq 8\Lambda^s + 12\max\{1-2\lambda, \Lambda\}^s \leq 8 * 0.35^s + 12 * 0.36^s \\ &< 8 * 0.35^{2.901} + 12 * 0.36^{2.901} \approx 1.000.\end{aligned}$$

By Theorem 4.1, $\dim_H(F) \leq 2.901$. Similarly, for any $s \leq 2.546$,

$$\begin{aligned}(||\mathbf{L}_\mathbf{k}||_s)^s &\geq 8\lambda^s + 12\min\{1-2\Lambda, \lambda\}^s \\ &\geq 8 * 0.32^s + 12 * 0.3^s \geq 8 * 0.32^{2.546} + 12 * 0.3^{2.546} \approx 1.000.\end{aligned}$$

By Theorem 4.1 again, $\dim_H(F) \geq 2.546$, provided F satisfies the uniform covering condition (2.4). As a result,

$$2.546 \leq \dim_H(F) \leq 2.901.$$

Example 5.10. *For each $\ell \geq 0$, let $\mathbf{k}_\ell = \left(k_\ell^{(1)}, k_\ell^{(2)}, \cdots, k_\ell^{(6)}\right)$ where*

$$\begin{aligned}k_\ell^{(1)} &= \frac{1}{3} + \frac{(-1)^\ell}{12(\ell+1)^2}, \quad k_\ell^{(2)} = 1 - k_\ell^{(1)}, \\ k_\ell^{(3)} &= \frac{1}{3} - \frac{(-1)^\ell}{6(\ell+1)^2}, \quad k_\ell^{(4)} = 1 - k_\ell^{(3)}, \\ k_\ell^{(5)} &= \frac{1}{3} + \frac{(-1)^\ell}{18(\ell+1)^2}, \quad k_\ell^{(6)} = 1 - k_\ell^{(5)}.\end{aligned}$$

Let F be the $\mathcal{F}$-limit set generated by $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^\infty, E_0)$. Then the third generation of the construction of F leads to an image like Figure 14.

In this case, we can still calculate the exact Hasudorff dimension of F. By direct computation,

$$\lim_{\ell\to\infty}(||\mathbf{U}_{\mathbf{k}_\ell}||_s)^s = \frac{20}{3^s} = \lim_{\ell\to\infty}(||\mathbf{L}_{\mathbf{k}_\ell}||_s)^s.$$

Thus, by Corollary 4.2, $\dim_H(F) = \frac{\log(20)}{\log(3)} \approx 2.7268$, since F satisfies the uniform covering condition according to Example 6.3.

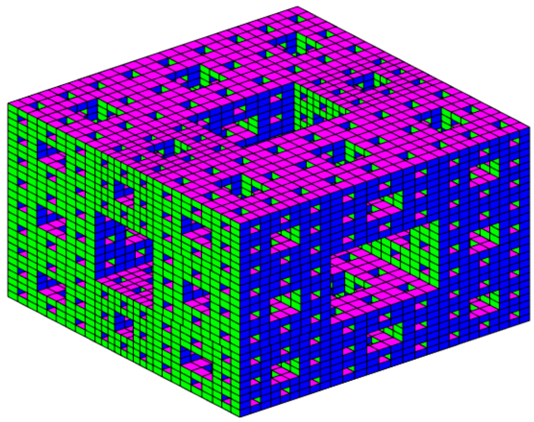


FIGURE 14. Generation 3 of random Menger sponge with controlled dimension

Remark 5.1. *Here, we discuss similarities and differences of this construction with $V-$variable fractals created by Barnsley, Hutchinson, and Stenflo in* [2], [3]*. These authors have described a similar approach to creating more generalized fractals that can take on a prescribed amount of randomness. In* [2] *and* [3]*, they describe a generating process for some fractals along with calculations of their dimensions. In essence, a V-variable fractal set has at most $V \in \mathbb{N}$ number of distinct patterns in each generation of the construction. This is done through the following process.*

Let (X, d) be a metric space, Λ an index set, $F^\lambda = \{f_1^\lambda, f_2^\lambda, \ldots, f_m^\lambda\}$ an IFS for each $\lambda \in \Lambda$, and P a probability distribution on some σ-algebra of subsets of Λ. Then denote $\mathbf{F} = \{(X, d), F^\lambda, \lambda \in \Lambda, P\}$ to be a family of IFSs (with at least two functions in each IFS) defined on (X, d). Assume that the IFSs F^λ are uniformly contractive and uniformly bounded, that is, for some $0 < r < 1$,

$$\sup_\lambda \max_m d\left(f_m^\lambda(x), f_m^\lambda(y)\right) \leq r d(x, y), \tag{5.9}$$

$$\sup_\lambda \max_m d\left(f_m^\lambda(a), a\right) < \infty \tag{5.10}$$

for all $x, y \in X$ and some $a \in X$.

A tree code is a map ω from the set of all finite sequences $\{1, \ldots, m\}$ to Λ. A tree code is V-variable if for each positive integer k, there are at most V distinct tree codes in the tree truncated at the k^{th} generation. For example, consider the Sierpinski triangle. We let F be the IFS that maps the triangle to three copies of 1/2 the size, as usual. Let G be the IFS that maps the initial triangle to three triangles that are 1/3 the size, with the vertices shared with the initial set being the fixed points of the maps. See Figure 15 for the image of the initial step of each. Thus, $\mathbf{F} = \{(\mathbb{R}^2, d), \{F, G\}, P = (1/2, 1/2)\}$ is the family $\{F, G\}$ with probability function uniformly choosing 1/2 for each IFS. Using these IFSs, three V-variable pre-fractals are given in Figure 16, being 1-variable, 2-variable, and 3-variable respectively.

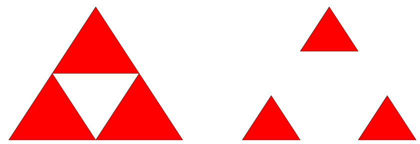


FIGURE 15. Initial steps of IFSs F and G respectively

Now, we express V-variable fractals in terms of $\mathcal{F}$-limit sets. Let $\mathcal{X}, \mathcal{M}, \mathcal{F}$ and E_0 be as in Section 5.2. Let $F = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \in \mathcal{M}$ and $G = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in \mathcal{M}$. We will use F and G to denote terms in the sequence $\{k_\ell\}_{\ell=1}^{\infty}$. Consider the third generation examples in Figure 16. Then from left to right we have the following:

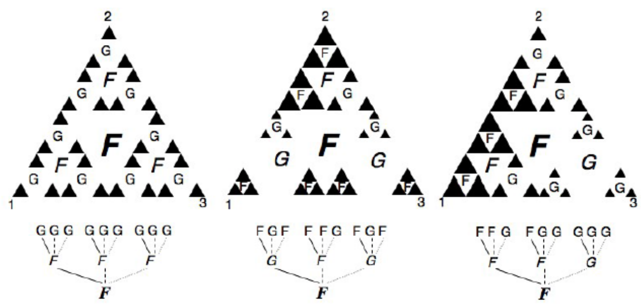


FIGURE 16. $n = 3$ generation prefractals that are 1, 2 and 3-variable respectively. Images from [3].

$$\begin{aligned} V = 1 \qquad \{k_\ell\}_{\ell=1}^{13} &= \{F, F, F, F, G, G, G, G, G, G, G, G, G\} \\ V = 2 \qquad \{k_\ell\}_{\ell=1}^{13} &= \{F, G, F, G, F, G, F, F, F, G, F, G, F\} \\ V = 3 \qquad \{k_\ell\}_{\ell=1}^{13} &= \{F, F, F, G, F, F, G, F, G, G, G, G, G\}. \end{aligned}$$

From these examples, we can see that if we want to create a V-variable fractal, for each generation we should choose at most V distinct triples from the set $\{(A, B, C) | A, B, C \in \{F, G\}\}$ and repeat those triples in any order.

When $V < \infty$, there are at most V distinct tree codes in the address of point in the set. We can create such a situation from our construction described earlier in section 3 by choosing blocks of $\{k_\ell\}$ that repeat across generations. In the case that $V = \infty$, the fractal is based off of a probability distribution for applying specific IFSs. In our construction we also can use a probability distribution to determine the contraction ratios within a generation (as in Examples 5.2, 5.4 and 5.8), but we do not require such a choice. We allow for deterministic sequences that also do not repeat any blocks, thus not falling into the category of V-variable.

6. Uniform Covering Condition

In previous sections, we have seen that the uniform covering condition (2.4) plays a vital role in computing a lower estimate for the Hausdorff dimension of a fractal. In this section we explore the sufficient conditions needed for a fractal to satisfy the uniform covering condition.

Proposition 6.1. *Let (X,d) be a metric space with the following property: For any $\epsilon > 0$, there exists a natural number N_ϵ such that for any $\rho > 0$, any closed ball in X of diameter ρ contains at most N_ϵ many disjoint balls of diameter $\epsilon\rho$. Clearly, any Euclidean space satisfies this property.*

Let $\mathcal{J} := \{J_\sigma : \sigma \in D\}$ be a collection of compact subsets of (X,d), and F be the limit set of $\mathcal{J}$ as given in (1.4). Suppose that $\mathcal{J}$ satisfies the following conditions:

(1) *there exists a number $r \in (0,1]$ such that for any $k \in \mathbb{N}$ and for each $\sigma \in D_k$,*
$$rc_k \le diam(J_\sigma) \le \frac{c_k}{r}$$
where $c_k := \min\{diam(J_{\bar\sigma}) : \bar\sigma \in D_{k-1}\}$.

(2) *there exists a number $\tau \in (0,1]$ such that for each $\sigma \in D$, the convex hull of J_σ contains a closed ball W_σ such that*
$$diam(W_\sigma) \ge \tau \cdot diam(J_\sigma)$$
and for each $k \in \mathbb{N}$, the collection $\{W_\sigma : \sigma \in D_k\}$ are pairwise disjoint.

Then F satisfies the uniform covering condition (2.4).

Proof. For any closed ball B in X, let k be the number such that
$$c_{k+1} \le diam(B) < c_k$$
where by convention, we set $c_0 = \infty$. Let
$$D_B := \{\sigma \in D_k : B \cap F \cap J_\sigma \ne \emptyset\}.$$
Note that
$$B \cap F = B \cap F \cap \bigcup_{\sigma\in D_k} J_\sigma \subseteq \bigcup_{\sigma \in D_B} J_\sigma.$$
Also for any $\sigma \in D_B$, since $diam(J_\sigma) \le \frac{c_k}{r}$ and $B \cap J_\sigma \ne \emptyset$, it follows that $J_\sigma \subseteq \bar{B}(x_0, \frac{r+2}{2r}c_k)$, where $x_0 \in X$ is the center of the ball B. Thus, $W_\sigma \subseteq \bar{B}(x_0, \frac{r+2}{2r}c_k)$. Let $\rho = \frac{r+2}{r}c_k$ and $\epsilon = \frac{r^2}{r+2}\tau$, then
$$diam(W_\sigma) \ge \tau \cdot diam(J_\sigma) \ge \tau r c_k = \epsilon\rho.$$
Since $\{W_\sigma : \sigma \in D_B\}$ are pairwise disjoint, the cardinality of D_B is at most $N := N_\epsilon$. On the other hand, for $\gamma = \frac{r^2}{N}$, it holds that
$$diam(B) \ge c_{k+1} \ge rc_k = \gamma N \frac{c_k}{r} \ge \gamma \sum_{\sigma\in D_B} \frac{c_k}{r} \ge \gamma \sum_{\sigma \in D_B} diam(J_\sigma). \tag{6.1}$$
As a result, $\mathcal{J}$ satisfies the condition (2.4) as desired. □

We now discuss some specific sufficient conditions concerning the types of examples provided in section 3.4. To start, let's first consider Cantor-like constructions. Let X be the family of closed intervals described in (5.1), $m = 2$, and $\mathcal{M} = [0,1]^2 \subseteq \mathbb{R}$.

Proposition 6.2. *Let $\{\mathbf{k}_\ell\}_{\ell=0}^\infty$ be a sequence in $\mathcal{M}$ with*
$$\sup\left\{k_\ell^{(1)} + k_\ell^{(2)} : \ell = 0,1,2,\cdots\right\} < 1,$$
and F be the $\mathcal{F}$-limit set generated by the triple $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^\infty, E_0)$. Then F satisfies the uniform covering condition (2.4).

Proof. Let $N = 1$ and

$$\gamma = \inf_{\ell} \left\{ 1 - k_{\ell}^{(1)} - k_{\ell}^{(2)} \right\} \in (0, 1]. \tag{6.2}$$

For any closed interval B in $\mathbb{R}$ with $B \cap F \neq \emptyset$, consider the set

$$\mathcal{L} := \{\ell(\sigma) : B \cap F \subseteq J_\sigma, \sigma \in D\},$$

where $\ell(\sigma)$ is given in (4.3). Note that $\mathcal{L}$ is nonempty because $B \cap F \subseteq J_0$ implies that $\ell(0) \in \mathcal{L}$. If $\mathcal{L}$ is an infinite set, then since $diam(J_\sigma) \to 0$ as $\ell(\sigma) \to \infty$, there exists $\sigma^* \in D$ such that $\ell(\sigma^*) \in \mathcal{L}$ and $diam(B) \geq diam(J_{\sigma^*}) \geq \gamma \cdot diam(J_{\sigma^*})$. If $\mathcal{L}$ is finite, let $\ell(\sigma^*)$ be the maximum number in $\mathcal{L}$ for some $\sigma^* \in D$. Then, $\ell(\sigma^*) \in \mathcal{L}$ but $\ell(\sigma^* * j) \notin \mathcal{L}$ for each $j = 1, 2$. This implies that $B \cap J_{\sigma^* * j} \neq \emptyset$ for both $j = 1, 2$ because $J_{\sigma^*} = J_{\sigma^* * 1} \cup J_{\sigma^* * 2}$. Since B is an interval, the gap $J \setminus (J_{\sigma^* * 1} \cup J_{\sigma^* * 2})$ between $J_{\sigma^* * 1}$ and $J_{\sigma^* * 2}$ is contained in B, which yields that

$$\begin{aligned} diam(B) &\geq diam\left(J \setminus (J_{\sigma^* * 1} \cup J_{\sigma^* * 2})\right) \\ &= diam(J) - diam(J_{\sigma^* * 1}) - diam(J_{\sigma^* * 2}) \\ &\geq diam(J_{\sigma^*}) \left(1 - k_{\ell(\sigma^*)}^{(1)} - k_{\ell(\sigma^*)}^{(2)}\right) \geq \gamma \cdot diam(J_{\sigma^*}). \end{aligned}$$

As a result, in both cases, the uniform covering condition (2.4) holds. □

Motivated by Proposition 6.2, we now consider a generalization of the above result.

Definition 6.1. *Let $n \geq 1$ and $\mathcal{H}$ be a collection of subsets of a metric space (X, d). Define*

$$\rho_n(\mathcal{H}) = \inf\{r : \textit{There exists a ball } B \textit{ in } X \textit{ of radius } r \textit{ that intersects at least } n+1 \textit{ elements in } \mathcal{H}\}. \tag{6.3}$$

Here $\rho_n(\mathcal{H})$ is a quantity describing the "gap" between $n+1$ elements of $\mathcal{H}$.

Definition 6.2. *Let $\mathcal{J} = \{J_\sigma : \sigma \in D\}$ be a collection of compact subsets of a metric space (X, d), and $n \geq 1$. Define*

$$\gamma_n(\mathcal{J}) := \inf \left\{ \frac{\rho_n(\{J_{\sigma * i} : \sigma \in R_k, i = 1, 2, \cdots, m\})}{\sum_{\sigma \in R_k} diam(J_\sigma)} : \textit{ for some } k \textit{ and } R_k \subseteq D_k \textit{ with } 1 \leq |R_k| \leq n \right\}, \tag{6.4}$$

where $|R_k|$ denotes the cardinality of the set R_k. Here $\gamma_n(\mathcal{J})$ is a quantity describing the relative size of the "gap" between $n+1$ children of a generation and the size of the parent sets.

Now we give some examples of calculations of these two quantities.

Example 6.1. *Let $\mathcal{J}$ be the collection of closed intervals used in the construction of a Cantor-like set given in (5.1). Then*

$$\begin{aligned}\gamma_1(\mathcal{J}) &= \inf\left\{\frac{\rho_1(\{J_{\sigma*i} : \sigma \in R_k, i=1,2\})}{diam(J_\sigma)} : \text{ for some } k \text{ and } R_k \subseteq D_k \text{ with } |R_k| = 1\right\}\\ &= \inf\left\{\frac{\rho_1(\{J_{\sigma*1}, J_{\sigma*2}\})}{diam(J_\sigma)} : \text{ for } \sigma \in D\right\}\\ &= \inf\left\{\frac{diam(J_\sigma) - diam(J_{\sigma*1}) - diam(J_{\sigma*2})}{diam(J_\sigma)} : \sigma \in D\right\}\\ &= \inf\left\{1 - \frac{diam(J_{\sigma*1})}{diam(J_\sigma)} - \frac{diam(J_{\sigma*2})}{diam(J_\sigma)} : \sigma \in D\right\},\end{aligned}$$

which agrees with the γ in (6.2), see Figure 17.

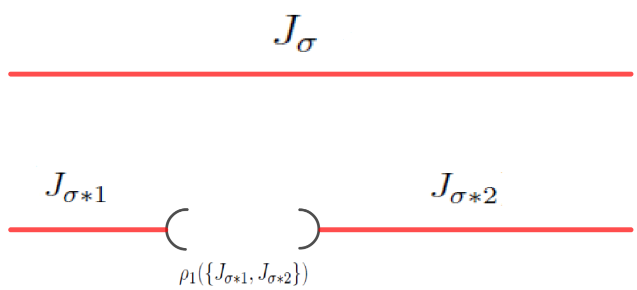


FIGURE 17. Illustration of $\rho_1(\{J_{\sigma*1}, J_{\sigma*2}\})$

Example 6.2. *Let $\mathcal{J}$ be the collection of triangles used in (5.3). In the following figures, we plot the smallest ball that intersects a certain number of children. The children that have non-empty intersection with the ball are colored red, while those that have empty intersection are light blue.*

First note that for any $\sigma \in D$, $\rho_1(\{J_{\sigma*1}, J_{\sigma*2}, J_{\sigma*3}\}) = 0$ since any pair of children share a vertex. At the intersection of the two children of J_σ one can construct a ball of arbitrarily small radius. See Figure 18.

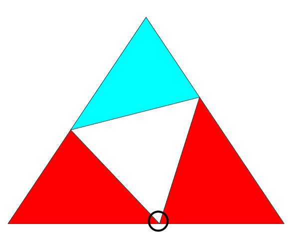


FIGURE 18. Illustration of $\rho_1(\{J_{\sigma*1}, J_{\sigma*2}, J_{\sigma*3}\}) = 0$

Moreover, $\rho_2(\{J_{\sigma*1}, J_{\sigma*2}, J_{\sigma*3}\}) > 0$ *because the radius of any ball that intersects all three children of* J_σ *is bounded below by the radius of the inscribed circle of the removed center triangle. In other words,* $\rho_2(\{J_{\sigma*1}, J_{\sigma*2}, J_{\sigma*3}\})$ *is equal to the radius of the inscribed circle. See Figure 19 for illustration.*

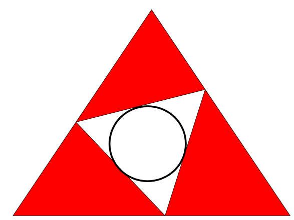


FIGURE 19. $\rho_2(\{J_{\sigma*1}, J_{\sigma*2}, J_{\sigma*3}\})$ = radius of inscribed circle

Now we may compute $\gamma_n(\mathcal{J})$ *as follows.*

Note that for $n = 1$*,*

$$\begin{aligned}\gamma_1(\mathcal{J}) &= \inf\left\{\frac{\rho_1(\{J_{\sigma*i} : \sigma \in R_k, i = 1,2,3\})}{diam(J_\sigma)} : \textit{ for some } k \textit{ and } R_k \subseteq D_k \textit{ with } |R_k| = 1\right\}\\ &= \inf\left\{\frac{\rho_1(\{J_{\sigma*1}, J_{\sigma*2}, J_{\sigma*3}\})}{diam(J_\sigma)} : \textit{ for } \sigma \in D\right\} = 0.\end{aligned}$$

On the other hand, when $n = 2$*, we have*

$$\gamma_2(\mathcal{J}) = \inf\left\{\frac{\rho_2(\{J_{\sigma*i} : \sigma \in R_k, i = 1,2,3\})}{diam(J_\sigma)} : \textit{ for some } k \textit{ and } R_k \subseteq D_k \textit{ with } |R_k| \le 2\right\}.$$

When $|R_k| = 1$*, this is reduced to the same case as Figure 19.*

When $|R_k| = 2$*, we use two parent triangles, and must find the ball with smallest radius that intersects three or more children. See Figure 20 for a few candidates for the ball with smallest radius.*

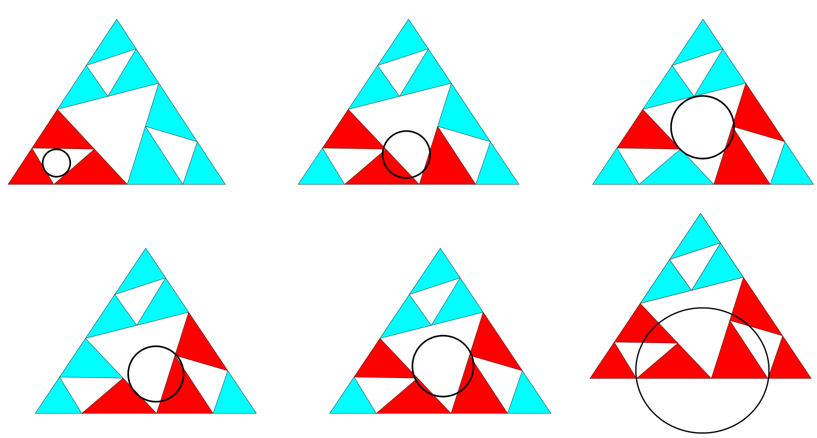


FIGURE 20. Various options for smallest radius ball

For each $R_k \subseteq D_k$ with $|R_k| \leq 2$, $\rho_2(\{J_{\sigma * i} : \sigma \in R_k, i = 1, 2, 3\}) > 0$. For some nice $\mathcal{J}$, one may expect $\gamma_2(\mathcal{J})$ to also be positive.

Theorem 6.3. *Let $\mathcal{J} := \{J_\sigma : \sigma \in D\}$ be a collection of compact subsets of (X, d) satisfying MSC(3) and*

$$\lim_{k\to\infty} \max \{diam(J_\sigma) : \sigma \in D_k\} = 0,$$

and let F be the limit set of $\mathcal{J}$ as given in (1.4). If there exists an N such that $\gamma_N(\mathcal{J}) > 0$, then F satisfies the uniform covering condition (2.4).

Proof. Let $\gamma = \gamma_N(\mathcal{J}) > 0$. For any closed ball B in X with $B \cap F \neq \emptyset$, let $g(k)$ be the number of elements σ in D_k such that $B \cap F \cap J_\sigma \neq \emptyset$. Then $g : \mathbb{N} \cup \{0\} \to \mathbb{N}$ is monotone increasing with $g(0) = 1$.

Case 1: If $g(k) \leq N$ for all $k = 0, 1, 2, \cdots$, that is, for each k, there exists an index set I_k with $|I_k| \leq N$ such that $B \cap F \subseteq \bigcup_{i\in I_k} J_{\sigma_i^{(k)}}$ for some $\sigma_i^{(k)} \in D_k$. Thus when k is large enough,

$$diam(B) > \gamma \cdot \sum_{i\in I_k} diam(J_{\sigma_i^{(k)}})$$

due to the fact that

$$0 \leq \lim_{k\to\infty} \sum_{i\in I_k} diam(J_{\sigma_i^{(k)}}) \leq N \cdot \lim_{k\to\infty} \max\{diam(J_\sigma) : \sigma \in D_k\} = 0.$$

Hence, equation (2.4) holds for B.

Case 2: There exists $k^* \geq 0$ such that $g(k^*) \leq N$ but $g(k^* + 1) > N$.

Since $g(k^*) \leq N$, there are $g(k^*)$ many elements $\sigma \in D_{k^*}$ such that $B \cap F \cap J_\sigma \neq \emptyset$. That is, there exists $R_{k^*} \subseteq D_{k^*}$ with $|R_{k^*}| \leq N$ such that $B \cap F \subseteq \bigcup_{\sigma\in R_{k^*}} J_\sigma$. On the other hand, since $g(k^* + 1) > N$, $B \cap F$ intersects at least $N + 1$ elements of D_{k^*+1}. Since $B \cap F \subseteq \bigcup_{\sigma\in R_{k^*}} J_\sigma$, all of these $N + 1$ elements must be children of $\{J_\sigma : \sigma \in R_{k^*}\}$. Then, by the definition of ρ_N in (6.3),

$$diam(B) \geq \rho_N(\{J_{\sigma*i} : \sigma \in R_{k^*}, i = 1, 2, \ldots m\}) \geq \gamma \cdot \sum_{\sigma\in R_{k^*}} diam(J_\sigma). \tag{6.5}$$

As a result, F satisfies the uniform covering condition (2.4). □

To show an application of Theorem 6.3, we now consider some examples provided in section 5.3. Let $\{\mathbf{k}_\ell\}_{\ell=0}^\infty$ be a sequence in $\mathcal{M}$ as defined in (5.6) and F be the $\mathcal{F}-$limit set generated by $(\mathcal{F}, \{\mathbf{k}_\ell\}_{\ell=0}^\infty, E_0)$ associated with $\mathcal{J}(\vec{k}) = \{J_\sigma : \sigma \in D\}$ as defined in Definition 3.3. Let $\mathcal{H} \subseteq \mathcal{J}_k := \{J_\sigma : \sigma \in D_k\}$ for some $k \geq 0$, and consider $\rho_8(\mathcal{H})$.

We now make the following observation: *Suppose there exists a ball B that intersects at least 9 elements of $\mathcal{H}$. Then $diam(B)$ is greater than or equal to the smallest edge length of the elements in $\mathcal{H}$.* Indeed, by considering the projections to the three coordinate axes, one can see that at least one coordinate contains three non-identical projected images of these 9 elements. As a result the ball B intersected with these 9 elements will have a diameter at least the length of the smallest side of the three projected images. This proves our observation.

Let

$$m_\ell = \min\{k_\ell^{(1)}, k_\ell^{(2)} - k_\ell^{(1)}, 1 - k_\ell^{(2)}, k_\ell^{(3)}, k_\ell^{(4)} - k_\ell^{(3)}, 1 - k_\ell^{(4)}, k_\ell^{(5)}, k_\ell^{(6)} - k_\ell^{(5)}, 1 - k_\ell^{(6)}\} \tag{6.6}$$

and
(6.7)

$$M_\ell = \max\{k_\ell^{(1)}, k_\ell^{(2)} - k_\ell^{(1)}, 1 - k_\ell^{(2)}, k_\ell^{(3)}, k_\ell^{(4)} - k_\ell^{(3)}, 1 - k_\ell^{(4)}, k_\ell^{(5)}, k_\ell^{(6)} - k_\ell^{(5)}, 1 - k_\ell^{(6)}\}.$$

For any $\sigma \in D$, direct calculation shows that

$$m_{\ell(\sigma)} \leq \frac{diam(J_{\sigma * i})}{diam(J_\sigma)} \leq M_{\ell(\sigma)} \tag{6.8}$$

where $\ell(\sigma)$ is given in (4.3). Thus, for any $\sigma = (i_1, i_2, \dots, i_k) \in D_k$, we have
(6.9)

$$m_{\ell((i_1))} m_{\ell((i_1,i_2))} \cdots m_{\ell((i_1,\dots,i_k))} \leq \frac{diam(J_\sigma)}{diam(J_\emptyset)} \leq M_{\ell((i_1))} M_{\ell((i_1,i_2))} \cdots M_{\ell((i_1,\dots,i_k))}.$$

Let $R_k \subseteq D_k$ for some k. Suppose $|R_k| \leq 8$. Then for any $\sigma \in R_k$, by the observation

$$\begin{aligned}
&\frac{\rho_8(\{J_{\sigma*i} : \sigma \in R_k, i = 1, 2, \dots 20\})}{\sum_{\sigma \in R_k} diam(J_\sigma)} \\
\geq\ & \frac{\text{smallest diameter of } J_{\sigma * i}}{8 \cdot \max\{diam(J_\sigma) : \sigma \in R_k\}} \\
\geq\ & \min_{\sigma = (i_1, i_2, \dots, i_k) \in R_k, i_{k+1} = 1, \dots, 20} \left\{ \frac{m_{\ell((i_1))} m_{\ell((i_1,i_2))} \cdots m_{\ell((i_1,\dots,i_{k+1}))} diam(J_\emptyset)}{M_{\ell((i_1))} M_{\ell((i_1,i_2))} \cdots M_{\ell((i_1,\dots,i_k))} diam(J_\emptyset)} \right\} \\
\geq\ & \frac{1}{8} \left(\prod_{i=1}^{\infty} \frac{m_i}{M_i} \right) \liminf_{i \to \infty} m_i,
\end{aligned}$$

where the last inequality follows from $0 \leq m_i \leq M_i$ for each i.

Example 6.3. *Using this observation, we show that the $\mathcal{F}-$limit set in Example 5.10 satisfies the uniform covering condition. In this example,*

$$m_\ell = \begin{cases} a_\ell & \ell \text{ even} \\ b_\ell & \ell \text{ odd} \end{cases} \text{ and } M_\ell = \begin{cases} b_\ell & \ell \text{ even} \\ a_\ell & \ell \text{ odd} \end{cases} \tag{6.10}$$

where

$$a_\ell = k_\ell^{(3)} = \frac{1}{3} - \frac{(-1)^\ell}{6(\ell+1)^2} \text{ and } b_\ell = 1 - 2k_\ell^{(3)} = \frac{1}{3} + \frac{(-1)^\ell}{3(\ell+1)^2}. \tag{6.11}$$

One may show that the product $\prod_{i=1}^{\infty} \frac{m_i}{M_i}$ is convergent, whose numerical value is 0.369761… and $\liminf_{i\to\infty} m_i = 1/3$. Thus $\gamma_8(\mathcal{J}) > 0$. Therefore, by Theorem 6.3, F satisfies the uniform covering condition.

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