Calc. Var. 19, 421–443 (2004)
Calculus of Variations
DOI: 10.1007/s00526-003-0222-0

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Intersection homology theory via rectifiable currents

Received: 21 April 2001 / Accepted: 15 April 2003 /
Published online: 1 July 2003 –

Abstract. Here is given a rectifiable currents' version of intersection homology theory on stratified subanalytic pseudomanifolds. This new version enables one to study some variational problems on stratified subanalytic pseudomanifolds. We first achieve an isomorphism between this rectifiable currents' version and the version using subanalytic chains. Then we define a suitably modified mass on the complex of rectifiable currents to ensure that each sequence of subanalytic chains with bounded modified mass has a convegent subsequence and the limit rectifiable current still satisfies the perversity condition of the approximating chains. The associated mass minimizers turn out to be almost minimal currents and this fact leads to some regularity results.

Introduction

The goal of this paper is to develop a setting for treating variational problems on stratified pseudomanifolds with singularities, such as complex projective varieties. Rather than using the ordinary homology theory on the base space, we will instead use a generalized "homology theory " – the intersection homology theory, introduced by MacPherson and Goresky in the early 1980's ([GM1,GM2]). Such a theory turns out to be more suitable than ordinary homology theory for pseudomanifolds with singularities (see [Kirwan] or [Borel] for details).

In variational problems, one needs to take various limits of e.g. minimizing sequences, but a basic problem is that a limit of geometric intersection chains [GM1] may fail to be a geometric chain; and even if it is, it may not satisfy the important perversity conditions of the approximating chains concerning intersection with singular set. This motivates our use of rectifiable currents with a suitably modified mass norm.

In Sect. 1, we present some necessary preliminaries on the categories of intersection homology theory, geometric measure theory, and subanalytic sets and chains.

In Sect. 2, for a compact stratified subanalytic pseudomanifold, we show how to express the intersection homology groups in terms of integer multiplicity rectifiable currents. These are then isomorphic to the usual intersection homology groups defined by geometric or subanalytic chains with the corresponding perversity conditions. The key idea here involves a technical modification of the proof of

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the Federer-Fleming's Deformation Theorem [Simon, Sect. 29] to accommodate the perversity condition of intersection homology theory. We study properties of a "*safety* function" used to quantify the perversity condition for each simplex of the singular locus.

In Sect. 3, we give a suitably modified mass on rectifiable currents such that all rectifiable currents with finite mass and finite boundary mass satisfy automatically the given perversity conditions. Also, by using the Lojasiewicz' inequality of subanalytic sets, we're able to show that all allowable subanalytic chains have finite (modified) mass and finite boundary mass. This fact ensures that our category of rectifiable currents with finite modified mass is still rich enough to contain all the "nice" chains one may consider. Moreover, this modified mass satisfies an important theorem – an analogue of the compactness theorem of Geometric Measure theory (see [F1, Simon]), which implies that each sequence of rectifiable currents with bounded modified mass and boundary mass will have a convergent subsequence and that the limit is a rectifiable current satisfying the perversity conditions of the approximating chains. This property of rectifiable currents overcomes the weakness of geometric chains stated above in the basic problem. The support of the currents we consider may intersect (in a controlled fashion) the singular locus of the pseudomanifold. A related problem with currents avoiding the vertex of a regular cone was studied by [Li].

In Sect. 4, we first achieve an existence theorem for modified mass minimizers. Moreover, we show that these mass minimizers are in fact almost minimizing currents [Bomb]. Thus, by a lemma of Almgren, we achieve a partial regularity theorem for these suitable mass minimizers.

We will, for notational convenience and clarity, restrict to manifolds and pseudomanifolds in $\mathbb{R}^N$, although most of our results carry over to more general contexts.

Acknowledgement. The author wants to thank his thesis advisor Robert Hardt for his uncountable help, fruitful discussions and endless encouragement.

1 Backgrounds

1.1 Intersection homology groups $IH^{\bar{p}}_{\cdot}(X)$

As in [Borel, Sect. 1.1], let X be a *topological stratified pseudomanifold* of dimension n with singular locus Σ and a given stratification

$$X = X_n \supset X_{n-2} (= \Sigma) \supset X_{n-3} \supset \cdots \supset X_0.$$

For a triangulation T of X, compatible with the stratification, let $C_*^T(X)$ be the complex of simplicial chains of T. Then an element $\xi \in C_i^T(X)$ is a linear combination

$$\xi = \sum_{\sigma \in T^{(i)}} \xi_\sigma \sigma, \ \xi_\sigma \in \mathbb{Z}.$$

For $\xi \in C_i^T(X)$, define $|\xi|$ (the *support* of ξ) to be the union of the closures of those i-simplices σ for which the coefficient of σ in ξ is non-zero. The complex

$C_i(X)$ of all *geometric chains* of X with integer coefficients is the direct limit of $C_*^T(X)$ under refinement over all such triangulations of X.

Let $\bar{p} = (p_2, p_3, \cdots, p_n)$, called the *perversity*, be any fixed nonnegative integers satisfying $p_2 = 0$, and $p_k \le p_{k+1} \le p_k + 1$ for all $2 \le k \le n$. A geometric chain ξ is said to be $(\bar{p}, i)$ *allowable* if

$$dim_{\mathbb{R}}(|\xi| \cap X_{n-k}) \le i - k + p_k, \text{ for all } k \ge 2.$$

The group $IC_i^{\bar{p}}(X)$ of intersection chains of dimension i and perversity $\bar{p}$ is the subgroup of geometric chains $\xi \in C_i(X)$ such that ξ is $(\bar{p}, i)$ allowable and $\partial\xi$ is $(\bar{p}, i-1)$ allowable.

Definition 1.1.1 *The intersection homology groups $IH_\cdot^{\bar{p}}(X)$ are defined to be the homology groups of the chain complex $IC_\cdot^{\bar{p}}(X)$.*

This definition is independent of the choice of the stratification. See [GM2]

1.2 Geometric measure theory

Let $U \subset \mathbb{R}^N$ be an open subset and $\mathcal{D}^i(U)$ be the set of all differential i-forms in U with compact support.

An i dimensional *current* T on U is a continuous linear functional on $\mathcal{D}^i(U)$. Let $\mathcal{D}_i(U)$ denote the set of all i dimensional currents on U (see [F1] or [Simon] for more details).

Given a sequence $\{T_j\} \in \mathcal{D}_i(U)$, we write $T_j \rightharpoonup T$ in U if $\{T_j\}$ converges *weakly* to $T \in \mathcal{D}_i(U)$ in the usual sense of distributions:

$$T_j \rightharpoonup T \Longleftrightarrow \lim T_j(\omega) = T(\omega), \forall \omega \in \mathcal{D}^i(U).$$

Given $T \in \mathcal{D}_i(U)$, the *support* of T is defined by

$$sptT = U \setminus \cup \{\, V \subset U \text{ open} : \text{spt}(\omega) \subset V \Rightarrow T(\omega) = 0\}.$$

The *mass* function on $\mathcal{D}_i(U)$ is defined by

$$M(T) = sup_{||\omega|| \le 1,\, \omega \in \mathcal{D}^i(U)} T(\omega).$$

More generally, for any open $W \subset U$, we define

$$M_W(T) = sup_{||\omega|| \le 1, \omega \in \mathcal{D}^i(U), spt\omega \subset W} T(\omega).$$

Clearly, $M(T)$ is lower semicontinuous under the weak convergence of currents.

An *integer multiplicity rectifiable current* T is a current coming from an oriented rectifiable set with integer multiplicities (see [F1] or [Simon]). Let $\mathcal{R}_i\left(\mathbb{R}^N\right)$ be the set of all i dimensional integer multiplicity rectifiable currents in $\mathbb{R}^N$ and for any subset $X \subset \mathbb{R}^N$, let

$$\mathcal{R}_i(X) = \left\{T \in \mathcal{R}_i\left(\mathbb{R}^N\right) \mid spt\,(T) \subset X\right\}.$$

1.3 Subanalytic sets and chains

According to [Hironaka, proposition 6.11], a subset A of a real-analytic space X is *subanalytic* if at any point $a \in A$, there exists an open neighborhood U of a in X, a real analytic manifold Y and a finite system of proper real-analytic maps $g_{ij} : Y \to U, 1 \leq i \leq p$ and $j = 1, 2$, such that $A \cap U = \cup_{i=1}^{p} (Im(g_{i1}) - Im(g_{i2}))$.

Examples of subanalytic subsets of $\mathbb{R}^N$:

(1)analytic varieties;

(2)polyhedrons;

(3)finite unions, intersections, proper projections of subanalytic subsets.

A really important inequality about a subanalytic set is the following (see [Hironaka, 9.5]):

Proposition 1.3.1 (Lojasiewicz' inequality) *Let f be a function on $\mathbb{R}^n$ with subanalytic graph. Then for each compact subset K of $\mathbb{R}^n$, we can find $N \in \mathbb{Z}_+$ and $C \in \mathbb{R}_+$ such that for all $x \in K$,*

$$C\,|f(x)| \geq dist_{\mathbb{R}^n}(x, f^{-1}(0))^N.$$

Definition 1.3.2 *An integer multiplicity rectifiable current $T \in \mathcal{R}_i(\mathbb{R}^N)$ is a subanalytic chain if spt(T) and spt(∂T) are i and $i-1$ dimensional subanalytic subsets of $\mathbb{R}^N$. That is, T is a geometric chain where all the supporting simplices are subanalytic sets.*

Definition 1.3.3 *A stratified subanalytic pseudomanifold X of dimension n in $\mathbb{R}^N$ is a subanalytic pseudomanifold with a stratification*

$$X = X_n \supset X_{n-2}(= \Sigma) \supset X_{n-3} \supset \cdots \supset X_0$$

with closed subanalytic subsets X_{n-k} of $\mathbb{R}^N$ such that $X_{n-k} \setminus X_{n-k-1}$ is empty or a subanalytic manifold of dimension $n-k$ and such that the local normal triviality holds in the subanalytic category. See [Hardt3] for details.

1.4 General setup

For the rest of the paper, we let $X \subset \mathbb{R}^N$ be a compact stratified subanalytic pseudomanifold with singular set Σ and a given stratification

$$X = X_n \supset X_{n-2}(= \Sigma) \supset X_{n-3} \supset \cdots \supset X_0.$$

Also, let $\bar{p}$ be a fixed perversity function (as in Sect. 1.1). A rectifiable current $T \in \mathcal{R}_i(X)$ is said to be $(\bar{p}, i)$ *allowable* if

$$\dim(spt\,(T) \cap X_{n-k}) \leq i - k + p_k, \text{ for all } k \geq 2.$$

Finally, we fix an integer $i \in \{0, 1, \cdots, n\}$ and consider various dimensional chains in X. Let

$$\mathcal{R}_i(X) = \{T \in \mathcal{R}_i(\mathbb{R}^N) | sptT \subset X\};$$

$$\mathcal{P}_i(X) = \{T \in \mathcal{R}_i(X) | T \text{ is } (\bar{p}, i) \text{ allowable and } \partial T \text{ is } (\bar{p}, i-1) \text{ allowable}\};$$

$$\mathcal{S}_i(X) = \{T \in \mathcal{P}_i(X) \text{ subanalytic chain}\}.$$

2 Intersection homology theory in rectifiable currents' version

Here we prove that intersection homology theory defined via rectifiable currents coincides with the usual definition [GM1] involving geometric chains or subanalytic chains.

Lemma 2.0.1 *There exists a triangulation $\mathcal{T}$ of (X, Σ), compatible with the given stratification, such that:*

For all open simplices $\sigma \subset \Sigma$ of $\mathcal{T}$, if $\sigma \subset X_{n-j} \backslash X_{n-j-1}$ for some $2 \leq j \leq n$, then $\partial\sigma \cap X_{n-j-1}$ is either empty or a face of $\partial\sigma$.

Proof. By [Hardt4], there exists a subanalytic triangulation of (X, Σ), compatible with the given stratification. Subanalytically subdividing this triangulation, one get a triangulation with the desired properties. □

From now on, we fix this triangulation $\mathcal{T}$ and denote

$$\mathcal{T}_\Sigma = \{\text{open simplices } \sigma \in \mathcal{T} : \sigma \subset \Sigma\}\,.$$

For notational convenience, we rephrase the perversity condition using the following

Definition 2.0.2 *The i-th safety function $s : \mathcal{T}_\Sigma \to \mathbb{R}$ is given by:*

$$s(\sigma) := i - j + p_j - 1, \textit{ if } \sigma \subset X_{n-j} \setminus X_{n-j-1}.$$

Remark 2.0.3. Now, $T \in \mathcal{P}_i$ if and only if for any $\sigma \in \mathcal{T}_\Sigma$,

$$\dim spt\,(T) \cap \sigma \leq s(\sigma) + 1 \text{ and } \dim spt\,(\partial T) \cap \sigma \leq s(\sigma) \tag{1}$$

Proposition 2.0.4 *If σ_1 is a face of σ_2 (denoted by $\sigma_1 \prec \sigma_2$), then $s(\sigma_1) \leq s(\sigma_2)$. This means the interior simplex is "safer" than the boundary.*

Proof. One may assume that $\sigma_1 \subset X_{n-j-1} \setminus X_{n-j-2}$ and $\sigma_2 \subset X_{n-j} \setminus X_{n-j-1}$ for some $2 \leq j \leq n$. Then

$$s(\sigma_1) - s(\sigma_2) = i - (j+1) + p_{j+1} - 1 - (i - j + p_j - 1) = p_{j+1} - p_j - 1 \leq 0. \square$$

Lemma 2.0.5 *Suppose $\sigma_1, \sigma_2 \in \mathcal{T}_\Sigma$. Let $\tau \in \mathcal{T}_\Sigma$ be the open simplex of minimum dimension such that $\sigma_1 \prec \bar{\tau}, \sigma_2 \prec \bar{\tau}$. Then*

$$s(\tau) = \max\left(s(\sigma_1), s(\sigma_2)\right).$$

Proof. Otherwise, by the Proposition 2.0.4, $s(\sigma_1) < s(\tau)$ and $s(\sigma_2) < s(\tau)$. This implies that if $\tau \subset X_{n-j} \setminus X_{n-j-1}$, then $\bar{\sigma}_1, \bar{\sigma}_2 \subset X_{n-j-1} \cap \partial\tau$. From the property 2.0.1, $\partial\tau \cap X_{n-j-1}$ is a unique closed simplex which is denoted by $\bar{\sigma}$. Therefore, we have $\sigma_1 \prec \bar{\sigma}$ and $\sigma_2 \prec \bar{\sigma}$, which contradicts to the minimum dimension property of τ. □

For an open simplex $\sigma \in \mathcal{T}$, let

$$st(\sigma) = \cup \{\tau \in \mathcal{T} : \sigma \prec \tau\}$$

be the open star of σ and $St(\sigma) = \overline{st(\sigma)}$ denotes the closed star of σ.

For the rest of this section, we fix one rectifiable current $T \in \mathcal{P}_i(X)$. Our goal is to deform T to an allowable subanalytic chain $S \in \mathcal{S}_i(X)$ using allowable currents. To achieve this, we make the following technical definitions.

Definition 2.0.6 *Given an open simplex $\sigma \in \mathcal{T}_\Sigma$.*

1. *σ is absolutely good if $\dim \sigma \leq s(\sigma)$;*
2. *σ is good w.r.t. T of type (I) if σ is absolutely good;*
   σ is good w.r.t. T of type (II) if $\dim(\sigma) = s(\sigma) + 1$ and $spt(\partial T) \cap st(\sigma) = \emptyset$;
   σ is good w.r.t. T of type (III) if $spt(T) \cap st(\sigma) = \emptyset$.
3. *σ is bad w.r.t. T if σ is not good w.r.t. T, i.e.*

$$\dim \sigma > s(\sigma),\ spt(T) \cap st(\sigma) \neq \emptyset$$

and also $spt(\partial T) \cap st(\sigma) \neq \emptyset$ in case $\dim(\sigma) = s(\sigma) + 1$.

Note that $\sigma \in \mathcal{T}_\Sigma$ being absolutely good trivially gives the perversity condition (1) for any $\tilde{T} \in \mathcal{R}_i(X)$.

To make the inductive argument in our deformation theorem 2.0.12, we will first fix any open simplex σ_0 of minimum dimension in the family

$$\{\tau : s(\tau) = \min\{s(\sigma) : \sigma \text{ is bad w.r.t. } T\}\}.$$

Lemma 2.0.7 *Any face σ_1 of $\partial\sigma_0$ is good w.r.t. T of types I or II.*

Proof. By the Proposition 2.0.4, $s(\sigma_1) \leq s(\sigma_0)$. Since $\dim \sigma_1 < \dim \sigma_0$, by the minimum of σ_0, σ_1 is good w.r.t. T. On the other hand, the fact $st(\sigma_0) \subset st(\sigma_1)$ implies $spt(T) \cap st(\sigma_1) \supseteq spt(T) \cap st(\sigma_0) \neq \emptyset$. i.e. σ_1 is not good w.r.t. T of types III. Thus, σ_1 is good w.r.t. T of types either I or II. □

Proposition 2.0.8 *For T and σ_0 as above, there exists a $T_1 \in \mathcal{P}_i$ and $R \in \mathcal{P}_{i+1}$, $L \in \mathcal{P}_i$ such that*

(a) $T = T_1 + \partial R + L$;
(b) $\{\sigma : \sigma \text{ is bad w.r.t. } T_1\} \subsetneqq \{\sigma : \sigma \text{ is bad w.r.t. } T\}$.
(c) $L = 0$ if $\partial T = 0$;
(d) $L \in \mathcal{S}_i$ if $\partial T \in \mathcal{S}_{i-1}$.

Proof. We'll obtain T_1 as $p_\# T$ for a suitable map p constructed differently in the two possible cases:

Case 1: $\dim \sigma_0 > s(\sigma_0) + 1$.

Since

$$\dim \sigma_0 > s(\sigma_0) + 1 \geq \dim spt(T) \cap \bar{\sigma}_0,$$

there exists a point $a_0 \in \sigma_0 \backslash spt\,(T)$. Let

$$p : St\,(\sigma_0) \setminus a_0 \to \partial\,(st\,(\sigma_0))$$

be the "radial retraction" of $St\,(\sigma_0)$ with a_0 as origin (see [Simon, p. 166]). Outside the closed star $St\,(\sigma_0)$, one may extend it to be the identical map.

Now, let's show that $p_{\#}T \in \mathcal{P}_i$. It's sufficient to show that for any $\sigma \in \mathcal{T}_\Sigma \cap St\,(\sigma_0)$ which is not absolutely good,

$$\dim spt\,(p_{\#}T) \cap \sigma \leq s(\sigma) + 1 \text{ and } \dim spt\,(\partial p_{\#}T) \cap \sigma \leq s(\sigma). \qquad (2)$$

In fact, if σ is a face of $\partial\sigma_0$, then by the Lemma 2.0.7, σ is good w.r.t. T of type III; i.e. $\dim \sigma = s\,(\sigma) + 1$ and $spt\,(\partial T) \cap st\,(\sigma) = \emptyset$. Thus, $\dim spt\,(p_{\#}T) \cap \sigma \leq \dim \sigma \leq s(\sigma) + 1$ and $spt\,(\partial p_{\#}T) \cap \sigma = spt\,(p_{\#}\partial T) \cap \sigma = spt\,(\partial T) \cap \sigma = \emptyset$. Therefore, σ satisfies the identity (2).

If σ belongs to the open star $st\,(\sigma_0)$, then by the definition of radial retraction, $\sigma \cap spt(p_{\#}T) = \emptyset$ and hence σ satisfies (2).

If σ belongs to $\partial st\,(\sigma_0) \setminus \partial\sigma_0$, then there are three subcases:

Subcase 1: σ is good w.r.t. T of type III . i.e. $st\,(\sigma) \cap spt\,(T) = \emptyset$. This is a trivial case, because $st\,(\sigma) \cap spt\,(p_{\#}T) = \emptyset$ is still true and hence σ satisfies (2) and is good w.r.t. $p_{\#}T$ of type II.

Subcase 2: σ is good w.r.t. T of type II . i.e. $\dim \sigma = s\,(\sigma) + 1$ and $spt\,(\partial T) \cap st\,(\sigma) = \emptyset$. This is also a trivial case, because one still has $\dim \sigma = s\,(\sigma) + 1$ and $st\,(\sigma) \cap spt\,(p_{\#}\partial T) = \emptyset$. Hence σ satisfies (2) and is good w.r.t. $p_{\#}T$ of type II.

Subcase 3: σ is bad w.r.t. T. Then, $s\,(\sigma) \geq s\,(\sigma_0)$ and we choose $\tau \in \mathcal{T}_\Sigma$ to be the open simplex of minimum dimension such that $\sigma_0 \prec \tau$ and $\sigma \prec \tau$. Since $s\,(\sigma_0) \leq s\,(\sigma) \leq s\,(\tau)$, the Lemma 2.0.5 implies $s\,(\sigma) = s\,(\tau)$. Now, note that

$$\begin{aligned} \dim spt(p_{\#}T) \cap \sigma &\leq \max\,(\dim spt\,(T) \cap \tau, \dim spt\,(T) \cap \sigma) \\ &\leq s\,(\tau) + 1 = s\,(\sigma) + 1; \\ \dim spt(\partial\,(p_{\#}T)) \cap \sigma &= \dim spt((p_{\#}\partial T)) \cap \sigma \\ &\leq \max(\dim spt\,(\partial T) \cap \tau, \dim spt\,(\partial T) \cap \sigma) \leq s\,(\tau) = s\,(\sigma)\,. \end{aligned}$$

Therefore, σ satisfies (2).

This shows that $p_{\#}T \in \mathcal{P}_i$. □

Case 2: $\dim \sigma_0 = s\,(\sigma_0) + 1$. In this case, $spt\,(\partial T) \cap st\,(\sigma_0) \neq \emptyset$.

Here we will use a different formula for p. Since $T \in \mathcal{P}_i$, by the identity (1), $\dim\,(spt\,(\partial T) \cap \sigma_0) \leq s\,(\sigma_0) < \dim \sigma_0$. We choose a point $a_0 \in \sigma_0 \setminus spt\,(\partial T)$ with $dist\,(a_0, \sigma_0 \setminus spt\,(\partial T)) > \epsilon$ for some small positive constant ϵ (It is possible that $a_0 \in spt\,(T)$). Define a help function $f : [0, +\infty) \to [0, +\infty)$ by

$$f\,(t) = \begin{cases} \frac{t}{\epsilon}, \; 0 \leq t < \epsilon \\ 1, \; \epsilon \leq t \leq 1 \\ t, \;\; t > 1 \end{cases}.$$

Now, for any $x \in st(\sigma_0)$, let $\gamma_x : [0,1] \to X$ be the "radial arc" starting at a_0, passing through x and ending at a point $\gamma_x(1)$ on $\partial st(\sigma_0)$. Now we define $p : X \to X$ by

$$p(x) = \begin{cases} \gamma_x\left(f\left(\frac{dist(x,a)}{dist(\gamma_x(1),a)}\right)\right), & \text{for } x \in St(\sigma_0)\,; \\ x & \text{for } x \in X \setminus St(\sigma_0). \end{cases}$$

To describe this map geometrically, let U_ϵ be the small star neighborhood of the point a, similar to $st(\sigma_0)$ but scaled down by a factor ϵ. Then the map p fixes all points outside $st(\sigma_0)$, maps U_ϵ homothetically to $st(\sigma_0)$ and projects the remaining "annular region" radially to $\partial(St\sigma_0)$. Clearly, this map is Lipschitz (with Lipschitz constant $\frac{C}{\epsilon}$). We need to show $p_\# T \in \mathcal{P}_i$.

As in case 1, it's sufficient to show (2) for any $\sigma \in \mathcal{T}_\Sigma \cap St(\sigma_0)$ which is not absolutely good. We treat all the possible subcases as before:

If σ belongs to the open star $st(\sigma_0)$ i.e. $\sigma_0 \prec \sigma$, then since $spt(\partial T) \subset X \setminus U_\epsilon$, we have $spt(\partial p_\# T) = spt(p_\# \partial T) \subset X \backslash st(\sigma_0)$. Thus, the fact $st(\sigma) \subset st(\sigma_0)$ implies $spt(\partial p_\# T) \cap st(\sigma) = \emptyset$. On the other hand,

$$\dim spt(p_\# T) \cap \sigma = \dim spt(T) \cap \sigma \le s(\sigma) + 1.$$

Therefore, σ also satisfies (2).

If σ belongs to the boundary $\partial\sigma_0$, then by the Lemma 2.0.7, σ is good w.r.t. T of type II; i.e. $\dim\sigma = s(\sigma)+1$ and $spt(\partial T) \cap st(\sigma) = \emptyset$. Now, $\dim spt(p_\# T) \cap \sigma \le \dim\sigma = s(\sigma) + 1$ and $spt(\partial p_\# T) \cap \sigma = \emptyset$. Hence, σ satisfies (2).

If σ belongs to $\partial st(\sigma_0) \setminus \partial\sigma_0$, then we choose $\tau \in \mathcal{T}_\Sigma$ to be the open simplex of minimum dimension such that $\sigma_0 \prec \tau$ and $\sigma \prec \tau$. Since $s(\sigma_0) \le s(\sigma) \le s(\tau)$, the Lemma 2.0.5 implies $s(\sigma) = s(\tau)$. Thus,

$$\begin{aligned} \dim spt(p_\# T) \cap \sigma &\le \max(\dim spt(T) \cap \tau, \dim spt(T) \cap \sigma) \\ &\le s(\tau) + 1 = s(\sigma) + 1 \end{aligned}$$

and

$$\dim spt(\partial p_\# T) \cap \sigma = \dim spt(p_\# \partial T) \cap \sigma$$

$$\le \max(\dim spt(\partial T) \cap \tau, \dim spt(\partial T) \cap \sigma) \le s(\tau) = s(\sigma).$$

Therefore, σ satisfies (2).

This shows that also in case 2, $p_\# T \in \mathcal{P}_i$. □

Now, let $T_1 = p_\# T$ and let h be an "affine homotopy" from the identity to p, $R = h_\sharp([[0,1]] \times T)$, and $L = h_\sharp([[0,1]] \times \partial T)$. Then for any $\sigma \in \mathcal{T}_\Sigma$,

$$\dim(R \cap \bar{\sigma}) \le \dim(T \cap \bar{\sigma}) + 1 \le i - j + p_j + 1,$$

and

$$\dim(\partial R \cap \bar{\sigma}) \le \dim(\partial T \cap \bar{\sigma}) + 1 \le i - j + p_j,$$

so $R \in \mathcal{P}_{i+1}$. Also,

$$\dim L \leq \dim(\partial T \cap \bar{\sigma}) + 1 \leq (i-1) - j + p_j + 1 = i - j + p_j,$$

so $L \in \mathcal{P}_i$.

(a) now follows from the homotopy formula [Simon, pg 139].

To prove (b), note that if σ is good w.r.t. T, then σ is still good w.r.t. $p_{\#}T$. Also, there is one open simplex, namely σ_0, which is good w.r.t. $p_{\#}T$ but bad w.r.t. T. Thus, we have

$$\{\sigma : \sigma \text{ is bad w.r.t. } p_{\#}T\} \subsetneqq \{\sigma : \sigma \text{ is bad w.r.t. } T\}.$$

(c) and (d) readily follow from the definition of L. □

Corollary 2.0.9 *For any $T \in \mathcal{P}_i$, there exists a $T_1 \in \mathcal{P}_i$ and $R \in \mathcal{P}_{i+1}$, $L \in \mathcal{P}_i$ such that*

(a) $T = T_1 + \partial R + L$;
(b) *All open simplices $\sigma \in \mathcal{T}_\Sigma$ are good w.r.t. T_1;*
(c) $L = 0$ *if* $\partial T = 0$;
(d) $L \in \mathcal{S}_i$ *if* $\partial T \in \mathcal{S}_{i-1}$.

Proof. Since X is compact, the $\#\{\sigma : \sigma$ is bad w.r.t. $T\}$ is finite. After using Proposition 2.0.8, we may apply it a second time with T replaced by T_1 (and new choice of σ_0). Continuing inductively a finite number of times, one will get the desired results. □

Lemma 2.0.10 *Suppose $T \in \mathcal{P}_i$ with spt$T \subset \mathcal{T}_k$, the k-skeleton of $\mathcal{T}$, for some $k \geq i+1$. If all open simplices $\sigma \in \mathcal{T}_\Sigma$ are good w.r.t. T, then there exists $T_1 \in \mathcal{P}_i$ with spt$T_1 \subset \mathcal{T}_{k-1}$, $R \in \mathcal{P}_{i+1}$and $L \in \mathcal{P}_i$ such that*

(a) *all open simplices $\sigma \in \mathcal{T}_\Sigma$ are good w.r.t. T_1,*
(b) $T = T_1 + \partial R + L$
(c) $L = 0$ *if* $\partial T = 0$
(d) $L \in \mathcal{S}_i$ *if* $\partial T \in \mathcal{S}_{i-1}$

Proof. As in [Simon, Lemma 29.4], one can choose, for each k-simplex τ of $\mathcal{T}_k$, a suitable point $a_\tau \in \tau$ so that the radial retraction away from a_τ gives a locally Lipschitz map $\psi : \mathcal{T}_k \setminus \cup_\tau \{a_\tau\} \to \mathcal{T}_{k-1}$ along with a mass bound for $\psi_{\#}T$.

For any $\sigma \in \mathcal{T}_\Sigma$, we'll show that σ is also good w.r.t. $T_1 = \psi_{\#}T$. A basic fact about the map ψ is that $\psi^{-1}(x) \subset st(\sigma)$ for any $x \in \sigma$ and hence $\psi^{-1}(st(\sigma)) \subset st(\sigma)$. Therefore, if σ is absolutely good, then σ is automatically good w.r.t.T_1 of type I. If σ is good w.r.t. T of type II, i.e. $\dim(\sigma) = s(\sigma) + 1$ and $spt(\partial T) \cap st(\sigma) = \emptyset$, then $\dim T_1 \cap \sigma \leq \dim \sigma = s(\sigma) + 1$ and $spt(\partial T_1) \cap st(\sigma) \subset \psi(spt(\partial T) \cap st(\sigma)) = \emptyset$. Hence σ is good w.r.t. T_1 of type II. Finally, if σ is good w.r.t. T of type III, i.e. $spt(T) \cap st(\sigma) = \emptyset$, then $spt(T_1) \cap st(\sigma) \subset \psi(spt(T) \cap st(\sigma)) = \emptyset$. Hence σ is also good w.r.t. T_1 of type III. This shows that all open simplices $\sigma \in \mathcal{T}_\Sigma$ are still good w.r.t. T_1.

Now as usual, let h be an "affine homotopy" from the identity to ψ, $R = h_\sharp([[0,1]] \times T)$, and $L = h_\sharp([[0,1]] \times \partial T)$. One readily checks that $R \in \mathcal{P}_{i+1}$and $L \in \mathcal{P}_i$ have the desired properties. □

By applying this lemma repeatedly, we will eventually get $spt(T) \subset \mathcal{T}_i$, then we'll use the following:

Lemma 2.0.11 *Suppose $T \in \mathcal{P}_i$ with $sptT \subset \mathcal{T}_i$. If all open simplices $\sigma \in \mathcal{T}_\Sigma$ are good w.r.t. T, then there exists $S = \Sigma_{F \in \mathcal{T}_i} \beta_F [[F]] \in \mathcal{S}_i$ for some integers β_F such that $M(T - S) + M(\partial(T - S)) \leq cM(\partial T)$ for some constant c.*

Proof. As in [Simon, pp. 175–176], for any i-dimensional face F, one can find an integer β_F such that $M(T \llcorner F - \beta_F [[F]]) + M(\partial(T \llcorner F - \beta_F [[F]])) \leq cM(\partial T \llcorner F)$. Now, we'll show that $S = \Sigma_{F \in \mathcal{T}_i} \beta_F [[F]] \in \mathcal{P}_i$. In fact, for any $\sigma \in \mathcal{T}_\Sigma$ we know that σ is good w.r.t. T. We'll show that σ is also good w.r.t. S. If σ is absolutely good, then σ is automatically good w.r.t. S. If σ is good w.r.t. T of type II, i.e. $\dim(\sigma) = s(\sigma) + 1$ and $spt(\partial T) \cap st(\sigma) = \emptyset$. This implies that for any $F \in \mathcal{T}_i \cap st(\sigma)$, we have $\partial T \llcorner F = 0$. Therefore, $T \llcorner F = \beta_F [[F]]$ and hence $spt(\partial S) \cap st(\sigma) = spt(\partial T) \cap st(\sigma) = \emptyset$. Thus, σ is also good w.r.t. S of type II. Finally, if σ is good w.r.t. T of type III, i.e. $spt(T) \cap st(\sigma) = \emptyset$, then for any $F \in \mathcal{T}_i \cap st(\sigma), T \llcorner F = 0$ and hence $\beta_F = 0$. This also shows $spt(S) \cap st(\sigma) = \emptyset$ and σ is good w.r.t. S of type III. Therefore, all open simplices $\sigma \in \mathcal{T}_\Sigma$ are good w.r.t. S, which automatically implies $S \in \mathcal{P}_i$. By the construction of S, we know that S is a subanalytic chain. □

Theorem 2.0.12 (Deformation Theorem) *For any $T \in \mathcal{P}_i$, there exists a $S \in \mathcal{S}_i$ and $R \in \mathcal{P}_{i+1}$, $L \in \mathcal{P}_i$ such that*
(a) $T = S + \partial R + L$
(b) $L = 0$ if $\partial T = 0$
(c) $L \in \mathcal{S}_i$ if $\partial T \in \mathcal{S}_{i-1}$

Proof. First apply Corollary 2.0.9 to change T so that all the open simplices are good w.r.t. T, then apply Lemma 2.0.10 inductively, we may assume $sptT \subset \mathcal{T}_i$. At last, apply Lemma 2.0.11. □

Definition 2.0.13 *Let $IH_*^{subanalytic}(X)$, $IH_*(X)$ denote the homology groups of the chain complexes $\{\mathcal{S}_i\}, \{\mathcal{P}_i\}$ defined above respectively.*

Then, we have the following isomorphism theorem:

Theorem 2.0.14 (Isomorphism Theorem) *The inclusion map $j : IH_*^{subanalytic}(X) \hookrightarrow IH_*(X)$ is an isomorphism.*

Proof. (i) j is injective.

If $[S] = [0] \in IH_i(X)$, i.e. $S = \partial T$ for some $T \in \mathcal{P}_{i+1}(X)$, then by the deformation Theorem 2.0.12,

$$T = S' + \partial R' + L'$$

with L' subanalytic. So, $S = \partial(S' + L')$ and $S' + L' \in \mathcal{S}_{i+1}$. Hence, j is injective.

(ii) j is onto. For any $[T] \in IH_i(X)$, by the deformation Theorem 2.0.12, $T = S + \partial R$. Hence, $[T] = [S] = j([S])$, i.e. j is onto. □

3 A modified mass on the complex of rectifiable currents

The limit of a sequence of rectifiable currents with bounded mass and bounded boundary mass is a rectifiable current [Simon, Theorem 27.3]. However, the limit rectifiable current may fail to satisfy the allowability conditions of the approximating chains. This motivates us to modify the usual mass by adding some suitable mass modifiers. For any rectifiable current $T \in \mathcal{R}_i(X)$ and each singular stratum, we'll add a mass modifier for T corresponding to that stratum. To control the amount of mass modified, we choose and fix a small tolerance $\delta > 0$.

As in Sect. 1.4, we have a fixed perversity $\bar{p} = (p_2, \cdots, p_n)$ and a fixed dimension $i \in \{0, \cdots, n\}$. Now, for each singular stratum X_{n-k} with some integer $k \in \{2, \cdots, n\}$ and any rectifiable current $T \in \mathcal{R}_i(X)$, we'll define a k-th mass *modifier* $m_k^\delta(T)$ in the 3 possible cases as follows:

Case 1: $i - k + p_k \geq n - k$, i.e. $i + p_k \geq n$ (e.g. $i = n$).
In this case,

$$\dim(spt\,(T) \cap X_{n-k}) \leq dim(X_{n-k}) \leq n - k \leq i - k + p_k.$$

i.e. the perversity condition is automatically satisfied. Since it's unnecessary to make any modification on the mass, we here set $m_k^\delta(T) := 0$.

Case 2: $i - k + p_k < 0$, i.e. $i < k - p_k$. e.g. $i = 0$ or 1.
In this case, $\dim(spt\,(T) \cap X_{n-k}) \leq i - k + p_k$ iff $spt\,(T) \cap X_{n-k} = \emptyset$.
Here we define the *k-mass modifier*

$$m_k^\delta(T) := ln \frac{\delta}{dist(spt\,(T)\,,\ X_{n-k})} M(T \llcorner B\,(X_{n-k}, \delta)).$$

Before considering the remaining case, we first make two easy but important observations:

Lemma 3.0.15 $\dim(spt\,(T) \cap X_{n-k}) \leq i - k + p_k$ *iff* $m_k^\delta(T) < +\infty$.

Proof.

$$m_k^\delta(T) < +\infty \Leftrightarrow dist(spt\,(T)\,, X_{n-k}) > 0$$

$$\Leftrightarrow spt\,(T) \cap X_{n-k} = \emptyset \Leftrightarrow \dim(spt\,(T) \cap X_{n-k}) \leq i - k + p_k. \qquad \square$$

Lemma 3.0.16 *m_k^δ is lower-semi continuous with respect to the weak convergence of currents.*

Proof. If $T_j \rightharpoonup T$, then $M\,(T \llcorner B\,(X_{n-k}, \delta)) \leq \liminf_{j\to\infty} M\,(T_j \llcorner B\,(X_{n-k}, \delta))$ and

$$dist(spt\,(T)\,, X_{n-k}) \geq \limsup_{j\to\infty} dist(spt\,(T_j)\,, X_{n-k}).$$

Therefore,

$$ln\frac{\delta}{dist(spt\,(T)\,,\ X_{n-k})} \leq \liminf_{j\to\infty} ln\frac{\delta}{dist(spt\,(T_j)\,,\ X_{n-k})}$$

and

$$\begin{aligned} m_k^\delta(T) &\leq \liminf_{j\to\infty} ln\frac{\delta}{dist(spt(T_j), X_{n-k})} \liminf_{j\to\infty} M(T_j \llcorner B\,(X_{n-k},\delta)) \\ &\leq \liminf_{j\to\infty} ln\frac{\delta}{dist(spt\,(T_j)\,,\ X_{n-k})} M(T_j \llcorner B\,(X_{n-k},\delta)) = \liminf_{j\to\infty} m_k^\delta(T_j). \end{aligned}$$

$\therefore m_k^\delta$ is lower-semi continuous. □

Case 3: $0 \leq i-k+p_k < n-k$ i.e. $k-p_k \leq i < n-p_k$

Remark 1. If X has isolated singularities, i.e. $\dim(\Sigma) = 0$, then case 3 will not happen.

Set

$$G_k = \{\text{all } N-(i-k+p_k)-1 \text{ dimensional planes in } \mathbb{R}^N\}$$

with the standard measure μ, induced from G_k being an $(i-k+p_k)+1$ dimensional vector bundle over the *grassmannian* manifold $G(N, N-(i-k+p_k)-1)$ with its invariant measure.

Define $d_T : G_k \to [0,+\infty]$ by

$$d_T(H) = dist(spt\,(T) \cap H, X_{n-k})/\delta. \tag{3}$$

and define

$$u_k^T\,(t) := \frac{1}{t}\mu\left(d_T^{-1}\,[0,t]\right) = \frac{1}{t}\mu(\{H \in G_k : T \llcorner B\,(X_{n-k}, t\delta) \cap H \neq 0\}),$$

so $tu_k^T\,(t)$ is increasing and hence differentiable for a.e. $t \in [0,1]$.

The function u_k^T gives a normalized count (without multiplicity) of the number of planes intersecting the current T near the singular stratum X_{n-k}. It's similar to the *quermassintegrale* [Santalo, 13.8] of convex sets. We'll use its $\mathcal{L}^1$ norm to define the mass modifier in the Definition 3.0.19. Note that $\left\|u_k^T\right\|_{L^1([0,1])}$ may be infinite for some allowable rectifiable current T having infinite order contact with X_{n-k}, but it will be finite for allowable subanalytic chain by Theorem 3.0.23.

To obtain the lower semicontinuity of our modified mass (defined in 3.0.19), we need the following lemmas:

Lemma 3.0.17 *If* $\sup M\,(T_j) + M\,(\partial T_j) < \infty$ *and* $T_j \rightharpoonup T$, *then* $u_k^T(t) \leq \liminf u_k^{T_j}\,(t)$ *for all* $t \in (0,1]$.

Proof. The hypotheses imply, by [F1, 4.3.2], that for a.e. $H \in G_k, T_j \cap H \rightharpoonup T \cap H$. Thus,

$$\begin{aligned} tu_k^T(t) &= \mu(\{H \in G_k : M(T \llcorner B(X_{n-k}, t\delta) \cap H) > 0\}) \\ &\leq \mu(\{H \in G_k : \liminf_{j\to\infty} M(T_j \llcorner B(X_{n-k}, t\delta) \cap H) > 0\}) \\ &= \int \chi_{\{H\in G_k : \liminf M(T_j \llcorner B(X_{n-k}, t\delta)\cap H)>0\}} d\mu \\ &\leq \int \liminf_{j\to\infty} \chi_{\{H\in G_k : M(T_j \llcorner B(X_{n-k}, t\delta)\cap H)>0\}} d\mu \\ &\leq \liminf_{j\to\infty} \int \chi_{\{H\in G_k : M(T_j \llcorner B(X_{n-k}, t\delta)\cap H)>0\}} d\mu \text{ (by Fatou's lemma)} \\ &= \liminf_{j\to\infty} \mu(\{H \in G_k : M(T_j \llcorner B(X_{n-k}, t\delta) \cap H) > 0\}) \\ &= t \liminf_{j\to\infty} u_k^{T_j}(t). \end{aligned}$$

□

Lemma 3.0.18 *For any* $T \in \mathcal{R}_i(X)$*, if* $u_k^T(t) \in L^1([0,1])$*, then*
(1) $\mu\left(d_T^{-1}(0)\right) = 0$;
(2) $\dim(spt(T) \cap X_{n-k}) \leq i - k + p_k$;
(3) $\left\|u_k^T(t)\right\|_{L^1([0,1])} = \int_{d_T^{-1}([0,1])} ln(\frac{1}{d_T}) d\mu$

Proof. Since $tu_k^T(t) = \mu\left(d_T^{-1}([0,t])\right)$ is increasing in t and $u_k^T(t) \in L^1([0,1])$, we have

$$\mu\left(d_T^{-1}(0)\right) = \mu\left(\{H : H \cap spt(T) \cap X_{n-k} \neq \emptyset\}\right) = 0.$$

This implies that $spt(T) \cap X_{n-k}$ is a set of $(i - k + p_k) + 1$ dimensional integral geometric Favard measure zero (see [F2] or [Santalo, III. 14.7.1] for details). By [F2, Theorem 9], $\dim(spt(T) \cap X_{n-k}) \leq i - k + p_k$.

As for (3), a classical application of Fubini's theorem is the formula

$$\int_E f(x)\, dx = \int_0^\infty m\{x | f(x) > t\}\, dt$$

where f is a nonnegative measurable function on a μ measurable set E.

Since $d_T^{-1}(0)$ has μ measure 0, we can apply the above formula to the function $ln(\frac{1}{d_T}) : d_T^{-1}([0,1]) \to [0,\infty)$ as follows:

$$\begin{aligned} \int_{d_T^{-1}([0,1])} ln(\frac{1}{d_T}) d\mu &= \int_0^\infty \mu\left(\left\{H | ln(\frac{1}{d_T(H)}) > t\right\}\right) dt \\ &= \int_0^\infty \mu\left(\left\{H | 0 < d_T(H) < e^{-t}\right\}\right) dt \\ &= \int_0^1 \frac{\mu\left(\{H | 0 < d_T(H) < s\}\right)}{s} ds = \int_0^1 u_k^T(t)\, dt. \end{aligned}$$

□

All our above discussion leads to the following:

Definition 3.0.19 *For fixed $\delta > 0$ and any $T \in \mathcal{R}_i(X)$, we define the k-mass modifier of T to be*

$$m_k^\delta(T) \equiv \begin{cases} 0, & \text{if } p_k \geq n-i \\ \left\| u_k^T(t) \right\|_{L^1([0,1])} M\left(T \llcorner B\left(X_{n-k}, \delta\right)\right), & \text{if } k-i \leq p_k < n-i \\ ln \frac{\delta}{dist(spt(T), X_{n-k})} M(T \llcorner B\left(X_{n-k}, \delta\right)) & \text{if } p_k < k-i \end{cases}.$$

A new modified mass on $\mathcal{R}_i(X)$ is given by $\tilde{M}(T) = \tilde{M}^\delta(T) \equiv \sum_{k=2}^n m_k^\delta(T) + M(T)$ for any $T \in \mathcal{R}_i(X)$. Also, we set

$$\mathcal{I}_i(X) = \{T \in \mathcal{R}_i(X) : \tilde{M}(T) + \tilde{M}(\partial T) < +\infty\}$$

to be the set of all rectifiable currents with finite modified mass and finite modified boundary mass.

Proposition 3.0.20 *If $\tilde{M}^{\delta_0}(T) < +\infty$ for some $\delta_0 > 0$, then $\lim_{\delta \to 0+} \tilde{M}^\delta(T) = M(T)$.*

Proof. For each $k : k-i \leq p_k < n-i$, whenever $\delta < \delta_0$,

$$\begin{aligned} m_k^\delta(T) &= \left\| u_k^{T,\delta}(t) \right\|_{L^1([0,1])} M\left(T \llcorner B\left(X_{n-k}, \delta\right)\right) \\ &= \left\| \frac{\delta}{\delta_0} u_k^{T,\delta_0}\left(\frac{t\delta}{\delta_0}\right) \right\|_{L^1([0,1])} M\left(T \llcorner B\left(X_{n-k}, \delta\right)\right) \\ &\leq \left\| u_k^{T,\delta_0}(t) \right\|_{L^1([0,\frac{\delta}{\delta_0}])} M\left(T \llcorner B\left(X_{n-k}, \delta_0\right)\right) \\ &\to 0, \text{ as } \delta \to 0+. \end{aligned}$$

Similarly, if $p_k < k-i$, we also have $m_k^\delta(T) \to 0$ as $\delta \to 0$. Thus, $\lim_{\delta \to 0+} \tilde{M}^\delta(T) = M(T)$. □

Proposition 3.0.21 *$\tilde{M}$ mass is lower semi-continuous. i.e. if $T_j \rightharpoonup T$, then $\tilde{M}(T) \leq \liminf \tilde{M}(T_j)$.*

Proof. For $k-i \leq p_k < n-i$, if $T_j \rightharpoonup T$, then by the Lemma 3.0.17, $u_k^T(t) \leq \liminf u_k^{T_j}(t)$. Therefore, by the Fatou's lemma,

$$\begin{aligned} m_k^\delta(T) &= \left\| u_k^T(t) \right\|_{L^1([0,1])} M\left(T \llcorner B\left(X_{n-k}, \delta\right)\right) \\ &\leq \left\| \liminf_{j \to \infty} u_k^{T_j}(t) \right\|_{L^1([0,1])} M\left(T \llcorner B\left(X_{n-k}, \delta\right)\right) \\ &\leq \liminf_{j \to \infty} \left\| u_k^{T_j}(t) \right\|_{L^1([0,1])} \liminf_{j \to \infty} M\left(T_j \llcorner B\left(X_{n-k}, \delta\right)\right) \\ &\leq \liminf_{j \to \infty} \left[\left\| u_k^{T_j}(t) \right\|_{L^1([0,1])} M\left(T_j \llcorner B\left(X_{n-k}, \delta\right)\right) \right] \\ &= \liminf_{j \to \infty} m_k^\delta(T_j). \end{aligned}$$

This, along with Lemma 3.0.16 and the lower semi-continuity of the usual mass, implies the lower semi-continuity of $\tilde{M}$. □

The following proposition says that a rectifiable current with finite modified mass and finite modified boundary mass automatically satisfies the perversity condition:

Proposition 3.0.22 $\mathcal{I}_i(X) \subset \mathcal{P}_i(X)$

Proof. For any $T \in \mathcal{I}_i(X)$, by the Lemma 3.0.15 and the Lemma 3.0.18, we have $\dim(spt\,(T) \cap X_{n-k}) \leq i - k + p_k$ and $\dim(spt\,(\partial T) \cap X_{n-k}) \leq i - k + p_k - 1$ for each k. Thus, $T \in \mathcal{P}_i(X)$. □

Theorem 3.0.23 $\mathcal{S}_i\,(X) \subset \mathcal{I}_i\,(X) \subset \mathcal{P}_i\,(X)$

Proof. It is sufficient to show $\mathcal{S}_i\,(X) \subset \mathcal{I}_i\,(X)$.

For any subanalytic chain $T \in \mathcal{S}_i\,(X)$ and k with $k - i \leq p_k < n - i$, the graph of the function d_T defined in (3) is a subanalytic set because both $spt\,(T)$ and X_{n-k} are subanalytic. By the Lojasiewicz's inequality (see Proposition 1.3.1), there exists a constant $C > 0$ and $N > 0$ such that $Cd_T(H) \geq dist_{G_k}\left(H, d_T^{-1}\,(0)\right)^N$. Also, $T \in \mathcal{P}_i\,(X)$ implies $\mu\left(d_T^{-1}(0)\right) = 0$. Thus,

$$u_k^T\,(t) = \frac{1}{t}\mu(d_T^{-1}[0,t]) \leq \frac{1}{t}\mu(\left\{H : dist_{G_k}\left(H, d_T^{-1}\,(0)\right) \leq (Ct)^{\frac{1}{N}}\right\}) \leq C_1 t^{\beta-1}$$

for some $\beta > 0$. Therefore, $u_k^T\,(t) \in L^1\,[0,1]$ and hence $m_k^\delta\,(T) < +\infty$ for each k with $k - i \leq p_k < n - i$. Similar arguments yield $m_k^\delta\,(T) < +\infty$ for all other k's. This shows $\tilde{M}(T) < +\infty$. Also, one has $\tilde{M}(\partial T) < +\infty$ because $\partial T \in \mathcal{S}_{i-1}$. Thus we have $T \in \mathcal{I}_i$. □

Theorem 3.0.24 (Compactness theorem) *Any sequence $\{T_j\}$ in $\mathcal{I}_i$ with*

$$\liminf \tilde{M}(T_j) + \tilde{M}(\partial T_j) < +\infty,$$

contains a subsequence $\{T_{j_k}\}$ weakly convergent to some $T \in \mathcal{I}_i$.

Proof. Since $\liminf M(T_j) + M(\partial T_j) < +\infty$, by the usual compactness theorem of integer multiplicity rectifiable currents (c.f. [Simon, Theorem 27.3]), there exists a subsequence $\{T_{j_k}\}$ of $\{T_j\}$ such that $T_{j_k} \rightharpoonup T$ for some $T \in \mathcal{R}_i\,(X)$. By the lower semi-continuity of $\tilde{M}$, we have $\tilde{M}(T) + \tilde{M}(\partial T) \leq \liminf \tilde{M}(T_{j_k}) + \tilde{M}(\partial T_{j_k}) < +\infty$, and hence $T \in \mathcal{I}_i\,(X)$. □

A direct corollary of the previous two theorems is:

Corollary 3.0.25 *Any sequence of subanalytic chains $\{T_j\} \subset \mathcal{S}_i$ with*

$$\liminf \tilde{M}(T_j) + \tilde{M}(\partial T_j) < +\infty,$$

contains a subsequence weakly convergent to some rectifiable current $T \in \mathcal{I}_i$.

4 $\tilde{M}$-mass minimizing currents

Definition 4.0.26 *We say that $T \in \mathcal{P}_i(X)$ is $\tilde{M}-$mass minimizing if*

$$\tilde{M}(T) \leq \tilde{M}(S)$$

whenever $S \in \mathcal{P}_i(X)$ and $\partial T = \partial S$.

The following theorem says that there is a $\tilde{M}-$mass minimizer in each intersection homology class of $IH_i(X)$:

Theorem 4.0.27 (Existence theorem) *1. Suppose $E = \partial S$ for some $S \in \mathcal{I}_i(X)$, then there exists a rectifiable current $T \in \mathcal{I}_i(X)$ such that $\partial T = E$ and $\tilde{M}(T) \leq \tilde{M}(R)$ for any $R \in \mathcal{I}_i(X)$ with $\partial R = E$.*

2. For each homology class $\alpha \in IH_i(X)$, there exists $T \in \alpha \cap \mathcal{I}_i(X)$ such that $\tilde{M}(T) \leq \tilde{M}(T')$ for any $T' \in \alpha$.

Proof. (1) follows from the direct method [Morgan, 1.3] and the compactness Theorem 3.0.24.

To obtain a minimizing sequence in (2), we note, by the isomorphism Theorem 2.0.14, that each class $\alpha \in IH_i(X)$ contains at least one subanalytic representative S and $\tilde{M}(S) < +\infty$, by Theorem 3.0.23. Now, the compactness Theorem 3.0.24 ensures the existence of an $\tilde{M}-$mass minimizing in the nonempty subset $\alpha \cap \mathcal{I}_i(X)$. □

Now, let's consider the regularity of $\tilde{M}$-Mass Minimizing currents.

For any $T \in \mathcal{I}_i(X)$, let $a \in spt(T) \setminus (spt(\partial T) \cup \Sigma)$. Recall that a is a regular point of $spt(T)$ if there exists an open neighborhood U of a in $X \setminus \Sigma$ such that $U \cap sptT$ is an open i-manifold of class C^1.

Definition 4.0.28 *Let $\omega(t)$ be defined for $0 < t \leq \delta$, with $lim_{t\to 0}\omega(t) = 0$. Let Ψ be a positive integrand and let T be a rectifiable current with compact support in Ω, an open subset of some Euclidean space R^N. Recall that T is (Ψ, ω, δ)-minimal if*

$$\Psi[T \llcorner K] \leq \Psi[T \llcorner K + X] + \omega(r) M(T \llcorner K + X)$$

for all rectifiable X with compact support in $K \subset \Omega$ and with

$$\partial X = 0,\ diam(sptX) \leq r \leq \delta.$$

Remark 4.0.29. By an argument similar to [Hardt2], the above inequality can be replaced by $\Psi[T \llcorner K] \leq \Psi[T \llcorner K + X] + \omega(r) C$ for some constant $C > 0$.

Let's first recall a lemma of Almgren in [Bomb]:

Lemma 4.0.30 *Let Ψ be $\lambda-$elliptic and of class C^2, and let T be a $(\Psi, \omega, \delta)-$minimal current for some $\lambda < \infty, \delta > 0$ and some $\omega(t)$ with*

$$\int_0^\delta \frac{\omega(t)^{1/2}}{t} dt < \infty.$$

Then regular points are dense in $spt(T) \setminus spt(\partial T)$.

Such current T is often called *an almost minimizing current*. Note that this notion is very general. An oriented compact C^2 submanifold T is automatically locally almost minimizing where (Ψ, ω, δ) depend on T and the distance to $spt\,(\partial T)$.

Proposition 4.0.31 *A $\tilde{M}$-mass minimizing current T locally is an almost minimizing current (with (Ψ, ω, δ) depending on T and distance to $spt\,(\partial T) \cup \Sigma$).*

Proof. For any $a \in sptT \setminus (spt\,(\partial T) \cup \Sigma)$, let U_a be a very small open neighborhood of a of radius $r_a < \frac{1}{2} dist(a, \Sigma)$. Since T is a $\tilde{M}$-mass minimizing current, for any i-current $S \in \mathcal{R}_i(U_a)$ with $\partial S = 0$ and diam(spt S)$\leq r \leq r_a$, we have

$$\tilde{M}(T) \leq \tilde{M}(T + S).$$

For each $k = 2, \cdots, n$, let Φ_k be the nonnegative integrand whose total integral over a rectifiable current R is

$$\Phi_k(R) \equiv h_k M\left(R \llcorner B\left(X_{n-k}, \delta\right)\right),$$

where

$$h_k = \begin{cases} \left|\left|u_k^T\right|\right|_{L^1[0,1]}, & \text{if } dist\left(a, X_{n-k}\right) \leq \delta \text{ and } k - i \leq p_k < n - i \\ \ln \frac{\delta}{dist(spt(T), X_{n-k})}, & \text{if } dist\left(a, X_{n-k}\right) \leq \delta \text{ and } p_k \geq n - i \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

Lemma 4.0.32 *For each $k = 2, \cdots, n$ and r small enough,*

$$\begin{aligned} &\left[\Phi_k\left(T\right) - \Phi_k\left(T + S\right)\right] - \left[m_k^\delta\left(T\right) - m_k^\delta\left(T + S\right)\right] \\ &\leq C_k \frac{r}{dist\left(a, X_{n-k}\right)} M\left(\left(T + S\right) \llcorner B\left(X_{n-k}, \delta\right)\right) \end{aligned}$$

for some positive constant C_k independent of T, S and a.

Proof. If $dist\,(a, X_{n-k}) > \delta$, we may choose U_a small enough such that $dist\,(U_a, X_{n-k}) > \delta$, then $m_k^\delta\,(T) = m_k^\delta\,(T + S)$ .Thus,

$$\left[\Phi_k\left(T\right) - \Phi_k\left(T + S\right)\right] - \left[m_k^\delta\left(T\right) - m_k^\delta\left(T + S\right)\right] = [0 - 0] - 0 = 0.$$

Now, we assume $dist\,(a, X_{n-k}) \leq \delta$. In this case, $\Phi_k\,(T) = m_k^\delta\,(T)$, so it is sufficient to show

$$m_k^\delta\left(T + S\right) - \Phi_k\left(T + S\right) \leq C_k \frac{r}{dist\left(a, X_{n-k}\right)} M\left(\left(T + S\right) \llcorner B\left(X_{n-k}, \delta\right)\right)$$

1. If $p_k \geq n - i$, then $m_k^\delta\,(T + S) - \Phi_k\,(T + S) = 0 - 0 = 0$.

2. If $k-i \le p_k < n-i$, then for any $t \in (0,1]$, we consider

$$\begin{aligned}
& t\left(u_k^{T+S}(t) - u_k^T(t)\right) \\
&= \mu\{H \in G_k : spt\,(T+S) \cap B\,(X_{n-k}, t\delta) \cap H \neq \emptyset\} \\
&\quad -\mu\{H \in G_k : spt\,(T) \cap B\,(X_{n-k}, t\delta) \cap H \neq \emptyset\} \\
&\le \mu\{H \in G_k : spt\,(T+S) \cap B\,(X_{n-k}, t\delta) \cap H \neq \emptyset \\
&\quad \text{ and } spt\,(T) \cap B\,(X_{n-k}, t\delta) \cap H = \emptyset\} \\
&= \mu\{H \in G_k : spt\,(S) \cap B\,(X_{n-k}, t\delta) \cap H \neq \emptyset \\
&\quad \text{ and } spt\,(T) \cap B\,(X_{n-k}, t\delta) \cap H = \emptyset\} \\
&\le \mu\{H \in G_k : U_r \cap B\,(X_{n-k}, t\delta) \cap H \neq \emptyset\} \\
&\le \begin{cases} 0, & \text{if } t\delta \le dist\,(U_a, X_{n-k}) \\ C_k r^{(i-k+p_k)+1} & \text{if } t\delta \ge dist\,(U_a, X_{n-k}) \end{cases}
\end{aligned}$$

where $U_r \subset U_{r_a}$ is an Euclidean ball of radius r that contains S and C_k is a constant depends only on N and $(i-k+p_k)+1$, the dimension of the moving planes. Note the last inequality follows from the formula ([Santalo, 13.46]) for the quermassintegrale of a ball. Thus, we have

$$\begin{aligned}
& \left|\left|u_k^{T+S}\right|\right|_{L^1[0,1]} - \left|\left|u_k^T\right|\right|_{L^1[0,1]} \\
&= \int_0^1 u_k^{T+S}(t) - u_k^T(t)\, dt \\
&\le \int_{dist(U_a, X_{n-k})/\delta}^1 \frac{C_k}{t} r^{(i-k+p_k)+1} dt \\
&\le C_k r^{(i-k+p_k)+1} \ln \frac{\delta}{dist\,(U_a, X_{n-k})} \\
&\le C_k r \frac{2\delta}{dist\,(a, X_{n-k})}
\end{aligned}$$

Hence,

$$\begin{aligned}
& m_k^\delta (T+S) - \Phi_k (T+S) \\
&= \left[\left|\left|u_k^{T+S}\right|\right|_{L^1[0,1]} - \left|\left|u_k^T\right|\right|_{L^1[0,1]}\right] M\left((T+S) \llcorner B\,(X_{n-k}, \delta)\right) \\
&\le 2\delta C_k \frac{r}{dist\,(a, X_{n-k})} M\left((T+S) \llcorner B\,(X_{n-k}, \delta)\right).
\end{aligned}$$

3. If $p_k < k - i$, then when r small enough,

$$
\begin{aligned}
&\ln \frac{\delta}{dist\left(spt\left(T+S\right), X_{n-k}\right)} - \ln \frac{\delta}{dist\left(spt\left(T\right), X_{n-k}\right)} \\
&= \begin{cases} 0, & \text{if } dist\left(a, X_{n-k}\right) > dist\left(spt\left(T\right), X_{n-k}\right) \\ \ln \frac{dist(spt(T), X_{n-k})}{dist(spt(T+S), X_{n-k})} & \text{if } dist\left(a, X_{n-k}\right) = dist\left(spt\left(T\right), X_{n-k}\right) \end{cases} \\
&\leq \begin{cases} 0, & \text{if } dist\left(a, X_{n-k}\right) > dist\left(spt\left(T\right), X_{n-k}\right) \\ \ln \frac{dist(spt(T), X_{n-k})}{dist(spt(T), X_{n-k})-r} & \text{if } dist\left(a, X_{n-k}\right) = dist\left(spt\left(T\right), X_{n-k}\right) \end{cases} \\
&\leq \frac{2r}{dist\left(a, X_{n-k}\right)}.
\end{aligned}
$$

Therefore, by the definition of h_k in (4),

$$
\begin{aligned}
&m_k^{\delta}\left(T+S\right) - \Phi_k\left(T+S\right) \\
&= \left(\ln \frac{\delta}{dist\left(spt\left(T+S\right), X_{n-k}\right)} - h_k\right) M\left(\left(T+S\right) \llcorner B\left(X_{n-k}, \delta\right)\right) \\
&\leq \frac{2r}{dist\left(a, X_{n-k}\right)} M\left(\left(T+S\right) \llcorner B\left(X_{n-k}, \delta\right)\right).
\end{aligned}
$$

□

Now, consider the positive integrand $\Psi = \sum_{k=2}^{n} \Phi_k + \Phi$, where Φ is the area integrand. Since T is $\tilde{M}-$mass minimizer, by the above Lemma 4.0.32,

$$
\begin{aligned}
&\Psi(T) - \Psi(T+S) \\
&\leq \Psi(T) - \Psi(T+S) + \tilde{M}\left(T+S\right) - \tilde{M}\left(T\right) \\
&= \sum_{k=2}^{n} \left[\Phi_k\left(T\right) - \Phi_k\left(T+S\right)\right] - \left[m_k^{\delta}\left(T\right) - m_k^{\delta}\left(T+S\right)\right] \\
&\leq \sum_{k=2}^{n} C_k \frac{r}{dist\left(a, X_{n-k}\right)} M\left(\left(T+S\right) \llcorner B\left(X_{n-k}, \delta\right)\right) \\
&\leq \sum_{k=2}^{n} C_k \frac{r}{dist\left(a, \Sigma\right)} M\left(\left(T+S\right) \llcorner B\left(\Sigma, \delta\right)\right) \\
&= \frac{C}{dist\left(a, \Sigma\right)} r M(T \llcorner B\left(\Sigma, \delta\right) + S)
\end{aligned}
$$

where $C = \sum_{k=2}^{n} C_k$ is a constant. Thus,

$$
\Psi(T \llcorner U_a) \leq \Psi(T \llcorner U_a + S) + \frac{C}{dist\left(a, \Sigma\right)} r M(T \llcorner B\left(\Sigma, \delta\right) + S).
$$

This implies T is $(\Psi, \frac{C}{dist(a,\Sigma)} r, r_a)$-minimal at a. □

Corollary 4.0.33 *(Regularity Theorem) Let T be a $\tilde{M}$-mass minimizing current of X, then the regular points are dense in $spt(T) \setminus (spt(\partial T) \cup \Sigma)$.*

5 Some simple examples

Example 1 Consider the stratified space $\mathbb{R}^2 \supset \{0\}$ with a single singular point $\{0\}$. Suppose $p, q \in \mathbb{R}^2 \setminus \{0\}$ such that 0 is on the line segment $\overline{pq}$. Then under the usual mass of $\mathbb{R}^2$, the minimal path from p to q is the line segment $\overline{pq}$ which passes through the singular point $\{0\}$ and hence does not satisfy the allowability condition. Given any small number $\delta > 0$, one can easily check that the minimum path from p to q under the modified mass is the minimum path from p to q in the domain $\mathbb{R}^2 \setminus B_\delta(0)$. A typical example looks like the following graph:

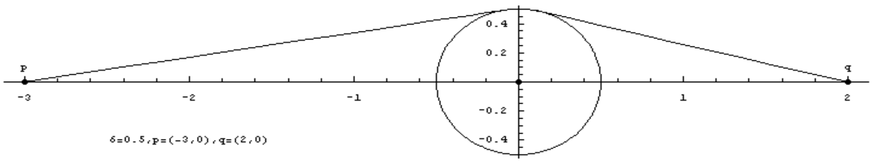


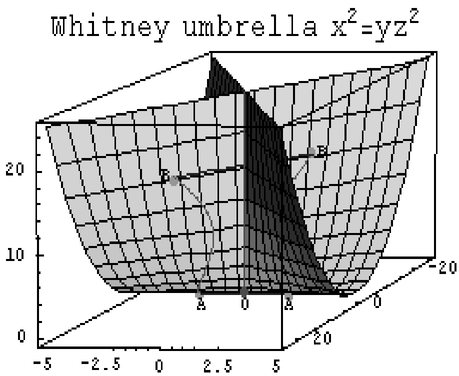
Fig. 1. $p = (-3, 0), q = (2, 0)$ and $\delta = 0.5$

Example 2 Consider Whitney umbrella $W : x^2 - zy^2 = 0$ in $\mathbb{C}^3$ with its Whitney stratification: $W \supseteq W_2 \supseteq W_0$, where W_2 is the complex z-axis defined by $x = y = 0$ and W_4 is the point $\{(0, 0, 0)\}$.

Note that for any nonzero complex number z_0, there are two associated real 2 dimensional planes $x^2 = z_0 y^2$. These two planes coincide at 0 to the plane $x = z = 0$.

Now, let's consider some simple variational problems on this famous stratified space.

Case 1: $i = 1$. Since $W \setminus W_2$ is path connected, any two points $p, q \in W \setminus W_2$ can be joined by some path inside $W \setminus W_2$. Let $\gamma_{pq} \subset W$ be a length minimizer from p to q under the usual mass. If γ_{pq} does not intersect the singular part W_2, i.e. γ_{pq} is $(\bar{p}, 1)$-allowable, then when δ is small enough, γ_{pq} will also be a minimizer from p to q under the modified mass. If γ_{pq} intersects W_2, then there exists a modified



mass minimizer $\bar{\gamma}_{pq}$ from p to q which does not intersect the singular part W_2. The shape of $\bar{\gamma}_{pq}$ looks like the graph in the previous example.

Case 2: $i = 2$. In this case, the crucial number $i - j + p_j$ in the perversity condition is given by

$$i - j + p_j = \begin{cases} 0, & j = 2 \text{ and } p_2 = 0 \\ -2, & j = 4 \text{ and } p_4 = 0 \\ -1, & j = 4 \text{ and } p_4 = 1 \\ 0, & j = 4 \text{ and } p_4 = 2 \end{cases}.$$

This means that $(\bar{p}, 2)$-allowable chains are allowed to intersect W_2 with 0 dimensional set but are not allowed to intersect W_4 unless $p_4 = 2$.

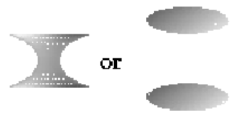
Now, let's consider two $(\bar{p}, 1)$-allowable circles in W:

$A = \left\{ \left(0, e^{i\theta}, 0\right) : \theta \in [0, 2\pi] \right\}$, the unit circle on the plane $x = z = 0$ (i.e. on the complex $y-$axis);

$B = \left\{ \left(z_0 e^{i\theta}, e^{i\theta}, z_0^2\right) : \theta \in [0, 2\pi] \right\}$, a circle around $\left(0, 0, z_0^2\right)$ on one of the two planes $x^2 = z_0^2 y^2$ associated to some $z_0 \neq 0$.

Under the usual mass, the minimal surface having boundary $A - B$ will be either a catenoid or union of two disjoint unit disks, depending on the location of z_0. In the later situation, the centers of these two unit disks are $(0, 0, 0) \in W_4$ and $\left(0, 0, z_0^2\right) \in W_2 \setminus W_4$, lying on the singular sets. Therefore, when $p_4 = 0$ or 1, the union of two unit disks does not satisfy the desired perversity conditions.

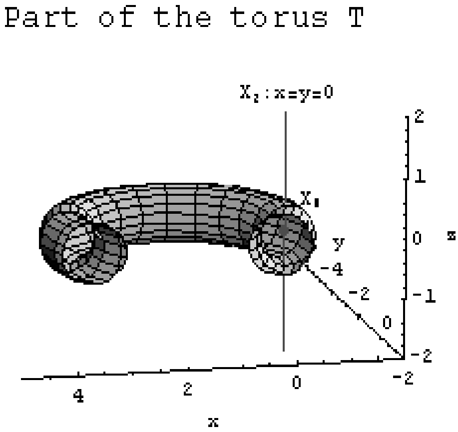
Under the modified mass, the corresponding minimal surface with boundary $A - B$ will still be either a catenoid or union of two disjoint slightly curved disks. However, in the later situation, the centers of the disks are no longer touching W_4, they all lie on $W_2 \setminus W_4$ now. This makes the minimal surface to be $(\bar{p}, 2)$-allowable as desired. Their graphs are shown in the following diagram:

$i = 2$	$p_4 = 0$ or 1	$p_4 = 2$
minimizer from A to B		

Example 3 Consider the stratified space $X \supseteq X_2 \supseteq X_0$ with $X = \mathbb{R}^4 = \{(x, y, z, t)\}$, $X_2 = \{(x, y, z, t) \,|x = y = 0\}$ and $X_0 = \{(0, 0, 0, 0)\}$. Let

$$T = \left\{ \left(2 + \cos\theta \left(2 + \frac{\cos\alpha}{2}\right), \sin\theta \left(2 + \frac{\cos\alpha}{2}\right), \frac{\sin\alpha}{2}, 0\right) : \theta, \alpha \in [0, 2\pi] \right\}$$

be a torus inside X. Note that T is $(\bar{p}, 2)$-allowable for any perversity $\bar{p}$. By using the convex hull property of mass minimizers, one easily see that the 3-dimensional volume minimizer having boundary T (under the usual mass) is simply the solid torus S. Now, $S \cap X_2 = \left\{(0, 0, z, 0) : z \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}$ has dimension 1 and passes through X_0. Thus, when $p_4 = 1$ or 2, S is $(\bar{p}, 3)$-allowable. However, when $p_4 = 0$, S becomes not allowable. In this case, one can use our modified mass to get a modified mass minimizer S' in X, having boundary T and is $(\bar{p}, 3)$-allowable. In other words, this modified mass minimizer doesn't intersect the singular point $\{(0, 0, 0, 0)\}$.



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