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AN APPLICATION OF OPTIMAL TRANSPORT PATHS TO URBAN TRANSPORT NETWORKS

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Abstract. In this article, we provide a model to study urban transport network by means of optimal transport paths recently studied by the author. Under this model, we can set up an optimal urban transport network of finite total cost which provide access to all residents from their home to their destinations. The quality of the road depends on the traffic density it carries, which make it necessary to build a large highway for heavy traffic. Moreover, we provide a reasonable pricing system for an optimal transport network, under which all residents will travel to their destinations by the network. We also studied the problem of expanding and modifying a given network.

1. **Introduction.** People travel everyday from their home to their working places or other destinations. Thus, building an efficient urban transportation network becomes essential. Many mathematical models have been given to this problem, usually by means of the graph theory. In [1][2], Buttazzo, Oudet and Stepanov gave a new approach to this problem, by means of Monge-Kantorovich mass transportation. For reader's convenience, we briefly recall their approach as follows.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set standing for the city, μ^+ be a probability measure on $\bar{\Omega}$ representing the density of the homes, and μ^- be another probability measure on $\bar{\Omega}$ representing the density of the working places. Let $\mathcal{H}^1$ denote the one dimensional Hausdorff measure. For each closed and connected set Σ with $\mathcal{H}^1(\Sigma) < +\infty$, one can study the following Monge-Kantorovich problem with the Dirichlet constraint Σ:

$$\text{minimize } I_\Sigma(\gamma) = \int_{\bar{\Omega}\times\bar{\Omega}} d(x,y) \wedge (dist(x,\Sigma) + dist(y,\Sigma))\, d\gamma(x,y)$$

among all transport plans $\gamma \in \mathcal{P}\left(\bar{\Omega}\times\bar{\Omega}\right)$ satisfying

$$\pi^+_{\#}\gamma - \pi^-_{\#}\gamma = \mu^+ - \mu^- \text{ over } \bar{\Omega}\setminus\Sigma,$$

where $d(x,y)$ denotes the geodesic distance from x to y in $\bar{\Omega}$. This problem always has a solution, and we let $MK(\mu^+,\mu^-,\Sigma)$ denote the minimum value of this problem. Then, an optimal urban transport network in the BOS model [1] [2] is a solution to the following minimizing problem:

$$\min\ \left\{MK\left(\mu^+,\mu^-,\Sigma\right) : \Sigma \subset \bar{\Omega} \text{ closed and connected}, \mathcal{H}^1(\Sigma) \le \Lambda\right\} \qquad (1)$$

where $\Lambda > 0$ is a given length constraint.

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They are able to get the existence of an optimal transportation network Σ under any given length constraint Λ. An optimal transport network Σ is a closed connected 1-set of assigned length, enjoying some nice topological and geometrical properties. However, it becomes complicated in studying the regularity properties of optimal transport network. It is not clear the asymptotic behavior of optimal transport network when Λ approaches to ∞. Moreover, highways and ordinary streets cannot be distinguished in their model.

In this article, we will give a new approach to the optimal urban transportation problem by means of the optimal transport paths build by the author in [3] and [4]. This model is closely related to the above BOS model in the sense that when the length constraint Λ of (1) becomes large enough (or more precisely, larger than the d_0-distance between μ^+ and μ^-, given by the author in [3]), the optimal transport network desired in the BOS model will be the optimal transport path in the sense of the author [3] with $\alpha = 0$. Thus, the regularity properties of the optimal transport network in the BOS model follows from the interior regularity of optimal transport path, achieved by the author in [4]. However, in general,the d_0 distance between μ^+ and μ^- may not be finite.

There are several advantages of this new model. When our parameter $\alpha \in (1/2, 1)$, which is possibly true in reality, we can build an optimal urban transport network of finite total cost which provides access to all residents from their home to their destinations. Thus, we do not need a length constraint Λ here. The restriction of Σ being connected in the BOS model is also not needed here. Also, in this model, the quality of the road will depend on the traffic density it carries. Thus, it is natural to have a larger highway for heavier traffic density. Moreover, in this model, it is easy to setup a reasonable pricing system for an optimal transport network. Under this pricing system, all residents will travel by the network because the ticket price is cheaper than the total cost of travelling by their own means.

The article is organized as follows. We first express the urban transport network problem in the framework of the optimal transport paths. Instead of viewing a urban transport network as a closed connected one dimension set of assigned length as in the BOS model, we view a urban transport network as a transport path from μ^+ to μ^- , which is a real coefficient rectifiable $1-$current. Then we will restate the main results of [3][4] without proof in terms of urban transport networks. A crucial step is to introduce the $\mathbf{M}_\alpha$ cost of urban transport networks which will force nearby items to travel on a common route. After that, we will set up a reasonable price system for an optimal urban transportation network. Under that price system, it will be advantageous for any resident to travel their destinations by the network.

In reality, most cities may already have some urban transport networks. When we build a new network, we should take advantage of the old one as much as possible. How to build a new efficient network from the old one? In section 4, we formulate the problem and prove the existence theorem. Away from the old system, the new system will still have the desired regularity.

2. **Urban transport networks.** In building an efficient urban transport network, we keep the following considerations in our mind. First, a reasonable network should be able to provide transportation for every resident to its destination. Second, to be realistic, the total cost of building the network should be finite, and as small as possible. Moreover, after building the network, the total travelling expenses for the whole population (e.g. travelling time, cost on fuels etc.) should be as small as possible.

Let $X \subset \mathbb{R}^2$ be a bounded region where people lived. It can be a city, a state, a country, or any region. For simplicity, we assume X to be convex. Suppose μ^+ is a probability measure in X representing the population density in X, while μ^- is another probability measure representing the density of destinations (e.g. business areas, landscapes, etc.).

Roughly speaking, a urban transport network corresponding to μ^+ and μ^- is a countable union of roads of various sizes connecting μ^+ to μ^-. In mathematical language, a urban transport network is a real coefficient rectifiable $1-$current of the form

$$T = \sum_i \theta_i \left[\left[l_i\right]\right]$$

with

$$\partial T = \mu^+ - \mu^-,$$

where $\theta_i \in (0,1]$ and l_i denotes an oriented segment in X. In other words, T is a countable weighted directed graph from μ^+ to μ^-, satisfying the "Kirchhoff's law" of electric currents at every interior vertices. Let

$$Path\left(\mu^+, \mu^-\right)$$

be the family of all urban transport networks from μ^+ to μ^-.

Now, let us consider the total cost of the network, which arises as the sum of the total construction cost for building the network and the total cost of consuming the network. We want to build a reasonable mathematical model for it.

For a road of length l with traffic density θ, what should be the cost of the road? As for the construction cost c_b for building the road, c_b should be proportional to l, the length of the road. Also, it should be an increasing function of the θ, the traffic density. The heavier the traffic is, the higher quality of road is needed, and thus the higher the cost of building the road is. However, c_b needs not to be proportional to θ, instead it should be a concave function of θ. The reason is that when two separate roads of density θ_1 and θ_2 are nearby, it is cheaper to build a common road supporting traffic density $\theta_1 + \theta_2$. Building a "Y shaped" road is cheaper than building a "V shaped" road, for at least the former occupies less land.

As for the travelling cost c_t for the road, c_t is also proportional to l. Moreover, it is also an increasing but concave function of θ. The higher is the density, the higher quality is the road, and thus, the less is the travelling time.

Therefore, the total cost of the road is an increasing but concave function of the traffic density θ and proportional to the length l. For simplicity, we model it by

$$\theta^\alpha l$$

for some parameter $0 \leq \alpha < 1$. When $\alpha = 0$, the quality of the road is independent of the traffic density. Thus, it is no need to build a highway. In reality, the value of the parameter α depends on many factors such as the ratio of the actual cost of building roads of different qualities, the cost of land, and so on. Usually $\alpha \in (0,1)$, which makes it necessary to build a large highway for heavy traffic.

Now, for any urban transport network $T = \sum_i \theta_i \left[\left[l_i\right]\right] \in Path\left(\mu^+, \mu^-\right)$, its total cost is given by

$$\mathbf{M}_\alpha\left(T\right) = \sum_i \left(\theta_i\right)^\alpha \text{length}\left(l_i\right). \tag{2}$$

To find an optimal urban network is to solve the following variational problem:

$$\text{Minimize } \mathbf{M}_\alpha\left(T\right)$$

among all urban transport networks $T \in Path(\mu^+, \mu^-)$.

In [3][4], the author solved this problem. For convenience, we restate the main results here:

Proposition 1 (Existence). *Suppose* $\alpha \in (1/2, 1)$. *For any two probability measures* $\mu^+, \mu^- \in \mathcal{P}(X)$*, there exists an optimal urban transport network*

$$T = \sum_i \theta_i [[l_i]]$$

with $\partial T = \mu^+ - \mu^-$ *under the cost function (2).*

Note that this result follows from a combination of the existence theorem 3.1 of [3] and the rectifiability theorem of [4, theorem 2.7]. Please see in [4, section 4.1] for more details.

Proposition 2 (Distance). *Suppose we set*

$$d_\alpha\left(\mu^+, \mu^-\right) = \min\left\{\mathbf{M}_\alpha(T) : T \in Path\left(\mu^+, \mu^-\right)\right\}.$$

Then, we have

- *(*[3, theorem 4.1]*)* d_α *gives a distance on the space* $\mathcal{P}(X)$ *of all probability measures in* X.
- *(*[3, theorem 4.2]*)The distance* d_α *metrizies the weak * topology of* $\mathcal{P}(X)$.
- *(*[3, theorem 5.1]*)*$(\mathcal{P}(X), d_\alpha)$ *forms a length space.*

Proposition 3 (Finite Cost). *(*[3, theorem 3.1]*)For each* $\alpha \in (1/2, 1)$

$$d_\alpha\left(\mu^+, \mu^-\right) \leq \frac{1}{2^{2\alpha} - 2} diameter(X).$$

Proposition 4 (Approximation). *(*[3, corollary 4.3]*) Suppose* $\mathrm{A}_n(\mu^+)$ *and* $\mathrm{A}_n(\mu^-)$ *are the* n^{th} *dyadic approximations of* μ^+ *and* μ^- *respectively. Suppose* $\alpha \in (1/2, 1)$. *Let* $G_n \in Path(A_n(\mu^+), A_n(\mu^-))$ *be an optimal transport network. Then, under flat metric of real flat* $1-$*chains,* $\{G_n\}$ *subsequently converges to an optimal transport network* T *between* μ^+ *and* μ^-. *Moreover,*

$$|\mathbf{M}_\alpha(T) - \mathbf{M}_\alpha(G_n)| \leq \frac{diam(X)}{2^{2\alpha-1} - 1}\left(2^{1-2\alpha}\right)^n \to 0$$

as $n \to \infty$.

Proposition 5 (Regularity). *(*[4, theorem 4.10]*) Suppose* $T \in Path(\mu^+, \mu^-)$ *is an optimal transport network of finite total* $\mathbf{M}_\alpha$ *cost. Then, at any point* p *on the support of* T *but not on the support of* $\mu^+ - \mu^-$*, there exists an open ball neighborhood* B_p *of* p *such that the support of* $T\lfloor B_p$ *is a cone which consists of finitely many segments with suitable multiplicities. The angles between the segments are determined by the parameter* α *as well as the multiplicities carried by these segments. Moreover, for an optimal transport network* T*, the support of* T *contains no cycles on* $X \setminus \{spt(\mu^+) \cup spt(\mu^-)\}$.

In the end of this section, we want to point out some links between this model and the BSO model.

Assume $d_0(\mu^+, \mu^-) < +\infty$, then there exists an optimal transport network T in the sense of this model such that $\mathcal{H}^1(T) = d_0(\mu^+, \mu^-) < +\infty$. For any $\Lambda \geq d_0(\mu^+, \mu^-)$, we claim that $spt(T)$ is also an optimal transport network in the sense of Buttazzo, Oudet and Stepanov [1] [2] if $spt(T)$ is connected. The reasoning is very simple. Since $\partial T = \mu^+ - \mu^-$,

$$dist\,(x, spt\,(T)) = 0$$

for all $x \in spt\,(\mu^+) \cup spt\,(\mu^-)$. Thus,

$$MK\left(\mu^+, \mu^-, spt\,(T)\right) = 0$$

and $spt\,(T)$ is a minimizer in the BOS model. In the case that $spt(T)$ is not connected, then each connected component $\tilde{\Sigma}$ of $spt(T)$ corresponds to an optimal transport network in the sense of BOS. To see this, we note that

$$\partial(T\lfloor\tilde{\Sigma}) = \mu^+\lfloor\tilde{\Sigma} - \mu^-\lfloor\tilde{\Sigma}.$$

A similar argument as above says that the connected component $\tilde{\Sigma}$ is an optimal transport network from $\mu^+\lfloor\tilde{\Sigma}$ to $\mu^-\lfloor\tilde{\Sigma}$. Note that, in general, $d_0\,(\mu^+, \mu^-)$ may be ∞.

3. **A reasonable pricing structure.** Even before constructing a urban transport network, the company would like to estimate its potential revenue. Besides any possible supports from outside sources, the company mainly earns its revenue from charging its customers directly or indirectly. Thus, it becomes necessary for the company to build a reasonable pricing struture for tolls. That is, how to set the price so that the company can maximize the profit but still attracting enough residents for using the network? To ensure residents will use the network, any reasonable pricing structure needs to satisfy the following constraint:

Constraint 1. It is cheaper to travel on the network than to travel through any alternative method.

We now give a reasonable pricing struture as follows. Suppose we have built an optimal urban transport network

$$T = \sum_i \theta_i\,[[l_i]]$$

corresponding to μ^+ and μ^- under some cost function $\mathbf{M}_\alpha$ for some $\alpha \in (\frac{1}{2}, 1)$. Each road l_i here associates with a traffic density θ_i. For this fixed parameter α, we consider a positive decreasing function

$$p(x) = x^\alpha - (x-1)^\alpha$$

for any $x \in [1, +\infty)$.

Now, for any $\theta > 0$, let $C(\theta)$ denote the minimum travelling expense for any θ residents travelling a unit distance among all posissible alternative methods. Then, we set the price for these residents to travel on each road l_i to be

$$p(\frac{\theta_i}{\theta})C(\theta) \tag{3}$$

per unit distance.

Since $p(x)$ is a decreasing function of x, the price decreases as θ_i increases. As a result, the price for using a highway is cheaper than the price for using an ordinary street. Also, the following proposition says that it is cheaper to travel in a group than as an individual.

Proposition 6. *For any* $\lambda \geq 1$ *and any* θ, θ_i *as above, we have*

$$p(\frac{\theta_i}{\lambda\theta})C(\lambda\theta) \leq \lambda p(\frac{\theta_i}{\theta})C(\theta)$$

whenever $C(\lambda x) \leq \lambda C(x)$*.*

Proof. Since $p(x)$ is decreasing,

$$p(\frac{\theta_i}{\lambda\theta})C(\lambda\theta) \leq p(\frac{\theta_i}{\theta})C(\lambda\theta) \leq \lambda p(\frac{\theta_i}{\theta})C(\theta).$$

□

Moreover, we have the following result:

Theorem 1. *The pricing system given in (3) satisfies the constraint 1.*

Proof. Suppose there are θ residents travelling from a point $P \in spt\,(\mu^+)$ to a point Q on the support of the network Σ. We need to show that the total price for travelling on the network is less than $|PQ|\,C\,(\theta)$, the expense of travelling by their own ways. On the optimal transport network T, there exists a unique route from P to Q on the system. Assume this route is given by

$$\gamma_{PQ} = \sum_i \theta_i\,[[L_i]]\,.$$

Note that since T is an optimal transport network,

$$\mathbf{M}_\alpha\,(T) \leq \mathbf{M}_\alpha\,(T - \gamma_{PQ} + \theta\,[[PQ]])\,,$$

which means

$$\sum_i (\theta_i)^\alpha \,\text{length}\,(L_i) \leq \sum_i (\theta_i - \theta)^\alpha \,\text{length}\,(L_i) + \theta^\alpha\,|PQ|\,.$$

Therefore, we have

$$\sum_i \left[\left(\frac{\theta_i}{\theta}\right)^\alpha - \left(\frac{\theta_i}{\theta} - 1\right)^\alpha\right] \text{length}\,(L_i) \leq |PQ|\,.$$

Thus, under the price system 3, the total price for them to travel on the network is less than $|PQ|\,C\,(\theta)$, which is the travel expense by their own means. As a result, residents would prefer to use the network. □

As a result, all residents will travel to their destinations by the network under the above pricing system. In other words, it is cheaper for any resident to travel by the network than by any alternative method. This price system is reasonable for general setting. Any optimal urban transport network can adapt this price system. One can using this price system to estimate the potential revenue before building the network.

4. **Modifying a given urban system.** Assume the region X has already an old urban transportation system, represented by a compact set Σ with $\mathcal{H}^1\,(\Sigma) < +\infty$. We want to build a new urban transport system from it. The constructing cost for building roads on Σ and outside of Σ is different. To distinguish them, we use two cost functionals $\mathbf{M}_\alpha$ and $\mathbf{N}$ to measure the cost of a transport network on Σ and outside of Σ. That is, for each transport network $T \in Path\,(\mu^+, \mu^-)$, we let

$$\mathbf{H}_\alpha\,(T) = \mathbf{M}_\alpha\,(T\lfloor(X \setminus \Sigma)) + \mathbf{N}\,(T\lfloor\Sigma)$$

for some $\alpha \in (1/2, 1)$. Here $\mathbf{N}$ is some given weakly lower semicontinuous functional on real rectifiable 1-currents. Since the cost for maintaining a road on Σ is usually cheaper than building a similar road outside of Σ, we usually assume

$$\mathbf{N}(T\lfloor\Sigma) \leq \mathbf{M}_\alpha(T\lfloor\Sigma)$$

for each $T \in Path(\mu^+, \mu^-)$. A typical example of the functional $\mathbf{N}$ is $\mathbf{M}_\beta$ for some $1 > \beta \geq \alpha$. In some situation, $\mathbf{N}$ may even be identically zero. Also, since both μ^+ and μ^- are probability measures, the density function θ of each $T \in Path(\mu^+, \mu^-)$ is always not greater than 1. Thus, we have

$$\mathbf{N}(T\lfloor\Sigma) \leq \mathbf{M}_\alpha(T\lfloor\Sigma) \leq \mathcal{H}^1(\Sigma) < +\infty.$$

Therefore,

$$\mathbf{H}_\alpha(T) \leq \mathbf{M}_\alpha(T) \leq \mathbf{H}_\alpha(T) + \mathcal{H}^1(\Sigma)$$

for each $T \in Path(\mu^+, \mu^-)$.

Proposition 7. *For each given $\alpha \in (1/2, 1)$, there exists a $T \in Path(\mu^+, \mu^-)$ with the least $\mathbf{H}_\alpha(T)$ cost among all $T \in Path(\mu^+, \mu^-)$. Moreover, T enjoys the same regularity of an optimal transport network on the region $X \setminus \Sigma$ as in proposition 5.*

Proof. Let T_0 be an optimal transport path in $Path(\mu^+, \mu^-)$ minimizing the $\mathbf{M}_\alpha$ cost functional. Since $\alpha > 1/2$, we know $\mathbf{M}_\alpha(T_0)$ is finite. Let $\{T_i\} \subset Path(\mu^+, \mu^-)$ be a minimizing sequence of $\mathbf{H}_\alpha$. Then,

$$\begin{aligned}\liminf_{i\to\infty} \mathbf{M}_\alpha(T_i) &\leq \liminf_{i\to\infty} \mathbf{H}_\alpha(T_i) + \mathcal{H}^1(\Sigma) \\ &\leq \mathbf{H}_\alpha(T_0) + \mathcal{H}^1(\Sigma) \\ &\leq \mathbf{M}_\alpha(T_0) + \mathcal{H}^1(\Sigma) < +\infty.\end{aligned}$$

Thus $\{\mathbf{M}_\alpha(T_i)\}$ is uniformly bounded. By the compactness theorem of real coefficient rectifiable $1-$currents under $\mathbf{M}_\alpha$ cost (see [4, theorem 2.7]), T_i subsequently converges to a rectifiable $1-$current T in $Path(\mu^+, \mu^-)$. Since $T\lfloor(X \setminus \Sigma)$ is also an $\mathbf{M}_\alpha$ minimizer, by the interior regularity theorem ([4, Theorem 4.3]), T enjoys the same regularity as in proposition 5 on $X \setminus \Sigma$. Also, by the lower semicontinuity of $\mathbf{H}_\alpha$, T is an optimal urban transport network for $\mathbf{H}_\alpha$. □

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