On functional algebra*

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The study of the consequences that may be deduced about the roots of an nth degree polynomial from knowledge of its coefficients ranks among the oldest problems in algebra and has been developed primarily by Descartes, Newton, Fourier, Sturm, Cauchy, Hermite, Laguerre and others into a fertile theory. These days, however, the algebra in this beautiful theory, which could be called functional algebra, has made way to analysis; that is on the one hand a result of the natural march of evolution, and on the other hand is due to the fact that problems for which a new point of view does not present itself end up being considered as closed. This is natural, and one need not turn back the clock of this evolution; and yet I would like to show a connection with a classical problem of analysis where one finds extended blank regions on the functional algebra map. This will involve the formulation of a simple point of view that isn't entirely new, then a few simple initial results that suggest themselves, and then a few questions that arise in this direction.

The problem that I have mentioned is the central problem of the analytic theory of numbers: the Riemann hypothesis, which, while remaining in the background, managed nonetheless to provide inspiration for works by Laguerre, Jensen, Pólya, I. Schur and others. Denoting by $s = \sigma + it$ a complex variable, we will be concerned with the zeta function $\zeta(s)$ of Riemann, defined in the half-plane $\sigma > 1$ by the series

$$\zeta(s) = \sum_{\nu=1}^{\infty} \nu^{-s}.$$

It is well-known that this function can be continued analytically in the entire complex plane except for a simple pole at the point $s = 1$, and that the nontrivial zeros, located in the strip $0 < \sigma < 1$, have an intimate relationship with the prime numbers. We know as well the important number-theoretic consequences which would follow if the above-mentioned Riemann hypothesis, according to which the nontrivial zeros of $\zeta(s)$ are located on the line $\sigma = 1/2$, were to be proved. And we know that a large portion of those consequences would already follow if there existed a $0 < \vartheta < 1/2$ such that all the nontrivial zeros were located in the strip

$$\vartheta \leq \sigma \leq 1 - \vartheta. \tag{1}$$

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Whether there exists such a ϑ has not been settled as of yet. Introducing in place of $\zeta(s)$ the function

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

and in place of s the variable $z = x + iy$ with $s = \frac{1}{2} + iz$, we obtain in $\xi(\frac{1}{2} + iz) = \Xi(z)$ a transcendental entire function whose zeros would all be real, or would be located in the strip $|y| \leq \frac{1}{2} - \vartheta$, if the original Riemann hypothesis, or, respectively, its weaker version (1), were true.

The question that now presents itself is, What is the analytic representation most suitable for showing that the zeros of the function thus represented are located in a strip, or even all lie on the real line? The most obvious representation is the Taylor series. But the fact that this is not suitable for our purposes can be seen on the one hand from the lack of any *simple* sufficient condition on the Taylor coefficients of an entire function implying the reality of its zeros; and on the other hand, from the fact that finding *necessary and sufficient* conditions on the Taylor coefficients for the zeros to all be real presents difficulties which appear insurmountable. The same difficulties are already present in the case of a finite series, that is, when the function in question is a polynomial. Since the case of polynomials contains the essence of the difficulties in the general case, let us consider the question, Which representation of a polynomial is best suited for showing that its zeros are all real or are located inside a strip?

The following heuristic reasoning shall guide us. Writing the polynomial in the form

$$V(z) = \sum_{\nu=0}^{n} a_\nu z^\nu, \tag{2}$$

we may conclude from the coefficients *in a simple manner* that certain circular regions contain all the zeros of $V(z)$. It must be so, since the level curves of the functions $z \mapsto z^\nu$ are concentric circles. If we wish to find *in a simple manner* a *strip* containing all the zeros of the polynomial, we have to rearrange the polynomial, expanding it in polynomials whose level curves are "essentially" straight lines running parallel to the real axis. Such are for example the Hermite polynomials, defined by the formula

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\left(e^{-z^2}\right). \tag{3}$$

We have therefore arrived at the idea that in questions concerning the reality of zeros of a polynomial it is probably useful to consider its Hermite expansion

$$H(z) = \sum_{\nu=0}^{n} b_\nu H_\nu(z). \tag{4}$$

We can formulate this idea in a more general fashion. Given a domain K, whose boundary consists for example of a finite numbre of Jordan arcs, and a point $P \in K$, enlarge the domain K around the central point P with an enlargement coefficient λ varying from 0 to ∞. We thus obtain a family of domains K_λ. To characterize the fact that all the zeros of a polynomial are located in a domain K_{λ_0} and related questions, it would make sense to begin with the representation

$$\Phi(z) = \sum_{\nu=0}^{n} c_\nu \phi_\nu(z) \tag{5}$$

of the polynomial, where the level curves of the polynomials $\phi_\nu(z)$ are "essentially" the boundaries of the domains K_λ.

The idea that, in order to get simple criteria for the reality of the zeros, instead of the expansion (2) of the polynomial—which I will name the Vieta expansion—another expansion is perhaps more suitable, has already appeared in the literature. We find the simplest criterion of this type if we consider the expansion

$$T(z) = \sum_{\nu=0}^{n} d_\nu T_\nu(z) \tag{6}$$

of the polynomial, where $T_\nu(z)$ denotes the νth Chebyshev polynomial of the first kind, defined by the formula

$$T_\nu(\cos\vartheta) = \cos\nu\vartheta.$$

We see easily that if, for example,

$$|d_n| > \sum_{\nu=0}^{n-1} |d_\nu|,$$

then all the zeros of $T(z)$ are real and each of the intervals

$$\cos\frac{\nu\pi}{n} < z < \cos\frac{(\nu-1)\pi}{n} \qquad (\nu = 1, 2, \ldots, n)$$

contains exactly one zero. Or I am thinking of the deeper theorem of Pólya and Szegő, according to which, if

$$0 < d_0 < d_1 < \ldots < d_n, \tag{7}$$

then all the zeros of $T(z)$ are real and each interval

$$\cos\frac{\left(\mu+\frac{1}{2}\right)\pi}{n+\frac{1}{2}} < z < \cos\frac{\left(\mu-\frac{1}{2}\right)\pi}{n+\frac{1}{2}} \qquad (\mu = 1, 2, \ldots, n)$$

contains exactly one zero of $T(z)$. However, as far as I know there have been no attempts to place these theorems within the framework of a theory and to construct a functional algebra involving other representations than that of Vieta, except for one area, which is that of the rule of Descartes;[1] but from the research done by Schoenberg, Obrechkoff and others we see that, in short, the rule of signs based on the Vieta representation (2) gives a better evaluation for the number of positive zeros than the rule of signs based on other representations. But I have shown in a recent short note[2] that this point of view can

[1] Translator's note: see the section "What is the Basis of Descartes' Rule of Signs" (Part Five, Chapter 1, §7, pp. 50–51) in the book Problems and Theorems in Analysis II by G. Pólya and G. Szegő, Springer, 1998.

[2] Translator's note: Turán is referring here to his paper "On Descartes-Harriot's rule" (*Bull. Amer. Math. Soc.* **55** (1949), 797–800).

produce something conceptually new on this topic. It is a common feature of the rules of signs based on different representations of a polynomial that they give an upper bound for the number of zeros located in a real interval based on knowledge of sign changes of a sequence formed by the coefficients and their partial sums. If we consider now the Laguerre polynomials defined by the formula

$$L_\nu(z) = \frac{e^z}{\nu!}\frac{d^\nu}{dz^\nu}\left(e^{-z}z^\nu\right)$$

it is well-known that their level curves are essentially parabolas whose focus is at the origin and whose axis is the positive axis. By virtue of the heuristic principle that we have just found, the Laguerre representation

$$L(z) = \sum_{\nu=0}^{n} e_\nu L_\nu(z) \tag{8}$$

of a polynomial is the one to consider when one wishes to study the positive zeros. I have shown in my abovementioned note that the sequence formed by successive differences of the coefficients in the Laguerre representation

$$e_0, (e_0 - e_1), (e_0 - 2e_1 + e_2), \ldots, \left(e_0 - \binom{n}{1}e_1 + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}e_n\right)$$

has the property that its number of sign changes gives a *lower* bound for the number of positive zeros.

Our first question will therefore be whether the Hermite representation is indeed suitable for the study of questions on the reality of the zeros, respectively on bounding the *imaginary* part of the zeros, through the coefficients b_ν in a simple and yet effective manner, or, quite simply, in a "natural" manner. The results that I will mention first give an affirmative answer to the second part of the question. We have for the Vieta representation the classical estimate of Cauchy: if

$$\max_{0<\nu<n-1} |a_\nu| = A, \tag{9}$$

then all the zeros of the polynomial are located in the disk

$$|z| \leq 1 + \frac{A}{|a_n|}. \tag{10}$$

By analogy with this theorem, I have shown for the Hermite representation that if we have

$$\max_{0<\nu<n-1} |b_\nu| = B, \tag{11}$$

then all the zeros of the polynomial are located in the strip

$$|\operatorname{Im}(z)| \leq \frac{1}{2}\left(1 + \frac{B}{|b_n|}\right). \tag{12}$$

The bound (10) cannot be replaced by a disk

$$|z| \leq \vartheta\left(1 + \frac{A}{|a_n|}\right)$$

for any $0 < \vartheta < 1$. Similarly, we cannot replace the strip (12) with the strip

$$|\operatorname{Im}(z)| \leq \frac{\vartheta}{2}\left(1 + \frac{B}{|b_n|}\right),$$

as can be shown with a suitable example.

Considering the fact that (10) gives in particular a bound for $|\operatorname{Im}(z)|$, one might ask if (10) doesn't imply (12). It is easy to see through the example $V(z) = H(z) = H_n(z)$ that that is not the case, since (12) gives in that case (B being 0) that the zeros of the polynomials are located in the strip $|\operatorname{Im}(z)| \leq \frac{1}{2}$ (they are all real, as we know). We know however that the largest positive zero of $H_n(z)$ is asymptotically equal to $\sqrt{2n+1}$ for n large; the radius of the disk in (10) is therefore at least $\sqrt{2n+1}$; that is to say, the bound on the imaginary part of the zeros given by (10) is surely weaker than that of $\operatorname{Im}(z)| \leq 2n+1$, which is essentially weaker than that of (12).

There are however other parallel theorems between the reprsentations of Vieta and Hermite. I will mention two of those. By a theorem of Walsh concerning the Vieta representation, all the zeros of the polynomial are located in the disk

$$|z| \leq \left|\frac{a_0}{a_n}\right|^{\frac{1}{n}} + \left|\frac{a_1}{a_n}\right|^{\frac{1}{n-1}} + \ldots + \left|\frac{a_{n-1}}{a_n}\right| = A_1. \tag{13}$$

The corresponding theorem for the Hermite representation states that all the zeros of the polynomial are located in the strip

$$|\operatorname{Im}(z)| \leq \left|\frac{b_0}{a_n}\right|^{\frac{1}{n}} + \left|\frac{b_1}{a_n}\right|^{\frac{1}{n-1}} + \ldots + \left|\frac{b_{n-1}}{a_n}\right| = B_1. \tag{14}$$

The example of the polynomial

$$H(z) = H_n(z) + \varepsilon H_{n-1}(z) + \varepsilon^2 H_{n-2}(z) + \ldots + \varepsilon^n H_0(z), \tag{15}$$

where ε is a small positive number, shows that this bound is eventually better than that of (12), which gives the bound $|\operatorname{Im}(z)| \leq \frac{1}{2}$ for the zeros, while the estimate (14) bounds their imaginary part by $B_1 = n\varepsilon$, which goes to 0 with ε. On the other hand, by a remark of Kakeya, if the coefficients a_ν of the Vieta representation are positive, then the zeros of the polynomial are all located in the disk

$$|z| \leq \max\left(\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_{n-1}}, \ldots, \frac{a_0}{a_1}\right) = A_2. \tag{16}$$

Similarly, if the coefficients b_ν of the Hermite representation are positive, then the zeros of the polynomial are all located in the strip

$$|\operatorname{Im}(z)| \leq \max\left(\frac{b_{n-1}}{b_n}, \frac{b_{n-2}}{b_{n-1}}, \ldots, \frac{b_0}{b_1}\right) = B_2. \tag{17}$$

In the case of the polynomial (15), this gives the strip $|\operatorname{Im}(z)| \leq \varepsilon$, which is a better estimate than that of (14).

All these results give a hint that the Hermite representation really is more suitable than the Vieta representation for the task of bounding the imaginary part of the zeros. Before moving on to results of a different type I will mention two more specialized results in this vein. We see easily that the Hermite representation of the function $\Xi(z)$ of Riemann is of the form

$$\Xi(z) \sim \sum_{\nu=0}^{\infty} (-1)^{\nu} b'_{2\nu} H_{2\nu}(z) \tag{18}$$

where the coefficients $b'_{2\nu}$ are positive. That leads us to the idea of studying the Hermite representation of even polynomials of the form

$$H^*(z) = \sum_{\nu=0}^{k} b^*_{2\nu} H_{2\nu}(z) \tag{19}$$

and, as a further specialization, those of polynomials of the form

$$H^{**}(z) = \sum_{\nu=0}^{k} (-1)^k b^{**}_{2\nu} H_{2\nu}(z) \tag{20}$$

with $b^{**}_{2\nu} > 0$, corresponding to the representation (18). For the polynomials $H^*(z)$, the bound (12) can be considerably improved; the zeros of $H^*(z)$ are all located in the strip

$$|\operatorname{Im}(z)| < \frac{\lambda}{\sqrt{k}} \left(1 + \frac{B^*}{|b^*_{2k}|}\right) = \frac{\lambda}{\sqrt{k}} \left(1 + \frac{B}{|b^*_{2k}|}\right) \tag{21}$$

where λ is a numerical constant. The same theorem states in another form that if we have

$$\max_{\nu=0,1,\ldots,(k-1)} \frac{|b^*_{2\nu}|}{\sqrt{(\nu+1)(\nu+2)\cdots k}} = B^*_0, \tag{22}$$

then all the zeros of $H^*(z)$ are located in the strip

$$|\operatorname{Im}(z)| \le 2\sqrt{2}\left(1 + \frac{B^*_0}{|b^*_{2k}|}\right). \tag{23}$$

The constant $2\sqrt{2}$ can be replaced by a smaller one by making a more careful estimate. The other result I mentioned bounds the zeros of $H^*(z)$ within a region inscribed by the arcs of a hyperbola. Denoting the complex variable by

$$z = x + iy \tag{24}$$

the zeros of $H^*(z)$ are located in the region

$$|xy| \le \frac{3}{4}\left(1 + \frac{B^*}{|b^*_{2k}|}\right). \tag{25}$$

Returning to our previous discussion, we must see now that with the help of the coefficients of the Hermite representation we can give *in a simple manner* criteria involving the reality of the zeros. One can show that all the zeros of $H(z)$ are real if the coefficients b_ν are real and

$$\max(|b_0|, |b_1|, \ldots, |b_{n-2}|) \le |b_n|\sqrt{2n-2}. \tag{26}$$

We have the same conclusion if the weaker condition

$$\sum_{\nu=0}^{n-2} |b_\nu|^2 2^\nu \nu! < |b_n|^2 2^n (n-1)! \tag{27}$$

holds. Writing $H(z)$ in the form

$$H(z) = \sum_{\nu=0}^{n} \frac{\beta_\nu}{\sqrt{2^\nu \nu!}} H_\nu(z), \tag{28}$$

all the zeros of $H(z)$ are real if we have

$$n \sum_{\nu=0}^{n-2} |\beta_\nu|^2 < |\beta_n|^2. \tag{29}$$

In view of the form (18) of the function $\Xi(z)$, the theorems concerning the more specialized polynomials $H^{**}(z)$ become interesting. We can show that if the coefficients satisfy[3]

$$\frac{b^{**}_{2\nu+2}}{b^{**}_{2\nu}} > \frac{1}{4} \qquad (\nu = 0, 1, \ldots, (k-1)), \tag{30}$$

then all the zeros of $H^{**}(z)$ are real and simple. Reasoning involving the function $\Xi(z)$ leads us to consider the polynomial of the form

$$H^{***}(z) = \beta'_0 H_0(z) - \frac{\beta'_2}{2^2 \cdot 2!} H_2(z) + \\ + \frac{\beta'_4}{2^4 \cdot 4!} H_4(z) - \ldots + (-1)^k \frac{\beta'_{2k}}{2^{2k}(2k)!} H_{2k}(z). \tag{31}$$

I have not succeeded in deciding whether the condition

$$0 < \beta'_0 < \beta'_2 < \beta'_4 < \ldots < \beta'_{2k} \tag{32}$$

for $H^{***}(z)$ implies the reality of all the zeros of $H^{***}(z)$. Because of certain analogies with trigonometric series, it is of some interest to note that the sum of the series

$$H^{****}(z) = \sum_{\nu=0}^{\infty} \frac{\beta_{2\nu}}{(2\nu)!} H_{2\nu}(z) \tag{33}$$

is nonnegative on the real axis, in fact it is even a totally monotone function of z on the positive real axis, provided that

$$\beta''_0 \ge \beta''_2 \ge \beta''_4 \ge \ldots \ge 0. \tag{34}$$

[3]The inequality (30) in the original French version has $b^*_{2\nu}$ in the denominator rather than $b^{**}_{2\nu}$, but this appears to be a typographical error.

Theorem (30) gives rise to an interesting question. Its proof involves showing that if the condition (30) only holds for $\nu = 0, 1, 2, \ldots, k-1$ where $\ell < k$, then the polynomial has at least 2ℓ sign changes. This remains true if the coefficients $b_{2\ell+2}, b_{2\ell+4}, \ldots, b_{2k}$ vary in an arbitrary manner provided they remain real. This reminds one immediately of the results in the flavor of the ideas of Picard and Landau concerning the Vieta representation, according to which at least p zeros of the polynomial $V(z)$ are located in a disk whose radius depends only on a_0, a_1, ..., a_{p-1}, a_{p+h}, h and p $(h \geq 0, p+h \leq n)$. Our earlier results suggest that for $b_0, b_1, \ldots, b_{p-1}, b_{p+h}$, h and p $(h \geq 0, p+h \leq n)$ fixed, the *imaginary parts* of at least p zeros of the polynomial (4) remains bounded as the other coefficients vary. I have not succeeded in proving this fact so far, except for the case $p = 1$, where a considerably more general theorem is still valid if we admit a dependence on n as well.

Let us return presently to our hypothesis—substantiated already to some extent—that the Hermite representation is a more suitable means for recognizing if the zeros of a polynomial are real or are located in a strip symmetric around the real axis. We can establish this hypothesis in yet another way. Let us consider the polynomials

$$V(z) = \sum_{\nu=0}^{n} a_\nu z^\nu, \tag{2}$$

and

$$H(z) = \sum_{\nu=0}^{n} a_\nu H_\nu(z) \tag{35}$$

in parallel. It is not hard to show that if the zeros of $V(z)$ are real, then the same holds for $H(z)$. That means that if we find in the $(n+1)$-dimensional space of the coefficients $(a_0, a_1, \ldots, a_n)$ a region in which the zeros of the corresponding polynomials $V(z)$ are all real, then those of the polynomias (35) $H(z)$ corresponding to the points of the same domain are also real. That is to say, it is *easier* to find a domain in the space of coefficients whose points are associated with polynomials with real zeros for the polynomials (35) than for those of (2). This proposition can be generalized in two directions; the corresponding proofs are a bit more difficult. One can show first of all that the zeros of $H(z)$ can be be restricted in a narrower strip that is symmetric around the real axis than those of $V(z)$. More precisely, if the zeros of $V(z)$ are located in the strip $|y| \leq C$, then those of $H(z)$ are located in the strip $|y| \leq \frac{1}{2}C$. The second generalization concerns polynomials with real coefficients a_ν and states that in that case $H(z)$ has at least as many real zeros as $V(z)$ (naturally counting zeros according to their multiplicities). Both theorems illustrate the tendency that in replacing a monomial z^ν with the polynomial $H_\nu(z)$, the zeros of a polynomial approach the real axis. I have not been successful in deciding whether the following statement (which generalizes the previous two generalizations) holds true: the polynomial $V(z)$ has at least as many zeros in the strip $|y| \leq D$ as the polynomial $H(z)$ has in the strip $|y| \leq \frac{1}{2}D$. By contrast, I have succeeded in showing that in replacing $V(z)$ by $H(z)$, not only the number of real zeros does not decrease, but that the number of sign changes does not decrease either.

However, like the severed heads of the hydra of Lerne, a problem resolved gives birth to two new ones; the question hence presents itself whether the correspondence we found between the representations of Vieta and Hermite with the same coefficients is an isolated phenomenon or not. It is easy to show that such a correspondence exists between $V(z)$ and the polynomials $T(z)$ (6) in the interval $-1 \le x \le 1$, that is to say, $T(z)$ has at least as many real zeros as $V(z)$. The situation remains similar when we compare $V(z) = \sum_{k=0}^{n} a_k z^k$ instead of $T(z)$ with the polynomial

$$U(z) = \sum_{\nu=0}^{n} a_\nu U_\nu(z) \qquad \left(U_\nu(\cos\vartheta) = \frac{\sin((n+1)\vartheta)}{\sin\vartheta} \right).$$

For a family $\phi_0(x), \phi_1(x), \ldots$ of normalized orthogonal polynomials with respect to a weight function $\psi(x)$ that is integrable and bounded from below by a positive number in $-1 \le x \le 1$, everything depends on the behavior of the function

$$G(x,w) = \sum_{n=0}^{\infty} \phi_\nu(x) w^\nu,$$

which is holomorphic for $|w| < 1$ as shown by a heuristic argument. By virtue of this argument, the number of zeros of $V(z)$ (see (2)) between -1 and $+1$ does not exceed that of the zeros of

$$\Phi(z) = \sum_{k=0}^{n} a_k \phi_k(z)$$

in the same interval, provided that the Wronskian determinant of the functions

$$G(x_1, w), G(x_2, w), \ldots, G(x_j, w)$$

does not vanish for $-1 < w < 1$ and its sign depends only on j, where $x_1, x_2, \ldots, x_j$ denote for any natural number j numbers such that

$$1 > x_1 > x_2 > \ldots > x_j > -1.$$

I have not yet been able to prove this.[4]

Returning to the parallels between the representations of Hermite and Vieta, there is a point where the Vieta representation leads to sufficiently simple results concerning the reality of the zeros. I am thinking here first of all of the pretty theorem of Laguerre according to which, if all the zeros of $V(z)$ are real, then all the zeros of the polynomial

$$V_*(z) = \sum_{\nu=0}^{n} a_\nu g(\nu) z^\nu$$

are also real, where $g(t)$ is an arbitrary polynomial having only negative zeros. Is there a similar theorem, simple and elegant, for the Hermite representation? The answer is yes; if one has

$$H(z) = \sum_{k=0}^{n} a_k H_k(z),$$

$$H_*(z) = \sum_{k=0}^{n} a_k g(k) H_k(z),$$

[4] Translator's note: a later paper by Iserles and Saff (A. Iserles and E. B. Saff, Zeros of expansions in orthogonal polynomials, *Math. Proc. Camb. Phil. Soc.* **105** (1989), 559–573) contains results that can be considered as rigorous versions of the heuristic idea described here by Turán.

where all the zeros of $H(z)$ are real, then the same holds for the zeros of $H_*(z)$, provided that $g(t)$ is a polynomial that has only negative zeros. The proof is not difficult, although the theorem does not follow from the theorem of Laguerre.

As we can see, the comparative theory of different representations of polynomials is still nascent. This is the situation even for the theory of the Hermite representation, being the representation that most interests us, as it seems a device more suited than the Vieta representation for dealing with questions concerning the reality of the zeros. We have so far not dealt—beyond the unresolved questions mentioned above—with the problem of giving explicit necessary and sufficient conditions on the coefficients in the Hermite representation such that all the zeros of the polynomial be real. It is not impossible that such conditions take their simplest form not in the case of the Hermite representation, but for that of some other representation. We do not yet know either the *true* analogues of the Newton-Waring formulas, however it is easy to transcribe the original Newton-Waring formulas into a form where the coefficients of the Hermite expansion figure and where in place of the sums of the powers of the zeros appear the numbers

$$\sigma_\nu = H_\nu(z_1) + H_\nu(z_2) + \ldots + H_\nu(z_n).$$

To all these problems and to questions of an arithmetic nature in the theory that I have not explicitly articulated, I wish to return at a later point.