HERMITE-expansion and strips for zeros of polynomials

ALEXANDER OSTROWSKI to his 60th anniversary

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1. In a previous paper[1]) I sketched the reasons why it seems desirable to extend some classical questions of the theory of the algebraic equations (a theory which owes very much to A. OSTROWSKI) to the case of other representations, different from the usual

$$f(z) = a_0 + a_1 z + \dots + a_n z^n . \tag{1.1}$$

I discussed in this respect without proofs in particular the role of the HERMITE-expansion

$$f(z) = \sum_{m=0}^{n} b_m H_m(z) , \tag{1.2}$$

where the m-th HERMITE-polynomial $H_m(z)$ of degree m is defined by

$$H_m(z) = (-1)^m e^{z^2} \frac{d^m}{dz^m} (e^{-z^2}) . \tag{1.3}$$

It turned out as characteristic that one can obtain with the same ease results *for strips* containing all zeros using the representation (1.2) as for *circles* containing all the zeros using representation (1.1). In this note I shall confine myself to the first and simplest results in this trend.

2. We quote the following results.

a) [CAUCHY]. *If*

$$\max_{\nu=0,1,\dots,(n-1)} | a_\nu | = M , \tag{2.1}$$

then all the zeros of the equation $f(z) = 0$ *lie in the circle*

$$| z | \leqq 1 + \frac{M}{| a_n |} . \tag{2.2}$$

The circle (2.2) cannot be replaced by a circle

$$| z | \leqq 1 + \vartheta \frac{M}{| a_n |} \tag{2.3}$$

1) Sur l'algèbre fonctionelle. Compt. Rend. du prem. Congr. des Math. Hongr. p. 279—290 (1950).

with a universal $0 < \vartheta < 1$ as the simple example

$$f_0(z) = z^n - Mz^{n-1}$$

shows if only M is sufficiently large.

b) [WALSH]. *All the zeros of $f(z) = 0$ lie in the circle*

$$|z| \leqq \sqrt[n]{\left|\frac{a_0}{a_n}\right|} + \sqrt[n-1]{\left|\frac{a_1}{a_n}\right|} + \dots + \sqrt{\left|\frac{a_{n-2}}{a_n}\right|} + \left|\frac{a_{n-1}}{a_n}\right| .$$

Equality is e. g. attained for $n = 1$.

3. Correspondingly we state the following theorems. We denote the complex variable by $z = x + iy$ throughout this paper.

I) If

(3.1) $$\max_{m=0,1,\dots,(n-1)} |b_m| = M^* ,$$

then all the zeros of $f(z) = 0$ lie in the strip

(3.2) $$|y| \leqq \tfrac{1}{2}\left(1 + \frac{M^*}{|b_n|}\right) .$$

II) *All the zeros of $f(z) = 0$ lie in the strip*

(3.3) $$|y| \leqq \tfrac{1}{2}\left(\sqrt[n]{\left|\frac{b_0}{b_n}\right|} + \sqrt[n-1]{\left|\frac{b_1}{b_n}\right|} + \dots + \sqrt{\left|\frac{b_{n-2}}{b_n}\right|} + \left|\frac{b_{n-1}}{b_n}\right|\right) .$$

4. The basis for all these results is a simple inequality for HERMITE-polynomials. Owing to the well-known identity [2])

(4.1) $$H'_m(z) = 2\, m\, H_{m-1}(z)$$

we can write

(4.2) $$\frac{H_{m-1}(z)}{H_m(z)} = \frac{1}{2\,m}\,\frac{H'_m(z)}{H_m(z)} = \frac{1}{2\,m}\sum_{j=1}^{m}\frac{1}{z - z_{jm}}$$

where z_{jm} denote the zeros of $H_m(z)$. Hence we have

(4.3) $$\left|\frac{H_{m-1}(z)}{H_m(z)}\right| \leqq \frac{1}{2\,m}\sum_{j=1}^{m}\frac{1}{|z - z_{jm}|} .$$

It is well-known that all the z_{jm}-zeros of $H_m(z)$ are real; thus

$$\frac{1}{|z - z_{jm}|} \leqq \frac{1}{|y|}$$

and from (4.3)

(4.4) $$\left|\frac{H_{m-1}(z)}{H_m(z)}\right| \leqq \frac{1}{2\,|y|} .$$

[2]) See e. g. G. SZEGÖ, Orthogonal polynomials. Amer. Math. Soc. Coll. **Publ. 23**, 102 (1939).

From this we obtain for all $k \leqq n-1$ and arbitrary nonreal z

$$\left|\frac{H_k(z)}{H_n(z)}\right| = \prod_{m=k+1}^{n} \left|\frac{H_{m-1}(z)}{H_m(z)}\right| < \frac{1}{(2\,y_1)^{n-k}}. \tag{4.5}$$

This is the inequality we want.

5. Now to prove I) we remark that

$$|f(z)| = \left|\sum_{k=0}^{n} b_k H_k(z)\right| \geqq |b_k|\,|H_n(z)| \left\{1 - \sum_{k=0}^{n-1} \left|\frac{b_k}{b_n}\right| \left|\frac{H_k(z)}{H_n(z)}\right|\right\}. \tag{5.1}$$

Thus using (3.1) and (4.5) we have for

$$|y| > \tfrac{1}{2}\left(1 + \frac{M^*}{|b_n|}\right) \quad (> \tfrac{1}{2}) \tag{5.2}$$

the estimation

$$\begin{aligned} |f(z)| &\geqq |b_n|\,|H_n(z)| \left\{1 - \frac{M^*}{|b_n|} \sum_{m=1}^{n} \left(\frac{1}{2|y|}\right)^m\right\} > \\ &> |b_n|\,|H_n(z)| \left\{1 - \frac{M^*}{|b_n|} \cdot \frac{1}{2|y| - 1}\right\}. \end{aligned} \tag{5.3}$$

Taking in account that $H_n(z)$ does not vanish in the domain given in (5.2), it follows indeed that for such z-values $f(z) \neq 0$.

In order to show that the strip in I) cannot be replaced by a strip of the form

$$|y| < \frac{\vartheta}{2}\left(1 + \frac{M^*}{|b_n|}\right) \tag{5.4}$$

with a fixed $0 < \vartheta < 1$, we consider the polynomial

$$f_1(z) \equiv H_n(z) - i\,a\,H_{n-1}(z), \tag{5.5}$$

where a denotes a sufficiently large positive number. Thus $M^* = a$. The equation $f_1 = 0$ can be also written in the form

$$\frac{H_{n-1}(z)}{H_n(z)} = \frac{1}{i\,a} \tag{5.6}$$

or from (4.1)

$$\sum_{j=1}^{n} \frac{1}{z - z_{jn}} = \frac{2\,n}{i\,a}. \tag{5.7}$$

Suppose e. g. n being even, $n = 2\,m$; then (5.7) takes the form

$$z \sum_{j=1}^{m} \frac{1}{z^2 - z_{jn}^2} = \frac{2\,m}{i a} \quad (z_{jn} > 0).$$

If z runs on the positive imaginary axis then we have with $z = iy$ (y real)

$$y \sum_{j=1}^{m} \frac{1}{y^2 + z_{jn}^2} = \frac{2m}{a}$$

or

$$\sum_{j=1}^{m} \frac{1}{1 + \frac{z_{jn}^2}{y^2}} = \frac{2my}{a}. \tag{5.8}$$

Since m is fixed, we may choose a sufficiently large, so that

$$\sum_{j=1}^{m} \frac{1}{1 + \frac{z_{jn}^2}{\left(\frac{1+\vartheta}{4} a\right)^2}} > \frac{1+\vartheta}{2} m, \tag{5.9}$$

$$\frac{1+\vartheta}{4} a > \frac{\vartheta}{2} (1+a). \tag{5.10}$$

In this case the left-side of (5.8) is for $y = \frac{1+\vartheta}{4} a$ greater than the right-side but for sufficiently large positive y the opposite is true. Hence the equation (5.8) has a real zero with $y > \frac{1+\vartheta}{4} a$, which means that the equation (5.5) has a zero iy_0 satisfying

$$|y_0| > \frac{1+\vartheta}{4} a > \frac{\vartheta}{2}(1+a) = \frac{\vartheta}{2}(1+M^*),$$

which shows indeed that (5.4) is false.

6. In order to prove II) we have using (4.5) and (5.1)

$$|f(z)| \geqq |b_n| \, |H_n(z)| \left\{ 1 - \sum_{k=0}^{n-1} \left| \frac{b_k}{b_n} \right| \left(\frac{1}{2|y|} \right)^{n-k} \right\} =$$

$$= |b_n| \, |H_n(z)| \left\{ 1 - \sum_{k=0}^{n-1} \left(\frac{1}{2|y|} \sqrt[n-k]{\left| \frac{b_k}{b_n} \right|} \right)^{n-k} \right\}.$$

Hence if z is not in the strip (3.3), then all quotients

$$\left| \frac{b_k}{b_n} \right|^{\frac{1}{n-k}} \frac{1}{2|y|}$$

are $\leqq 1$, i. e.

$$|f(z)| \geqq |b_n| \, |H_n(z)| \left\{ 1 - \sum_{k=0}^{n-1} \left| \frac{b_k}{b_n} \right|^{\frac{1}{n-k}} \frac{1}{2|y|} \right\} > 0, \quad \text{q. e. d.}$$

Also the strip (3.3) is best-possible in a certain sense. Taking with a small positive ε

$$f(z) = H_n(z) + \varepsilon H_{n-1}(z) + \varepsilon^2 H_{n-2}(z) + \dots + \varepsilon^n H_0(z),$$

then according to (3.3) all the zeros are in the strip $|y| \leqq \frac{n}{2}\varepsilon$, which shrinks to the real axis if $\varepsilon \to 0$.

7. We may add the following remarks. Replacing in (1.2) $H_n(z)$ by $g(z)$, then all the zeros of $g(z)$ are real and according to (4.1) $f(z)$ takes the form

$$f(z) = b_n g(z) + \frac{b_{n-1}}{2n} g'(z) + \frac{b_{n-2}}{2^2 n(n-1)} g''(z) + \dots + \frac{b_{n-k}}{2^k n(n-1)\dots(n-k+1)} g^{(k)}(z) + \\ + \dots + \frac{b_0}{2^n n!} g^{(n)}(z).$$

Hence the results I) II) are special cases of results concerning

$$(z) = \sum_{k=0}^{n} c_k G_m^{(k)}(z) \tag{7.1}$$

where $G_m(z)$ is a polynomial of m-th degree with real zeros only. To (4.4) it corresponds

$$\left|\frac{G'_m(z)}{G_m(z)}\right| \leqq \frac{m}{|y|},$$

to (4.5) the inequality

$$\left|\frac{G_m^{(k)}(z)}{G_m(z)}\right| \leqq \frac{m!}{(m-k)!} \cdot \frac{1}{|y|^k}. \tag{7.2}$$

Thus operating as before we obtain the following theorems.

I') *If for $F(z)$ in (7.1) we have*

$$\max_{k=1,2,\dots,n} |c_k| \frac{m!}{(m-k)!} = \overline{M}, \tag{7.3}$$

then all the zeros of $F(z)$ are in the strip

$$|y| \leqq 1 + \frac{\overline{M}}{|c_0|}. \tag{7.4}$$

II') *All the zeros of $F(z)$ lie also in the strip*

$$|y| \leqq \sum_{k=1}^{n} \sqrt[k]{\left|\frac{c_k}{c_0}\right| \frac{m!}{(m-k)!}}.$$

As to the further results mentioned in [1]), I shall return elsewhere.

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