TO THE ANALYTICAL THEORY OF ALGEBRAIC EQUATIONS

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1. This paper contains a modest contribution to the subject in the title, to which Prof. L. Čakalov added many important contributions. Some results of a lecture* will be detailed in which the necessity of the study of polynomials *in their Hermite-development* was systematically discussed, i. e. in the form

$$(1.1)\qquad f(z)=\sum_{\nu=0}^{n} a_\nu H_\nu(z)$$

where $H_\nu(z)$, the νth Hermite-polynomial is, as usual, defined by

$$(1.2)\qquad e^{-z^2} H_\nu(z)=(-1)^\nu(e^{-z^2})^{(\nu)}$$
$$\nu=0, 1, \ldots$$

Many signs show that in the questions of reality of zeros or giving strips for the zeros the Hermite-development is a much more suitable tool than the Taylor-development

$$(1.3)\qquad f(z)=\sum_{\nu=0}^{\infty} b_\nu z^\nu .$$

The significance of all results in this direction would be greatly enhanced if the Hermite-coefficients of Riemann's Ξ-function could be given in a simple closed form. What is easy to show is that this expansion has the form**

$$(1.4)\qquad \Xi(t)\sim\sum_{\nu=0}^{\infty}(-1)^\nu d_{2\nu} H_{2\nu}(t)$$
$$d_{2\nu}>0,\quad \nu=0, 1, \ldots$$

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* Turan [1]. The results announced in this lecture were found in 1938—39.

** For the $d_{2\nu}$'s the following explicit representation can be given

$$d_{2\nu}=\frac{1}{2^{2\nu-2}(2\nu)!}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4}}\, t^{2\nu}\left\{\sum_{m=1}^{\infty}\left(2m^4\pi e^{\frac{9}{2}t}-3m^2 e^{\frac{5}{2}t}\right)e^{-m^2\pi e^{2t}}\right\}dt.$$

In lack of such forms we are not aiming best-possible results, rather we confine ourselves to obtain characteristical results with possibly short proofs.

The first result we are going to prove is

Theorem I. If $f(z)=\sum_{\nu=0}^{n} c_{2\nu} H_{2\nu}(z)$ with arbitrary complex coefficients and

$$\max_{\nu=0,1,\ldots,n-1} |c_{2\nu}| = M,$$

then all zeros of $f(z)$ lie in the strip

$$|\operatorname{Im} z| \leqq \frac{1}{2}\left(1+\frac{5}{\sqrt{2n-1}}\cdot\frac{M}{|c_{2n}|}\right). \tag{1.5}$$

We have to compare this theorem to the other one* according which all zeros of

$$g(z)=\sum_{\nu=0}^{n} a_\nu H_\nu(z)$$

with

$$\max_{\nu=0,1,\ldots,n-1} |a_\nu| = M_1$$

lie in the strip

$$|\operatorname{Im} z| \leqq \frac{1}{2}\left(1+\frac{M}{|a_n|}\right). \tag{1.6}$$

The constant $\frac{1}{2}$ in (1.6) is best-possible.

2. For the proof we need a simple inequality concerning Hermite-polynomials. We consider for $m \geqq 2$ the quotient

$$\frac{H_{m-2}(z)}{H_m(z)}.$$

Since the $x_{\nu m}$ – zeros of $H_m(z)$ are simple,

$$\frac{H_{m-2}(z)}{H_m(z)}=\sum_{j=1}^{m}\frac{H_{m-2}(x_{jm})}{H_m'(x_{jm})}\cdot\frac{1}{z-x_{jm}}.$$

As well-known**

* Turan [1] p. 283 and [2]. By a more careful estimation the strip (1.5) can be replaced by $|\operatorname{Im} z| \leqq \frac{\lambda}{\sqrt{n}}\left(1+\frac{M}{|c_{2n}|}\right)$ with a numerical λ -constant.

** This follows easily from the generator-function

$$\sum_{\nu=0}^{\infty}\frac{H_\nu(z)}{\nu!}w^\nu = e^{2zw-w^2}$$

easily after differentiation after z.

(2.1) $$H'_m(z) = 2mH_{m-1}(z),$$

i. e.

$$\frac{H_{m-2}(z)}{H_m(z)} = \frac{1}{2m}\sum_{j=1}^{m} \frac{H_{m-2}(x_{jm})}{H_{m-1}(x_{jm})} \cdot \frac{1}{z-x_{jm}}.$$

Using the recursion-formula*

(2.2) $$H_m(z) = 2zH_{m-1}(z) - 2(m-1)H_{m-2}(z),$$

we get

$$2x_{jm}H_{m-1}(x_{jm}) = 2(m-1)H_{m-2}(x_{jm})$$

i. e.

(2.3) $$\frac{H_{m-2}(z)}{H_m(z)} = \frac{1}{2m(m-1)}\sum_{j=1}^{m} \frac{x_{jm}}{z-x_{jm}}.$$

Since** for $j=1, 2, \ldots, m$

(2.4) $$-\sqrt{2m+1} \leqq x_{jm} \leqq \sqrt{2m+1},$$

we have

(2.5) $$\left|\frac{H_{m-2}(z)}{H_m(z)}\right| \leqq \frac{\sqrt{2m+1}}{2(m-1)} \frac{1}{|\operatorname{Im} z|}.$$

This is the required inequality. For $m=2n$ this gives

(2.6) $$\left|\frac{H_{2n-2}(z)}{H_{2n}(z)}\right| \leq \frac{\sqrt{4n+1}}{2(2n-1)} \cdot \frac{1}{\operatorname{Im} z},$$

further for $1 \leqq l \leqq n-2$

$$\left|\frac{H_{2l}(z)}{H_{2n}(z)}\right| = \left|\frac{H_{2l}(z)}{H_{2l+2}(z)}\right| \left|\frac{H_{2l+2}(z)}{H_{2l+4}(z)}\right| \cdots \left|\frac{H_{2n-2}(z)}{H_{2n}(z)}\right| \leqq$$

(2.7) $$\leqq \left(\frac{1}{2 \operatorname{Im} z}\right)\left(\frac{1}{2 \operatorname{Im} z}\right) \cdots \left(\frac{1}{2 |\operatorname{Im} z}\right) \times$$

$$\times \left(\frac{\sqrt{4n+1}}{2(2n-1)} \cdot \frac{1}{\operatorname{Im} z|}\right) = \left(\frac{1}{2 \operatorname{Im} z}\right)^{n-l} \cdot \frac{\sqrt{4n+1}}{2n-1}$$

and*** for $l=0$

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* This can be easily verified, since differentiation of (1.2) gives

$$H'_{m-1}(z) = 2zH_{m-1}(z) - H_m(z)$$

and then (2.1) gives (2.2).

** See e. g. G. Szegö [3], p. 125.

*** Using (2.6) with $n=1, 2, \ldots, N$ and multiplying we get the inequality

$$|H_{2N}(z)| \geq \frac{2^N \cdot 1 \cdot 3 \cdot 5 \ldots (2N-1)}{\sqrt{1 \cdot 5 \cdot 9 \ldots (4N+1)}} \cdot |\operatorname{Im} z|^N,$$

valid for $N=1, 2, \ldots$ and all complex z-values. This method of estimation can be greatly improved if necessary.

$$\left|\frac{H_0(z)}{H_{2n}(z)}\right| = \left|\frac{H_0(z)}{H_2(z)}\right| \cdot \left|\frac{H_2(z)}{H_{2n}(z)}\right| \leqq$$

(2.8)

$$\leqq \sqrt{5}\left(\frac{1}{2\,|\operatorname{Im} z|}\right)^n \frac{\sqrt{4n+1}}{2n-1}.$$

Thus we obtain

$$|f(z)| \geqq |H_{2n}(z)|\left(|c_{2n}| - M\sum_{l=0}^{n-1}\left|\frac{H_{2l}(z)}{H_{2n}(z)}\right|\right) >$$

$$> |H_{2n}(z)|\left(|c_{2n}| - \sqrt{5}\,\frac{\sqrt{4n+1}}{2n-1}\,M\sum_{l=0}^{n-1}\left(\frac{1}{2|\operatorname{Im} z|}\right)^{n-l}\right).$$

Suppose $|\operatorname{Im} z| > \frac{1}{2}$; then

$$|f(z)| > |H_{2n}(z)|\left(|c_{2n}| - M\sqrt{5}\,\frac{\sqrt{4n+1}}{2n-1}\cdot\left(\frac{1}{2|\operatorname{Im} z|-1}\right)\right)$$

Since for $n \geqq 1$ we have

$$4n+1 \leqq 5(2n-1),$$

we get

$$|f(z)| > |H_{2n}(z)|\left(|c_{2n}| - M\,\frac{5}{\sqrt{2n-1}}\cdot\frac{1}{2|\operatorname{Im} z|-1}\right) > 0,$$

if*

$$|\operatorname{Im} z| > \frac{1}{2}\left\{1 + \frac{5}{\sqrt{2n-1}}\cdot\frac{M}{|c_{2n}|}\right\}. \qquad \text{Q. e. d.}$$

3. Owing to the possible application to Riemann's Ξ-function it is of interest to replace strips by equilateral hyperbolas where the real axis is an asymotota. It holds the

Theorem II. *If* $z = x + iy$ *and* $g(z) = \sum_{\nu=0}^{n} c_{2\nu} H_{2\nu}(z)$ *with* $\max_{\nu=0,1,\ldots,n-1} |c_{2\nu}| = M$, *then all zeros of* $g(z)$ *lie in the hyperbola*

$$|xy| \leqq \frac{5}{4}\left(1 + \frac{M}{|c_{2n}|}\right).$$

For the proof we start from the identity (2.3). Taking in account the symmetry of the $x_{j,2\nu}$—zeros this gives

$$\frac{H_{2\nu-2}(z)}{H_{2\nu}(z)} = \frac{1}{2\nu(2\nu-1)}\sum_{j=1}^{\nu}\frac{x_{j,2\nu}^2}{z^2 - x_{j,2\nu}^2}$$

* Since (2.1) and (2.2) *characterize* the Hermite-polynomials, Theorem I. expresses a property *of the Hermite-expansion.* This was *not* the case with (1.5), since it uses *only* (2.1). One can prove however easily that property (2.1) *together with the orthogonality-property* along the real axis characterizes again the Hermite-polynomials „essentially".

and from (2.4)

$$\left|\frac{H_{2\nu-2}(z)}{H_{2\nu}(z)}\right| \le \frac{4\nu+1}{2\nu(2\nu-1)} \sum_{j=1}^{\nu} \frac{1}{z^2 - x^2_{j,2\nu}}.$$

Since

$$|z^2 - x^2_{j,2\nu}| = |x^2 - y^2 - x^2_{j,2\nu} + 2xyi| \ge 2|xy|,$$

we get for $\nu \ge 1$

$$\left|\frac{H_{2\nu-2}(z)}{H_{2\nu}(z)}\right| \le \frac{4\nu+1}{4(2\nu-1)} \cdot \frac{1}{|xy|} \le \frac{5}{4} \cdot \frac{1}{|xy|}$$

and thus for $k=0, 1, \ldots, n-1$

$$\left|\frac{H_{2k}(z)}{H_{2n}(z)}\right| \le \left(\frac{5}{4} \cdot \frac{1}{|xy|}\right)^{n-k}.$$

Hence again for $|xy| > \frac{5}{4}$

$$|g(z)| \ge |H_n(z)| \left\{ |c_{2n}| - \frac{M}{\frac{4}{5}|xy| - 1} \right\},$$

which proves the theorem.

4. Next we turn to the proof of the following

Theorem III. *If the coefficients of*

$$G(z) = \sum_{\nu=0}^{n} c_\nu H_\nu(z)$$

are real and

$$\sum_{\nu=0}^{n-2} 2^\nu \nu! \, c_\nu^2 < 2^n (n-1)! \, c_n^2 \tag{4.1}$$

is fulfilled, then all zeros of $G(z)$ are real and simple

For the proof of this theorem we shall need a simple formula. We start from the well-known formula of Christoffel-Darboux*

$$\sum_{m=0}^{n-1} \frac{H_m(x)H_m(y)}{2^m m!} = \frac{1}{2^n(n-1)!} \cdot \frac{H_n(x)H_{n-1}(y) - H_n(y)H_{n-1}(x)}{x-y}. \tag{4.2}$$

We shall denote by

$$x_1 > x_2 > \cdots > x_{n-1} \tag{4.3}$$

the zeros of $H_{n-1}(z)$. For an arbitrary integer l between 1 and $n-1$ putting $x = x_l$ into (4.2) we get

$$\sum_{m=0}^{n-2} \frac{H_m(x_l)}{2^m m!} H_m(y) = \frac{1}{2^n(n-1)!} H_n(x_l) \frac{H_{n-1}(y)}{x_l - y}.$$

* See e. g. G. Szegő [3]. p. 102.

If $y \to x_l$, this gives

$$\sum_{m=0}^{n-2} \frac{H_m^2(x_l)}{2^m \cdot m!} = -\frac{1}{2^n(n-1)!} H_n(x_l) H'_{n-1}(x_l). \tag{4.4}$$

But, as it follows from footnote(*), p. 125 for $x = x_l$,

$$H'_{n-1}(x_l) = -H_n(x_l).$$

Putting it into (4.4) we get for $l = 1, 2, \ldots, n-1$

$$\sum_{m=0}^{n-2} \frac{H_m^2(x_l)}{2^m \cdot m!} = \frac{1}{2^n(n-1)!} H_n^2(x_l). \tag{4.5}$$

This is the required formula.

5. We shall prove Theorem III. by showing that the zeros of $H_{n-1}(z)$ separate in our case the zeros of $G(z)$. Without loss of generality we may suppose the leading coefficient c_n in $G(z)$ is positive. Writing

$$x_0 = +\infty, \quad x_n = -\infty$$

we shall prove

$$\operatorname{sign} G(x_l) = (-1)^l, \tag{5.1}$$

$$l = 0, 1, \ldots, n,$$

Since

$$H_n(x) = 2^n x^n + \cdots,$$

this is for $l=0$ and $l=n$ trivial. In order to prove it for $l = 1, 2, \ldots, n-1$, we remark first, that as well known* the zeros of $H_{n-1}(z)$ separate those of $H_n(z)$,

$$\operatorname{sign} H_n(x_l) = (-1)^l, \qquad l = 0, 1, \ldots, n.$$

Hence

$$(-1)^l c_n H_n(x_l) = |c_n| |H_n(x_l)|$$

and thus for $l = 1, 2, \ldots, n-1$ we have

$$(-1)^l G(x_l) = |c_n| |H_n(x_l)| + \sum_{\nu=0}^{n-2} (-1)^l c_\nu H_\nu(x_l).$$

From this we obtain

$$(-1)^l G(x_l) \geqq |c_n| |H_n(x_l)| - \sum_{\nu=0}^{n-2} |c_\nu| |H_\nu(x_l)| =$$

$$= |c_n| |H_n(x_l)| - \sum_{\nu=0}^{n-2} 2^{\frac{\nu}{2}} \sqrt{\nu!} \, |c_\nu| \frac{|H_\nu(x_l)|}{2^{\frac{\nu}{2}} \sqrt{\nu!}}.$$

Cauchy's inequality gives

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* See e. g. G. Szegő [3], p. 45.

$$(-1)^l G(x_l) > c_n \, |H_n(x_l)| - \sqrt{\left(\sum_{\nu=0}^{n-2} 2^\nu \cdot \nu! \; c_\nu^{\;2}\right)\left(\sum_{\nu=0}^{n-2} \frac{H_\nu^2(x_l)}{2^\nu \cdot \nu!}\right)}$$

and using (4.5)

$$(-1)^l G(x_l) > |H_n(x_l)| \left\{ |c_n| - \frac{1}{2^{\frac{n}{2}} \sqrt{(n-1)!}} \sqrt{\sum_{\nu=0}^{n-2} 2^\nu \cdot \nu! \, |c_\nu|^2} \right\} > 0$$

indeed owing to (4.1). Since $x_1, x_2, \ldots, x_{n-1}$ are all simple zeros, also the simplicity of the zeros of $G(z)$ follows.

One can obtain of course a similar theorem for the reality of all zeros of $G_1(z) = \sum_{\nu=0}^{n} c_{2\nu} H_{2\nu}(z)$ with real $c_{2\nu}$'s. We shall do not go into details of its proof.

6. The condition (4.1) of the previous theorem is obviously fulfilled if the coefficients c_ν do not decrease „too quickly". Now we are going to prove as a counterpart of Theorem III that the same conclusion holds if the coefficients decrease sufficiently quickly. More exactly we shall prove the

Theorem IV. *If $f(z)$ has the form*

$$f(z) = \sum_{\nu=0}^{n} (-1)^\nu c_{2\nu} H_{2\nu}(z) \tag{6.1}$$

with positive $c_{2\nu}$'s and for $\nu = 1, 2, \ldots, n-1$ we have

$$c_{2\nu}^2 > 4 c_{2\nu-2} \, c_{2\nu+2}, \tag{6.2}$$

then all zeros of $f(z)$ are real.

7. For the proof of this theorem we need two lemmas.

Lemma I. If the coefficients $c_{2\nu}$ of

$$F(z) = \sum_{\nu=0}^{n} (-1)^\nu c_{2\nu} z^{2\nu}$$

satisfie the condition (6.2), then all zeros of $F(z) = 0$ are real and simple.

Proof: Let

$$\xi_j = \sqrt{\frac{c_{2j}}{2c_{2j+2}}}. \tag{7.1}$$

$$j = 0, 1, \ldots, n-1.$$

From (6.2) it follows evidently that

$$0 < \xi_0 < \xi_1 < \cdots < \xi_{n-1}. \tag{7.2}$$

We fix any of our j's with $1 \leq j \leq n-1$ and consider $F(\xi_j)$. Owing to (6.2) we have for $\nu < j$

$$\sqrt{\frac{c_{2\nu}}{c_{2\nu+2}}} \leq \sqrt{\frac{c_{2j-2}}{c_{2j}}} < \xi_j$$

i. e.

(7.3) $$c_0 < c_2\xi_j^2 < c_4\xi_j^4 < \cdots < c_{2j}\xi_j^{2j}$$

and for $\mu > j$

$$\xi_j < \sqrt{\frac{c_{2j}}{c_{2j+2}}} < \sqrt{\frac{c_{2\mu}}{c_{2\mu+2}}},$$

i. e.

(7.4) $$c_{2j}\xi_j^{2j} > c_{2j+2}\xi_j^{2j+2} > \cdots > c_{2n}\xi_j^{2n}.$$

Hence writing

$$(-1)^j F(\xi_j) = c_{2j}\xi_j^{2j} - (c_{2j-2}\xi_j^{2j-2} - c_{2j-4}\xi_j^{2j-4} + \cdots) -$$
$$-(c_{2j+2}\xi_j^{2j+2} - c_{2j+4}\xi_j^{2j+4} + \cdots),$$

we may observe that the terms in the brackets decrease monotonically i. e.

$$\frac{(-1)^j}{\xi_j^{2j-2}} F(\xi_j) \geqq c_{2j}\xi_j^2 - c_{2j-2} - c_{2j+2}\xi_j^4 =$$
$$= \frac{c_{2j}^2 - 4c_{2j-2}\, c_{2j+2}}{4c_{2j+2}} > 0.$$

Since $F(0) > 0$ and $(-1)^n F(+\infty) > 0$, we obtained that each of the intervals

$$(0, \xi_1), (\xi_1, \xi_2), \cdots, (\xi_{n-2}, \xi_{n-1}), (\xi_{n-1}, +\infty)$$

contain at least one zero and owing to the evenness of $F(z)$ the same holds for the intervals

$$(-\infty, -\xi_{n-1}), \ldots, (-\xi_2, -\xi_1), (-\xi_1, 0).$$

Since $F(z)$ is of $2n^{th}$ degree, Lemma I. is proved.

Lemma II. If the polynomial

$$\Phi(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

has only real zeros, then all zeros of

$$\gamma(z) = \sum_{\nu=0}^{n} a_\nu H_\nu(z)$$

are also real*.

Theorem IV follows evidently from lemmata I and II. A trivial limit-process shows that Theorem IV. holds under condition (6.2) also for infinite Hermite-developments.

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* This was stated in my quoted lecture. Quite recently I observed that it was stated, as an incoherent remark, in a paper of G. Pólya [4], in part p. 242, though with another normalisation of $H_n(x)$. Since the necessary changes are obvious, we omit the proof.

8. The condition (4.1) is, as easy to see, certainly fulfilled if $f(z)$ has the form (6.1) and

$$\frac{c_{2\nu+2}}{c_{2\nu}} > \frac{1}{4} \qquad \nu=0,\ 1, \ldots, n-1 \tag{8.1}$$

In this case we shall prove more.

Theorem V. If $f(z)=\sum_{\nu=0}^{n}(-1)^{\nu}c_{2\nu}H_{2\nu}(z)$ with positive $c_{2\nu}$'s and for an integer $1\leqq k\leqq n$ we have

$$\frac{c_2}{c_0}>\frac{1}{4},\quad \frac{c_4}{c_2}>\frac{1}{4},\ldots,\frac{c_{2k}}{c_{2k-2}}>\frac{1}{4}, \tag{8.2}$$

then $f(z)$ has at least $2k$ real zeros with odd multiplicities.

For the proof we shall use the well-known formula*

$$z^m=\frac{m!}{2^m}\sum_{r=0}^{\left[\frac{m}{2}\right]}\frac{1}{r!\,(m-2r)!}H_{m-2r}(z) \qquad m=0,\ 1, \ldots \tag{8.3}$$

We form the integrals for $\nu=0,\ 1, \ldots, k$

$$J_{2\nu}=\int_0^{\infty} e^{-x^2}x^{2\nu}f(x)dx\,. \tag{8.4}$$

Using (8.3) we get

$$J_{2\nu}=\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}x^{2\nu}f(x)\,dx=$$

$$=\frac{(2\nu)!}{2^{2\nu+1}}\sum_{r=0}^{\nu}\frac{1}{r!\,(2\nu-2r)!}\int_{-\infty}^{\infty} e^{-x^2}H_{2\nu-2r}(x)f(x)dx.$$

Since**

$$\int_{-\infty}^{\infty} e^{-x^2}H_{\mu}(x)H_{\nu}(x)dx=\begin{cases}0 & \text{if } \mu\neq\nu\\ 2^{\mu}\mu!\sqrt{\pi} & \text{if } \mu=\nu,\end{cases}$$

* For the sake of completeness we reproduce a proof of (8.3). Using the generator-function in footnote(*) p. 124 we have

$$\sum_{m=0}^{\infty}\frac{(2z)^m}{m!}w^m=e^{2zw}=e^{w^2}\cdot e^{2zw-w^2}=$$

$$=\left(\sum_{j=0}^{\infty}\frac{w^{2j}}{j!}\right)\left(\sum_{l=0}^{\infty}\frac{H_l(z)}{l!}w^l\right),$$

from which (8.3) easily follows.

** See e. g. G. Szegő [3], p. 101.

we obtain

$$J_{2\nu}=\frac{(2\nu)!}{2^{2\nu+1}}\sqrt{\pi}\sum_{r=0}^{\nu}(-1)^{\nu-r}c_{2\nu-2r}\frac{2^{2\nu-2r}}{r!},$$

or

$$(-1)^{\nu}J_{2\nu}=\frac{(2\nu)!\sqrt{\pi}}{2^{2\nu+1}}\left\{\frac{2^{2\nu}}{0!}c_{2\nu}-\frac{2^{2\nu-2}}{1!}c_{2\nu-2}+\cdots\right\}.$$

But the condition (2.2) together with the positivity of the $c_{2\nu}$'s results, that the terms in the bracket form a monotonically decreasing sequence with alternating signs and thus we get

$$(-1)^{\nu}J_{2\nu}>0$$

$$\nu=0,\ 1,\ldots,k$$

But then owing to a theorem of Fekete [5], and Fejér [6] $f(x)$ has in $(0,+\infty)$ at least k sign-changes and since $f(x)$ is an even function of x, it has on the real axis at least $2k$ sign=changes. Q. e. d.

9. Inserting in the integrals (8.4) e.g. a suitably chosen non-negative polynomial $h(x)$, Theorem V. can be greatly improved; a good choice might be suggested by the required simple form of the $d_{2\nu}$'s in (1.4). Having no such a guide we confine ourselves to the case $h(x)\equiv 1$. What is curious in Theorem V., is the fact, that apart from the reality of the coefficients and evenness of $f(z)$, restrictions are made only upon the first $k+1$ coefficients and by this the reality of at least $2k$ zeros of $f(z)$ is assured, *independently upon the other c_{2j}'s and upon n.* This reminds one to the theorems of Landau-Fejér-Montel-type, the most general of whose asserting that the polynomial

$$e_0+e_1z+\cdots+e_pz^p+e_{p+1}z^{n_{p+1}}+\cdots+e_kz^{n_k} \tag{9.1}$$

with $e_p\neq 0$ and integer

$$p<n_{p+1}<\cdots<n_k$$

has at least p zeros in a circle

$$|z|\leq\varrho_0=\varrho_0(e_0,\ldots,e_p,\ k)$$

and suggest the existence of a $\varrho_1=\varrho_1\ (c_0,\ldots,c_p)$ such that (perhaps if $c_p\neq 0$) any

$$f(z)=\sum_{\nu=0}^{n}c_\nu H_\nu(z)$$

polynomial with $n\geq p$ has at least p zeros in the strip $|\mathrm{Im}z|\leq\varrho_1$. The somewhat weaker assertion about the existence of a $\varrho_2=\varrho_2(c_0,c_1,\ldots,c_p,n)$ such that with $c_p\neq 0$ any

$$f(z)=\sum_{\nu=0}^{n}c_\nu H_\nu(z)$$

polynomial with $n \geqq p$ has at least p zeros in the strip $|\mathrm{Im} z| \leqq \varrho_2$, would follow from the above quoted theorem (9.1) of Montel with

$$\varrho_2 = \varrho_0(c_0, \ldots, c_p, n),$$

if the following assertion is true. For any $A \geqq 0$ the polynomial

$$F(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$$

has in the strip $|\mathrm{Im} z| \leqq A$ at most as many zeros (counted with multiplicity) as

$$f(z) = \sum_{\nu=0}^{n} c_\nu H_\nu(z).$$

As I mentioned in my quoted congress-lecture, I can prove this at the present only for $A=0$ and for such A's, for whose *all* zeros of $F(z)$ are contained in our strip $|\mathrm{Im} z| \leqq A$.

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