

Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #0: review of some elementary notions

1. Below is a list of basic formulas in complex analysis (“formulas you need to know like the back of your hand”). Review each of them, making sure that you understand what it says and why it is true (that is, if it is a theorem, prove it, or if it is a definition, make sure you understand that that is the case).

Note: in the formulas below, a, b, c, d, t, x, y denote arbitrary real numbers; w, z denote arbitrary complex numbers.

1. $(a + bi)(c + di)$
 $= (ac - bd) + (ad + bc)i$
2. $i^2 = -1$
3. $\frac{1}{i} = -i$
4. $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$
5. $\bar{z} = \operatorname{Re}(z) - i\operatorname{Im}(z)$
6. $\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$
7. $\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$
8. $|z|^2 = z\bar{z}$
9. $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$
10. $\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$
11. $\overline{w \cdot z} = \bar{w} \cdot \bar{z}$
12. $|wz| = |w| \cdot |z|$
13. $\left| |w| - |z| \right| \leq |w + z| \leq |w| + |z|$
14. $e^{x+iy} = e^x(\cos(y) + i\sin(y))$
15. $|e^z| = e^{\operatorname{Re}(z)}$
16. $|e^z| \leq e^{|z|}$
17. $e^{it} = \cos(t) + i\sin(t)$
18. $|e^{it}| = 1$
19. $\cos(t) = \frac{e^{it} + e^{-it}}{2}$
20. $\sin(t) = \frac{e^{it} - e^{-it}}{2i}$
21. $e^{\pi i} = -1$
22. $e^{\pm\pi i/2} = \pm i$
23. $e^{2\pi i} = 1$

2. Remind yourself of the definitions of the following terms in complex analysis, referring to the textbook (in particular, sections 1.2, 1.3, 2.1, 2.2) or online sources if necessary. (Try to spend some time thinking about the answers yourself before looking them up.)

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|--------------------------------|---|
| 1. real part | 13. Cauchy sequence |
| 2. imaginary part | 14. limit point |
| 3. complex conjugate | 15. accumulation point |
| 4. modulus | 16. continuous function |
| 5. argument | 17. differentiable function (of a complex variable) |
| 6. open set (in \mathbb{C}) | 18. holomorphic function |
| 7. closed set | 19. analytic function |
| 8. connected set | 20. entire function |
| 9. bounded set | 21. meromorphic function |
| 10. compact set | 22. harmonic function (of two variables) |
| 11. region | |
| 12. convergent sequence | |

3. For each of the following functions, determine for which z it is analytic

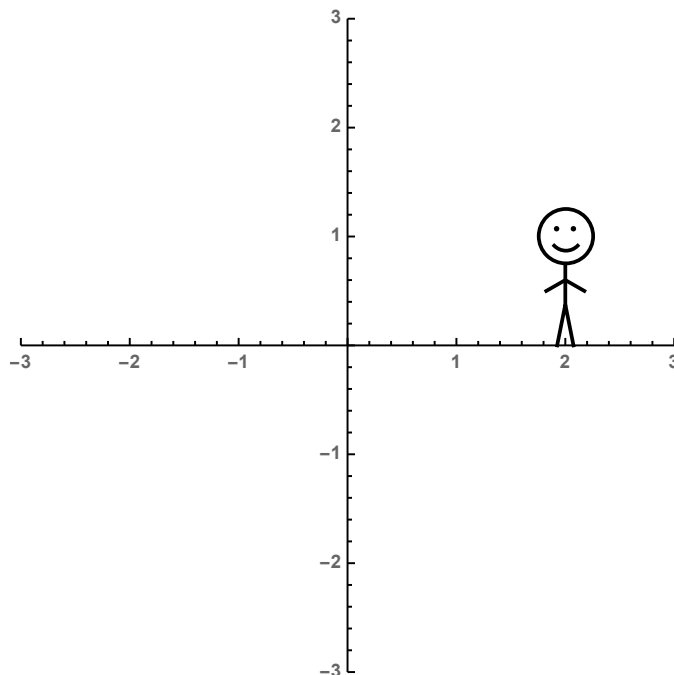
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|----------------------------------|---------------------|
| 1. $f(z) = z$ | 4. $f(z) = z ^2$ |
| 2. $f(z) = \operatorname{Re}(z)$ | 5. $f(z) = \bar{z}$ |
| 3. $f(z) = z $ | 6. $f(z) = 1/z$ |

4. For each of the following functions $u(x, y)$, determine if there exists a function $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is an entire function, and if so, find it, and try to find a formula for $f(z)$ directly in terms of z rather than in terms of its real and imaginary parts. (**Hint**¹)

- | | |
|--------------------------|---|
| 1. $u(x, y) = x^2 - y^2$ | 3. $u(x, y) = x^4 - 6x^2y^2 + 3x + y^4 - 2$ |
| 2. $u(x, y) = y^3$ | 4. $u(x, y) = \cos x \cosh y$ |

¹Use the well-known equations named after two pioneers of complex analysis.

5. Draw (approximately, or with as much precision as you can) the image in the w -plane of the following figure in the z -plane



under each of the maps $w = f(z)$:

1. $w = \frac{1}{2}z$

4. $w = (2 + i)z - 3$

2. $w = iz$

5. $w = 1/z$

3. $w = \bar{z}$

6. $w = z^2 - 1$

6. Prove that the complex numbers a, b, c form the vertices of an equilateral triangle if and only if $a^2 + b^2 + c^2 = ab + ac + bc$.

7. Prove that if $f = u + iv$ is an analytic function on a region Ω , then the level curves $L_c = \{(x, y) : u(x, y) = c\}$ and $K_d = \{(x, y) : v(x, y) = d\}$ of its real and imaginary parts are mutually orthogonal (that is, whenever L_c and K_d intersect for given values $c, d \in \mathbb{R}$, they meet at right angles to each other).

8. Illustrate the claim of the previous question by drawing the level curves of $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ for $f = z^2$, $f = e^z$.