Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #1

1. An immediate corollary of the Fundamental Theorem of Algebra (together with standard properties of polynomials, namely the fact that c is a root of $p(z)$ if and only if $p(z)$ is divisible by the linear factor $z - c$) is that any complex polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0,$$

(where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$), can be factored as

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k)$$

for some $z_1, \ldots, z_n \in \mathbb{C}$ (these are the roots of $p(z)$ counted with multiplicities). Use this to prove that any such polynomial where the coefficients $a_0, \ldots, a_n$ are *real* has a factorization

$$p(z) = a_n Q_1(z) Q_2(z) \ldots Q_m(z)$$

where each $Q_k(z)$ is a linear or quadratic monic polynomial (i.e., is of one of the forms $z - c$ or $z^2 + bz + c$) with real coefficients.

2. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ be a complex polynomial of degree n (that is, $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$), which as mentioned in question 1 above can be factored as

$$p(z) = a_n \prod_{k=1}^{n} (z - z_k)$$

where $z_1, \ldots, z_n$ are the roots of $p(z)$ counted with multiplicities. Assuming that $n \geq 2$, the derivative $p'(z)$ can be similarly factored as

$$p'(z) = n a_n \prod_{k=1}^{n-1} (z - w_k)$$

where $w_1, \ldots, w_{n-1}$ denote the roots of $p'(z)$. Prove that $w_1, \ldots, w_{n-1}$ all lie in the convex hull of $z_1, \ldots, z_n$ (see Figure 1 for an illustration). That is, each w_k can be expressed as a convex combination

$$w_k = \alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_n z_n,$$

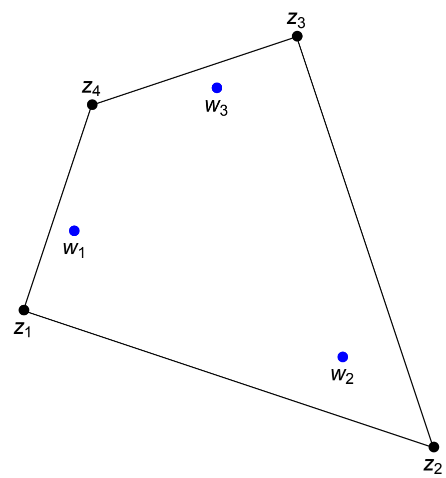

Figure 1: An example of the roots of a complex polynomial and of its derivative. Here $z_1 = 0$, $z_2 = 3 - i$, $z_3 = 2 + 2i$, $z_4 = \frac{1+3i}{2}$ and $w_1 \doteq 0.375 + 0.586i$, $w_2 \doteq 2.336 - 0.335i$, $w_3 \doteq 1.414 + 1.624i$.

where $\alpha_1, \ldots, \alpha_n$ are nonnegative real numbers and $\sum_j \alpha_j = 1$. (To be clear, there are different coefficients for each k.)

Hint. A complex number z is a root of $p'(z)$ that is not also a root of $p(z)$ if and only if $p'(z)/p(z) = 0$. (Note: the expression p'/p is known as the logarithmic derivative of p.) Find a way to make more explicit what this equation says.

3. Cardano's method for solving cubic equations. Let $p(z) = az^3 + bz^2 + cz + d$, with $a, b, c, d \in \mathbb{C}$, $a \neq 0$. We wish to solve the equation $p(z) = 0$, i.e., find the roots of the cubic polynomial $p(z)$.

(a) Show that the substitution $w = z - \frac{b}{3a}$ brings the equation to the simpler form

$$w^3 + pw + q = 0 \tag{1}$$

for some values of p, q (find them!) given as functions of a, b, c, d.

(b) Show that assuming a solution to (1) of the form $w = u + v$, the equation (1) for w can be solved by finding a pair u, v of complex numbers such that the equations

$$p = -3uv, \tag{2}$$
$$q = -(u^3 + v^3) \tag{3}$$

are satisfied.

(c) Explain why, in order to solve the pair of equations (2)–(3), one can alternatively solve

$$\frac{p^3}{27} = -RS, \tag{4}$$
$$q = -(R + S), \tag{5}$$

where we now denote new unknowns R, S defined by $R = u^3$, $S = v^3$. More precisely, any solution of (2)–(3) can be obtained from *some* (easily determined) solution of (4)–(5).

(d) Explain why the problem of solving (4)–(5) in the unknowns R, S is equivalent to solving the quadratic equation

$$t^2 + qt - \frac{p^3}{27} = 0 \tag{6}$$

in a (complex) unknown variable t.

(e) Using the above reductions, show that the three solutions of the simplified cubic (1) can be expressed as

$$w_1 = u + v,$$
$$w_2 = \zeta\, u + \overline{\zeta}\, v,$$
$$w_3 = \overline{\zeta}\, u + \zeta\, v,$$

where $\zeta = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$ (a cube root of unity) and u, v are properly chosen cube roots of R, S obtained as solutions to (6).

(f) Illustrate the above procedure by applying it to get formulas for the roots of the cubic equation

$$z^3 + 6z^2 + 9z + 3 = 0.$$

Bring the formulas to a form that makes it clear that the roots are real numbers.

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 real matrix. Prove the equivalence of the following three conditions:

(a) A as a linear map preserves orientation (that is, $\det A > 0$) and is conformal, that is
$$\frac{\langle Aw_1, Aw_2 \rangle}{|Aw_1|\,|Aw_2|} = \frac{\langle w_1, w_2 \rangle}{|w_1|\,|w_2|}$$
for all $w_1, w_2 \in \mathbb{R}^2$. (Here $\langle w_1, w_2 \rangle$ denotes the standard inner product in $\mathbb{R}^2$, and $|w| = \langle w, w \rangle^{1/2}$ is the usual two-dimensional norm of a vector in $\mathbb{R}^2$.)

(b) A takes the form $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$.

(c) A takes the form $A = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $r > 0$ and $\theta \in \mathbb{R}$. (That is, geometrically A acts by a rotation followed by a scaling.)

Note: this is the "conformality lemma" we formulated in class and which forms a key part of the claim that a continuously differentiable conformal, orientation-preserving map is holomorphic.

5. A function $f = u + iv$ of a complex variable $z = x + iy$ is traditionally thought of as a function of the two coordinates x and y. However, if we think of the equations
$$z = x + iy, \qquad \overline{z} = x - iy$$
as representing a formal change of variables from the "real coordinates" (x, y) to the "complex conjugate coordinates" $(z, \overline{z})$, then it may make sense to think of f as a function of the two variables z and $\overline{z}$ (pretending that those are two independent variables). Thus we may suggestively write $u = u(z, \overline{z})$ and $v = v(z, \overline{z})$, and consider operations such as taking the partial derivatives of f, u, v with respect to z and $\overline{z}$.

Show that, from this somewhat strange point of view, the Cauchy-Riemann equations
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

can be rewritten in the more concise equivalent form

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

assuming that it is okay to apply the chain rule from multivariable calculus; and moreover, that in this notation we also have the identity

$$f'(z) = \frac{\partial f}{\partial z}.$$

6. Let $f : \Omega \to \mathbb{C}$ be a function defined on a region Ω such that both the functions $f(z)$ and $zf(z)$ have real and imaginary parts that are harmonic functions (i.e., satisfy the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$). Prove that $f(z)$ is holomorphic on Ω.

Additional suggested exercises: exercises $5, 6, 7, 10, 14, 15$ in Chapter 1 of the Stein-Shakarchi textbook (pages 25–28). Exercise 7 in particular is an especially important fact in the theory of conformal mappings (which we will study later on in the course, or in 205B), and is worth thinking about.