Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #2

1. Uniform convergence on compact subsets

Given a sequence of functions $f_n : \Omega \to \mathbb{C}$, $n \geq 1$, defined on a region $\Omega \subset \mathbb{C}$, we say that f_n **converges uniformly on compact subsets** to a limiting function $f : \Omega \to \mathbb{C}$ if for any compact subset $K \subset \Omega$, $f_n(z) \to f(z)$ as $n \to \infty$, uniformly in $z \in K$.

(a) (Warm-up) Write this definition more precisely in ϵ-δ language.

(b) Prove that if f_n are holomorphic functions, $f_n \to f$ uniformly on compact subsets, $f_n' \to g$ uniformly on compact subsets, and g is continuous, then f is holomorphic and $f' = g$.

Hint. Fix some $z_0 \in \mathbb{C}$. Start by proving that for z in a sufficiently small neighborhood of z we have the two identities

$$f_n(z) = f_n(z_0) + \int_{z_0}^{z} f_n'(w)\,dw,$$
$$f(z) = f(z_0) + \int_{z_0}^{z} g(w)\,dw,$$

where the integral is over the line segment connecting z_0 to z.

(c) Prove that a power series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on compact subsets in its disk of convergence, and that the function it defines is continuous.

Hint. It is enough (why?) to prove uniform convergence on any closed disk of the form $\overline{D_r(0)}$ where $0 < r < R$ and R is the radius of convergence of the series.

(d) Deduce that power series are holomorphic functions that can be differentiated termwise (a fact we already proved in class in a more direct way; the above approach provides an alternative proof).

Remark. We will prove later as a consequence of Cauchy's theorem that in part (b) above the assumption that the sequence of derivatives f_n' converges to a limit can be dropped; that is, if a sequence of holomorphic functions converges uniformly on compact subsets, then the limiting function is automatically a holomorphic function whose derivative is the limit (in the sense of uniform convergence on compacts) of the sequence of derivatives of the original sequence. This is a surprising and nontrivial fact, as illustrated for example by the observation that the analogous statement in real analysis is false (e.g., by the Weierstrass approximation theorem, any continuous function on a closed interval is the uniform limit of a sequence of polynomials).

2. Solve exercise 25 on pages 30–31 of the Stein-Shakarchi textbook.

3. Cauchy's theorem and irrotational vector fields

Recall from vector calculus that a planar vector field $\mathbf{F} = (P, Q)$ defined on some region $\Omega \subset \mathbb{C} = \mathbb{R}^2$ is called **conservative** if it is of the form $\mathbf{F} = \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$ (the gradient of g) for some scalar function $g : \Omega \to \mathbb{R}$. By the fundamental theorem of calculus for line integrals, for such a vector field we have

$$\oint_\gamma \mathbf{F} \cdot \mathbf{ds} = 0$$

for any *closed* curve γ. Recall also that (as is easy to check) any conservative vector field is **irrotational**, namely satisfies

$$\operatorname{curl} \mathbf{F} = 0$$

(where in the context of two-dimensional vector fields, the curl is simply $\operatorname{curl} \mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$). The converse also holds under suitable conditions: if the region Ω is **simply-connected** (a concept we will discuss later in the course), then a theorem in vector calculus says that an irrotational vector field is also conservative.

Use these background results to show that if $f = u + iv$ is holomorphic on a simply-connected domain Ω, then

$$\oint_\gamma f(z)\, dz = 0$$

for any closed curve γ in Ω. (This is, of course, Cauchy's theorem.)

4. Bernoulli numbers

Define the function

$$f(z) = \begin{cases} \dfrac{z}{e^z - 1} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

(a) Convince yourself that $f(z)$ is analytic in a neighborhood of 0. Where else is it analytic? In particular, find the maximal radius R such that $f(z)$ is analytic on the disk $D_R(0)$.

(b) As we will see later, analytic functions have a power series expansion. The **Bernoulli numbers** are the numbers $(B_n)_{n=0}^{\infty}$ defined by the power series expansion

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = f(z).$$

For example, the first three Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$. Prove that the Bernoulli numbers satisfy the following identities:

1. $B_{2k+1} = 0$ for $k = 1, 2, \ldots$ (but not for $k = 0$).

   Hint. A function $g(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies $a_1 = a_3 = a_5 = \ldots = 0$ if and only if $g(z) = g(-z)$, i.e., $g(z)$ is an even function.

2. $(n+1)B_n = -\sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \quad (n \geq 2)$.

3. $(2n+1)B_{2n} = -\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k}, \quad (n \geq 2)$.

   Hint. Show that the function $g(z) = f(z) + z/2$ satisfies the equation

   $$g(z) - zg'(z) = g(z)^2 - z^2/4.$$

4. $\dfrac{z}{2} \coth\left(\dfrac{z}{2}\right) = \displaystyle\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}.$

(c) As we will also see later, the radius of convergence of the power series of an analytic function around $z = z_0$ is precisely the radius of the maximal disk around z_0 where f is analytic. Assuming this, deduce that

$$\limsup_{n\to\infty} \left| \frac{B_n}{n!} \right|^{1/n} = 1/R,$$

where R is the number you found in part (a). (Note: we will derive a much better estimate for the asymptotic rate of growth of the Bernoulli numbers later in the course.)

5. Bessel functions

The **Bessel functions** are a family of functions $(J_n)_{n=-\infty}^{\infty}$ of a complex variable,defined by

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n}.$$

(For example, note that $J_0(-2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}$, which is reminiscent of the exponential function and already seems like a fairly natural function to study.) Find the radius of convergence of the series defining $J_n(z)$, and prove that the Bessel functions satisfy the following properties:

(a) $J_{-n}(z) = (-1)^n J_n(z)$.

(b) Recurrence relation: $J_{n+1}(z) = \dfrac{2n}{z} J_n(z) - J_{n-1}(z)$.

(c) Differential equation: $z^2 J_n''(z) + z J_n'(z) + (z^2 - n^2) J_n(z) = 0$.

(d) Summation identity: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z}{2}\right)^n J_n(z) = 1$.

(e)* Other miscellaneous identities (for those who enjoy this sort of thing—

feel free to skip if you find these sorts of computations uninteresting):

$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)t^n,$$

$$\cos(z\sin t) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2nt),$$

$$\sin(z\sin t) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin((2n+1)t),$$

$$\cos(z\cos t) = J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2nt),$$

$$\sin(z\cos t) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z)\sin((2n+1)t),$$

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos\left(z\sin t - nt\right)\, dt.$$

Hint for the last equation: $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$.

Remark. The Bessel functions are very important functions in mathematical physics, and appear naturally in connection with various problems in diffusion, heat conduction, electrodynamics, quantum mechanics, Brownian motion, probability, and more. More recently they played an important role in some problems in combinatorics related to longest increasing subsequences (a subject I wrote a book about, available to download from my home page). Their properties as analytic functions of a complex variable are also a classical, though no longer very fashionable, topic of study.