Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #3

January 31, 2018

Instructions. This problem set is one of the two problem sets I will collect and grade as part of the course's requirements. Please hand in your solutions (in person or over email) by **Wednesday, February 14**.

Reminder. Please be mindful of the course ethics policy described in the syllabus. As long as you keep those guidelines in mind and adhere to them, I have no objections to you working on the problems collaboratively with others and/or consulting books and online resources. (As a general rule, I *recommend* trying to solve problems by yourself first.)

1. Show that Liouville's theorem ("a bounded entire function is constant") can be proved directly using the "simple" ($n = 0$) case of Cauchy's integral formula, instead of using the case $n = 1$ of the extended formula as we did in the lecture.

Hint. For an arbitrary pair of complex numbers $z_1, z_2 \in \mathbb{C}$, show that $|f(z_1) - f(z_2)| = 0$.

2. Show that Liouville's theorem can in fact be deduced even just from the mean value property of holomorphic functions, which is the special case of Cauchy's integral formula in which z is taken as the center of the circle around which the integration is performed.

Hint. Here it makes sense to consider a modified version of the mean value property (that follows easily from the original version) that says that $f(z)$ is the average value of $f(w)$ over a *disc* $D_R(z)$ (instead of a circle $C_R(z)$). That is,

$$f(z) = \frac{1}{\pi R^2} \iint_{D_R(z)} f(x + iy)\, dx\, dy,$$

where the integral is an ordinary two-dimensional Riemann integral. Explain why this formula holds, then use it to again bound $|f(z_1) - f(z_2)|$ from above by a quantity that goes to 0 as $R \to \infty$.

3. Prove the following generalization of Liouville's theorem: let f be an entire function satisfying the inequality

$$|f(z)| \leq A + B|z|^n \qquad (z \in \mathbb{C})$$

for some constants $A, B > 0$ and integer $n \geq 0$. Then f is a polynomial of degree $\leq n$.

4. If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + \ldots + a_0$ is a polynomial of degree n such that

$$|a_n| > \sum_{j=0}^{n-1} |a_j|,$$

prove that $p(z)$ has exactly n zeros (counting multiplicities) in the unit disc $|z| < 1$.

Hint. Use the fundamental theorem of algebra.

Note. This is a special case of the following less elementary fact, that we will prove in a future homework problem after learning Rouché's theorem: if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + \ldots + a_0$ is a polynomial of degree n such that for some $0 \leq k \leq n$ we have

$$|a_k| > \sum_{\substack{0 \leq j \leq n \\ j \neq k}} |a_j|,$$

then $p(z)$ has exactly k zeros (counting multiplicities) in the unit disk $|z| < 1$.

5. It is important to have computational facility with contour integrals in addition to a good grasp of the theory. To practice your contour integral skills, solve exercises 1–4 (pages 64–65) in Chapter 2 of [Stein-Shakarchi].

6. (Optional problem) The Cauchy integral formula is intimately connected to an important formula from the theory of the Laplace equation and harmonic functions called the **Poisson integral formula**. Solve exercises 11–12 (pages 66–67) in Chapter 2 of [Stein-Shakarchi], which explore this connection, and more generally the connection between holomorphic and harmonic functions.

7. (Optional problem) Spend at least 5–10 minutes thinking about the concept of a **toy contour**. Specifically, for the case of a **keyhole contour**

we discussed in the context of the proof of Cauchy's integral formula, think carefully about the steps that are needed to get a proof of Cauchy's theorem for the region enclosed by such a contour. Even better, sketch a proof of the key result that a function holomorphic in such a region (and therefore having the property that its contour integral along triangles and rectangles vanish) has a primitive.