Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #4

February 6, 2018

> **Instructions.** I will accept a submission of this homework problem set, or any of the planned subsequent ones (Problem Sets 5, 6, 7 and 8), as satisfying the course requirement for the second for-grading homework submission (in addition to Problem Set #3, which is due on Wednesday 2/14). Please hand in your solutions by **Wednesday, March 14**.
>
> You do not need to solve all the problems for an A grade in this submission. I will want to see that you've made a good effort — for example, if you solve problems 1–4 and 6 I will be happy; anything more and I will be quite impressed.

Notation. For the problems below, denote

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \text{the Riemann sphere},$$

$$\mathcal{K} = \text{the set of constant functions } z \mapsto c \in \mathbb{C},$$

$$\mathcal{L} = \text{the set of linear functions } z \mapsto az + b, \quad a, b \in \mathbb{C},$$

$$\mathcal{P} = \text{the set of complex polynomials } z \mapsto \sum_{k=0}^{n} a_k z^k,$$

$$\mathcal{R} = \text{the set of rational functions } z \mapsto \frac{p(z)}{q(z)}, \quad p, q \in \mathcal{P},$$

$$\mathcal{M} = \text{the set of Möbius transformations } z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}.$$

Note the containment relations $\mathcal{K} \subset \mathcal{L} \subset \mathcal{P} \subset \mathcal{R} \supset \mathcal{M} \supset \mathcal{L} \supset \mathcal{K}$.

1. (Warm-up problem) Prove that an entire function has a removable singularity at ∞ if and only if it is constant.

2. Prove that the set of entire functions $f : \mathbb{C} \to \mathbb{C}$ that have a nonessential singularity at ∞ is $\mathcal{P}$, the polynomials.

3. Prove that the set of meromorphic functions $f : \mathbb{C} \to \hat{\mathbb{C}}$ that have a nonessential singularity at ∞ is $\mathcal{R}$, the rational functions.

4. Prove that the set of meromorphic, one-to-one and onto functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is $\mathcal{M} \setminus \mathcal{K}$, the set of nonconstant Möbius transformations.

Hint. Use the characterization in exercise 3 above. Specifically, show that a rational function $f(z) = p(z)/q(z)$ that is one-to-one must be a Möbius transformation. For example (I'm not sure if this is the simplest argument): argue that if z_0 is a complex number such that $q(z_0) \neq 0$ and such that $p'(z_0)q(z_0) - p(z_0)q'(z_0) \neq 0$, and $w_0 = f(z_0)$, then the equation $f(z) = w_0$ must have more than one solution in z, unless $p(z), q(z)$ are linear functions.

5.* Prove that set of entire functions $f : \mathbb{C} \to \mathbb{C}$ that are one-to-one and onto is precisely $\mathcal{L} \setminus \mathcal{K}$, the set of nonconstant linear functions.

Hint. This is a slightly more advanced problem since it relies on both the Casorati-Weierstrass theorem and the open mapping theorem (although a more elementary solution may exist). See the guidance for exercise 14 on page 105 of [Stein-Shakarchi].

Remarks. Given a region $\Omega \subset \mathbb{C}$, or more generally a Riemann surface Σ, complex analysts are interested in understanding the structure of its set of holomorphic functions ($\mathbb{C}$-valued holomorphic functions on Σ); its set of meromorphic functions ($\hat{\mathbb{C}}$-valued holomorphic functions on Σ); and its set of holomorphic automorphisms (holomorphic, one-to-one and onto mappings from Σ to itself). Although we won't get into the general theory of Riemann surfaces, once one defines these concepts it easy to see that the above exercises essentially prove the following conceptually important results:

1. The constant functions are the only holomorphic functions on $\hat{\mathbb{C}}$.
2. The rational functions are the meromorphic functions on $\hat{\mathbb{C}}$.
3. The nonconstant linear functions are the holomorphic automorphisms of $\mathbb{C}$.
4. The nonconstant Möbius transformations are the holomorphic automorphisms of $\hat{\mathbb{C}}$.

Another related result that is not very difficult to prove is:

5. The holomorphic automorphisms of the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ are the Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. (Try to prove that any such map is indeed an automorphism of $\mathbb{H}$; the reverse implication that all automorphisms of $\mathbb{H}$ are of this form is a bit more difficult and requires a result known as the Schwarz lemma, which will likely be covered at some point in MAT205A/B.)

Note that the set of holomorphic functions on $\mathbb{C}$ (a.k.a. entire functions) and the set of meromorphic functions on $\mathbb{C}$ are much larger families of functions that do not have such a simple description as the functions in the relatively small families $\mathcal{L}, \mathcal{P}, \mathcal{R}, \mathcal{M}$. This is related to the fact that $\mathbb{C}$ is a non-compact Riemann surface.

6. (a) Let $z \in \mathbb{C} \setminus \mathbb{Z}$. Use the residue theorem to evaluate the contour integral

$$I_N := \oint_{\gamma_N} \frac{\pi \cot(\pi w)}{(w+z)^2}\, dw$$

over the contour γ_N going in the positive direction around the rectangle with the four vertices $(\pm(N+1/2), \pm N)$. Take the limit as $N \to \infty$ to deduce the well-known identity

$$\frac{\pi^2}{(\sin \pi z)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} \qquad (z \in \mathbb{C} \setminus \mathbb{Z}). \tag{$\star$}$$

Guidance. This is not a trivial exercise, but is not very difficult when broken down into the following elementary steps:

i) Start by identifying the location of the singularities of the function $w \mapsto f_z(w) = \frac{\pi \cot(\pi w)}{(w+z)^2}$ (considered as a function of w for a fixed z whose value is not an integer), and their residues. This provides some good practice with residue computations.

ii) Use the residue theorem to obtain an expression for the contour integral I_N defined above.

iii) Separately, obtain estimates for I_N that can be used to show that $I_N \to 0$ as $N \to \infty$. Specifically, show using elementary manipulations that

$$|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y,$$

use this to conclude that when $x = \pi(N + 1/2)$ and y is arbitrary,

$$|\cot(x+iy)| = \frac{\sinh^2 y}{1+\sinh^2 y} \le 1,$$

and that when $y = N$ and x is arbitrary,

$$|\cot(x+iy)| \le \frac{1+\sinh^2 N}{\sinh^2 N} \le 2 \quad (\text{if } N > 10);$$

then use these estimates to bound the integral.

iv) By comparing the two results about I_N, deduce ($\star$).

(b) Integrate the identity ($\star$) to deduce (using some additional fairly easy reasoning) the formulas

$$\pi\cot(\pi z) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \qquad (z \in \mathbb{C}\setminus\mathbb{Z}).$$

Remark. In next week's homework we will see how these formulas contain the secret to proving the famous formulas

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^4}{945}, \ \ldots$$

and to additional interesting results such as Wallis's infinite product for π,

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots.$$

7. Let $f(z) = p(z)/q(z)$ be a rational function such that $\deg q \ge \deg p + 2$ (where $\deg p$ denotes the degree of a polynomial p). Prove that the sum of the residues of $f(z)$ over all its poles is equal to 0.

8. Additional exercises — strongly recommended to practice your computational skills, especially for those who have not had recent practice in such things:

(a) Read section 2.1 of Chapter 3 (pages 77–83) of [Stein-Shakarchi] for examples of the use of the residue theorem for the evaluation of definite integrals.

(b) Solve some or all of problems 1–10 in Chapter 3, pages 103–104 of [Stein-Shakarchi].