Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #5

February 13, 2018

1. (Continuation of problem 4 from Problem Set #3) If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + \ldots + a_0$ is a polynomial of degree n such that for some $0 \le k \le n$ we have

$$|a_k| > \sum_{\substack{0\le j\le n \\ j\ne k}} |a_j|,$$

prove that $p(z)$ has exactly k zeros (counting multiplicities) in the unit disk $|z| < 1$.

2. Show how Rouché's theorem can be used to give yet another proof of the fundamental theorem of algebra. This proof is one way to make precise the intuitively compelling "topological" proof idea we discussed at the beginning of the course.

3. (a) Draw a simply-connected region $\Omega \subset \mathbb{C}$ such that $0 \notin \Omega$, $1, 2 \in \Omega$, and such that there exists a branch $F(z)$ of the logarithm function on Ω satisfying

$$F(1) = 0, \qquad F(2) = \log 2 + 2\pi i$$

(where $\log 2 = 0.69314\ldots$ is the ordinary logarithm of 2 in the usual sense of real analysis).

(b) More generally, let $k \in \mathbb{Z}$. If we were to replace the above condition $F(2) = \log 2 + 2\pi i$ with the more general condition $F(2) = \log 2 + 2\pi i k$ but keep all the other conditions, would an appropriate simply-connected region $\Omega = \Omega(k)$ exist to make that possible? If so, what would this region look like, roughly, as a function of k?

4. Continuation of last week's homework: the partial fraction expansion of the cotangent function and its consequences

In the previous homework problem set we outlined an approach to using residue calculus to prove the important identity

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad (z \in \mathbb{C} \setminus \mathbb{Z}), \qquad (*)$$

known as the **partial fraction expansion of the cotangent function**. We now derive some additional consequences from this identity.

(a) Show that (*) implies the following infinite-product representation for the sine function:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad (z \in \mathbb{C}). \tag{**}$$

Note that the function on the right-hand side is (or can be easily checked to be) an entire function of z with a simple zero at any integer $z = n \in \mathbb{Z}$, and whose Taylor expansion around $z = 0$ starts with $\pi z + O(z^3)$; thus it is a natural guess for an infinite product expansion of $\sin(\pi z)$, although the fact that this guess is correct is far from obvious; for example one can multiply the right-hand side by an arbitrary function of the form $e^{g(z)}$ and still have an entire function with the same set of zeros.

Hint. Compute the logarithmic derivatives of both sides of (**). You may want to review some basic properties of infinite products, as discussed for example on pages 140–142 of [Stein-Shakarchi]. (**Spoiler alert:** pages 142–144 contain a solution to this subexercise, starting with an independent proof of (*) and proceeding with a derivation of (**) along the same lines as I described above.)

(b) By specializing the value of z in (**) to an appropriate specific value, obtain the following infinite product formula for π, known as **Wallis' product** (first proved by John Wallis in 1655):

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots .$$

(c) By comparing the first terms in the Taylor expansion around $z = 0$ of both sides of (**), derive the well-known identities

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(d) More generally, one can use (**), or more conveniently (*), to obtain closed formulas for all the series

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \qquad (k = 1, 2, \ldots) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \ldots,$$

that is, the special values of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ at the positive even integers. To see this, first, rewrite (*) as

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right) \qquad (z \in \mathbb{C} \setminus \mathbb{Z}). \tag{***}$$

Expand both sides of (***) in a Taylor series around $z = 0$, making use of the expansion

$$\frac{z}{2} \coth\left(\frac{z}{2}\right) = \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n}$$

we proved in an earlier homework exercise (where $(B_n)_{n=0}^{\infty}$ are the Bernoulli numbers). Compare coefficients and simplify to get the formula

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}.$$

For example, using the first few values $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, we get

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$
$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$
$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945},$$
$$\zeta(8) = \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450},$$

where of course the first two values coincide with those found earlier.

(e) Show that $\zeta(2k) = 1 + O(2^{-2k})$ as $k \to \infty$, and deduce that the asymptotic behavior of the Bernoulli numbers is given by

$$B_{2k} = (1 + O(2^{-2k}))(-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}}, \qquad k \to \infty.$$

Note that this is consistent with our earlier (and much weaker) result that

$$\limsup_{k\to\infty}\left|\frac{B_{2k}}{(2k)!}\right|^{1/2k}=\frac{1}{2\pi}.$$

5. Suggested reading: go to the Mathematics Stack Exchange website (`http://math.stackexchange.com`) and enter "Rouche" into the search box, to get an amusing list of questions and exercises involving applications of Rouché's theorem to count zeros of polynomials and other analytic functions.