1. Prove the following properties satisfied by the gamma function:
   (a) Values at half-integers:
       \[ \Gamma \left( n + \frac{1}{2} \right) = \frac{(2n)!}{4^n n!} \sqrt{\pi} \quad (n = 0, 1, 2, \ldots). \]
   (b) The duplication formula:
       \[ \Gamma(s)\Gamma(s + 1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s). \]
   (c)* The multiplication theorem:
       \[ \Gamma(s)\Gamma \left( s + \frac{1}{k} \right) \Gamma \left( s + \frac{2}{k} \right) \cdots \Gamma \left( s + \frac{k-1}{k} \right) = (2\pi)^{(k-1)/2}k^{1/2-k}\Gamma(ks). \]

2. For \( n \geq 1 \), let \( V_n \) denote the volume of the unit ball in \( \mathbb{R}^n \). By evaluating the \( n \)-dimensional integral
   \[ A_n = \int \cdots \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} \sum_{j=1}^{n} x_j^2 \right) \, dx_1 \, dx_2 \cdots dx_n \]
   in two ways, prove the well-known formula
   \[ V_n = \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} + 1 \right)}. \]
   **Note.** This problem requires applying a small amount of geometric intuition (or, alternatively, having some technical knowledge of spherical coordinates in \( \mathbb{R}^n \)). The solution can be found on this Wikipedia page.

3. The beta function is a function \( B(s, t) \) of two complex variables, defined for \( \text{Re}(s), \text{Re}(t) > 0 \) by
   \[ B(s, t) = \int_0^1 x^{s-1}(1 - x)^{t-1} \, dx. \]
(a) (Warm-up) Convince yourself that the improper integral defining $B(s,t)$ converges if and only if $\text{Re}(s), \text{Re}(t) > 0$.

(b) Show that $B(s,t)$ can be expressed in terms of the gamma function as

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$ 

**Hint.** Start by writing $\Gamma(s)\Gamma(t)$ as a double integral on the positive quadrant $[0, \infty)^2$ of $\mathbb{R}^2$ (with integration variables, say, $x$ and $y$); then make the change of variables $u = x + y$, $v = x/(x+y)$ and use the change of variables formula for two-dimensional integrals to show that the integral evaluates as $\Gamma(s+t)B(s,t)$.

**Remark.** Note the similarity of the identity relating the gamma and beta functions to the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$; indeed, using the relation $\Gamma(m+1) = m!$ and the functional equation $\Gamma(s+1) = s\Gamma(s)$, we see using the above relation that for nonnegative, integer-valued arguments we have

$$B(n,m)^{-1} = \frac{nm}{n+m} \cdot \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)} = \frac{nm}{n+m} \left( \frac{n+m}{n} \right).$$

In other words, except for the correction factor $\frac{nm}{n+m}$, the inverse of the beta function can be thought of as a natural extension of binomial coefficients to real-valued arguments.

4. The **digamma function** $\psi(s)$ is the logarithmic derivative

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

of the gamma function, also considered as a somewhat important special function in its own right.

(a) Show that $\psi(s)$ has the convergent series expansions

$$\psi(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \frac{s}{n(n+s)}$$

$$= -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+s} \right) \quad (s \neq 0, -1, -2, \ldots).$$

where $\gamma$ is the Euler-Mascheroni constant.
(b) Equivalently, show that \( \psi(s) \) can be expressed as

\[
\psi(s) = -\lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k+s} - \log n \right).
\]

(c) Show that \( \psi(s) \) satisfies the functional equation

\[
\psi(s + 1) = \psi(s) + \frac{1}{s}.
\]

(d) Show that

\[
\psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k} \quad (n = 0, 1, 2, \ldots).
\]

That is, \( \psi(x) + \gamma \) can be thought of as extending the definition of the harmonic numbers \( H_n = \sum_{k=1}^{n} \frac{1}{k} \) to non-integer arguments.

(e) Show that \( \psi(s) \) satisfies the reflection formula

\[
\psi(1 - s) - \psi(s) = \pi \cot(\pi s).
\]

(f)* Here is an amusing application of the digamma function. Consider the sequence of polynomials

\[
P_n(x) = x(x - 1) \ldots (x - n) \quad (n = 0, 1, 2, \ldots)
\]

and their derivatives

\[
Q_n(x) = P'_n(x).
\]

Note that by Rolle’s theorem, \( Q_n(x) \) has precisely one root in each interval \((k, k+1)\) for \(0 \leq k \leq n - 1\). Denote this root by \( k + \alpha_{n,k} \), so that the numbers \( \alpha_{n,k} \) (the fractional parts of the roots of \( Q_n(x) \)) are in \((0, 1)\).

A curious phenomenon can now be observed by plotting the points \( \alpha_{n,k} \), \( k = 0, \ldots, n - 1 \) numerically, say for \( n = 50 \) (Figure 1(a)). It appears that for large \( n \) they approximate some smooth limiting curve. This is correct, and in fact the following precise statement can be proved.
Theorem. Let $t \in (0, 1)$. Let $k = k(n)$ be a sequence such that $0 \leq k(n) \leq n - 1$, $k(n) \to \infty$ as $n \to \infty$, $n - k(n) \to \infty$ as $n \to \infty$, and $k(n)/n \to t$ as $n \to \infty$. Then we have

$$\lim_{n \to \infty} \alpha_{n,k(n)} = R(t) := \frac{1}{\pi} \arccot \left( \frac{1}{\pi} \log \left( \frac{1-t}{t} \right) \right).$$

In the above formula, arccot$(\cdot)$ refers to the branch of the inverse cotangent function taking values between 0 and $\pi$. The limiting function $R(t)$ is shown in Figure 1(b).

Prove this.

Guidance. Take the logarithmic derivative of $P_n(x)$ to see when the equation $Q_n(x)/P_n(x) = 0$ (which is equivalent to $Q_n(x) = 0$) holds. This will give an equation with a sum of terms. Find a way to separate them into two groups such that the sum in each group can be related, in an asymptotic sense as $n \to \infty$, to the digamma function evaluated at a certain argument (using property (b) above). Take the limit as $n \to \infty$, then simplify using the reflection formula (part (c)).

5. Given two integrable functions $f, g : \mathbb{R} \to \mathbb{C}$ (of a real variable), their convolution is the new function $h = f \ast g$ defined by the formula

$$h(x) = (f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) \, dt \quad (x \in \mathbb{R}).$$

The convolution operation is extremely important in harmonic analysis, since it corresponds to a simple multiplication operation in the Fourier domain; in probability theory, where it corresponds to the addition of independent random variables; and in many other areas of mathematics, science and engineering.

For $\alpha > 0$ define the gamma density with parameter $\alpha$, denoted $\gamma_\alpha : \mathbb{R} \to \mathbb{R}$, to be the function

$$\gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} 1_{[0,\infty)}(x) \quad (x \in \mathbb{R})$$

(where $1_A(x)$ denotes the characteristic function of a set $A \subset \mathbb{R}$, equal to 1 on the set and 0 outside it). Note that $\gamma_\alpha(x)$ is the nonnegative function whose
integral equals $\Gamma(\alpha)$, except that it is divided by $\Gamma(\alpha)$ so that it becomes a probability density function. See Figure 2 for an illustration.

Show that for each $\alpha, \beta > 0$ we have

$$\gamma_\alpha \ast \gamma_\beta = \gamma_{\alpha + \beta}.$$ 

That is, the family of density functions $(\gamma_\alpha)_{\alpha > 0}$ is closed under the convolution operation. This fact is one of the reasons why the family of gamma densities plays a very important role in probability theory and appears in many real-life applications.
Figure 2: The gamma densities $\gamma_\alpha(x)$ for $\alpha = 1, 2, 3, 4, 5$. 