Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #6

February 20, 2018

1. Prove the following properties satisfied by the gamma function:

(a) Values at half-integers:

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \qquad (n = 0, 1, 2, \ldots).$$

(b) The duplication formula:

$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

(c)* The multiplication theorem:

$$\Gamma\left(s\right)\Gamma\left(s+\tfrac{1}{k}\right)\Gamma\left(s+\tfrac{2}{k}\right)\cdots\Gamma\left(s+\tfrac{k-1}{k}\right) = (2\pi)^{(k-1)/2}k^{1/2-ks}\Gamma(ks).$$

2. For $n \geq 1$, let V_n denote the volume of the unit ball in $\mathbb{R}^n$. By evaluating the n-dimensional integral

$$A_n = \int\!\!\int\!\ldots\!\int_{\mathbb{R}^n} \exp\left(-\tfrac{1}{2}\sum_{j=1}^{n} x_j^2\right) dx_1\, dx_2 \ldots dx_n$$

in two ways, prove the well-known formula

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Note. This problem requires applying a small amount of geometric intuition (or, alternatively, having some technical knowledge of spherical coordinates in $\mathbb{R}^n$). The solution can be found on this Wikipedia page.

3. The **beta function** is a function $B(s,t)$ of two complex variables, defined for $\mathrm{Re}(s), \mathrm{Re}(t) > 0$ by

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}\, dx.$$

(a) (Warm-up) Convince yourself that the improper integral defining $B(s,t)$ converges if and only if $\mathrm{Re}(s), \mathrm{Re}(t) > 0$.

(b) Show that $B(s,t)$ can be expressed in terms of the gamma function as

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Hint. Start by writing $\Gamma(s)\Gamma(t)$ as a double integral on the positive quadrant $[0,\infty)^2$ of $\mathbb{R}^2$ (with integration variables, say, x and y); then make the change of variables $u = x+y$, $v = x/(x+y)$ and use the change of variables formula for two-dimensional integrals to show that the integral evaluates as $\Gamma(s+t)B(s,t)$.

Remark. Note the similarity of the identity relating the gamma and beta functions to the formula $\binom{n}{k} = \frac{n}{k!(n-k)!}$; indeed, using the relation $\Gamma(m+1) = m!$ and the functional equation $\Gamma(s+1) = s\,\Gamma(s)$, we see using the above relation that for nonnegative, integer-valued arguments we have

$$B(n,m)^{-1} = \frac{nm}{n+m} \cdot \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)} = \frac{nm}{n+m}\binom{n+m}{n}.$$

In other words, except for the correction factor $\frac{nm}{n+m}$, the inverse of the beta function can be thought of as a natural extention of binomial coefficients to real-valued arguments.

4. The **digamma function** $\psi(s)$ is the logarithmic derivative

$$\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$$

of the gamma function, also considered as a somewhat important special function in its own right.

(a) Show that $\psi(s)$ has the convergent series expansions

$$\begin{aligned}\psi(s) &= -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \frac{s}{n(n+s)} \\ &= -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+s}\right) \qquad (s \neq 0, -1, -2, \ldots).\end{aligned}$$

where γ is the Euler-Mascheroni constant.

(b) Equivalently, show that $\psi(s)$ can be expressed as

$$\psi(s) = -\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+s} - \log n\right).$$

(c) Show that $\psi(s)$ satisfies the functional equation

$$\psi(s+1) = \psi(s) + \frac{1}{s}.$$

(d) Show that

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \qquad (n = 0, 1, 2, \ldots).$$

That is, $\psi(x) + \gamma$ can be thought of as extending the definition of the **harmonic numbers** $H_n = \sum_{k=1}^{n}\frac{1}{k}$ to non-integer arguments.

(e) Show that $\psi(s)$ satisfies the reflection formula

$$\psi(1-s) - \psi(s) = \pi\cot(\pi s).$$

(f)* Here is an amusing application of the digamma function. Consider the sequence of polynomials

$$P_n(x) = x(x-1)\ldots(x-n) \qquad (n = 0, 1, 2, \ldots)$$

and their derivatives

$$Q_n(x) = P_n'(x).$$

Note that by Rolle's theorem, $Q_n(x)$ has precisely one root in each interval $(k, k+1)$ for $0 \le k \le n-1$. Denote this root by $k + \alpha_{n,k}$, so that the numbers $\alpha_{n,k}$ (the fractional parts of the roots of $Q_n(x)$) are in $(0, 1)$.

A curious phenomenon can now be observed by plotting the points $\alpha_{n,k}$, $k = 0, \ldots, n-1$ numerically, say for $n = 50$ (Figure 1(a)). It appears that for large n they approximate some smooth limiting curve. This is correct, and in fact the following precise statement can be proved.

Theorem. Let $t \in (0,1)$. Let $k = k(n)$ be a sequence such that $0 \le k(n) \le n-1$, $k(n) \to \infty$ as $n \to \infty$, $n - k(n) \to \infty$ as $n \to \infty$, and $k(n)/n \to t$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \alpha_{n,k(n)} = R(t) := \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi}\log\left(\frac{1-t}{t}\right)\right).$$

In the above formula, $\operatorname{arccot}(\cdot)$ refers to the branch of the inverse cotangent function taking values between 0 and π. The limiting function $R(t)$ is shown in Figure 1(b).

Prove this.

Guidance. Take the logarithmic derivative of $P_n(x)$ to see when the equation $Q_n(x)/P_n(x) = 0$ (which is equivalent to $Q_n(x) = 0$) holds. This will give an equation with a sum of terms. Find a way to separate them into two groups such that the sum in each group can be related, in an asymptotic sense as $n \to \infty$, to the digamma function evaluated at a certain argument (using property (b) above). Take the limit as $n \to \infty$, then simplify using the reflection formula (part (c)).

5. Given two integrable functions $f, g : \mathbb{R} \to \mathbb{C}$ (of a *real* variable), their **convolution** is the new function $h = f * g$ defined by the formula

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \qquad (x \in \mathbb{R}).$$

The convolution operation is extremely important in harmonic analysis, since it corresponds to a simple multiplication operation in the Fourier domain; in probability theory, where it corresponds to the addition of independent random variables; and in many other areas of mathematics, science and engineering.

For $\alpha > 0$ define the **gamma density** with parameter α, denoted $\gamma_\alpha : \mathbb{R} \to \mathbb{R}$, to be the function

$$\gamma_\alpha(x) = \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1} 1_{[0,\infty)}(x) \qquad (x \in \mathbb{R})$$

(where $1_A(x)$ denotes the characteristic function of a set $A \subset \mathbb{R}$, equal to 1 on the set and 0 outside it). Note that $\gamma_\alpha(x)$ is the nonnegative function whose

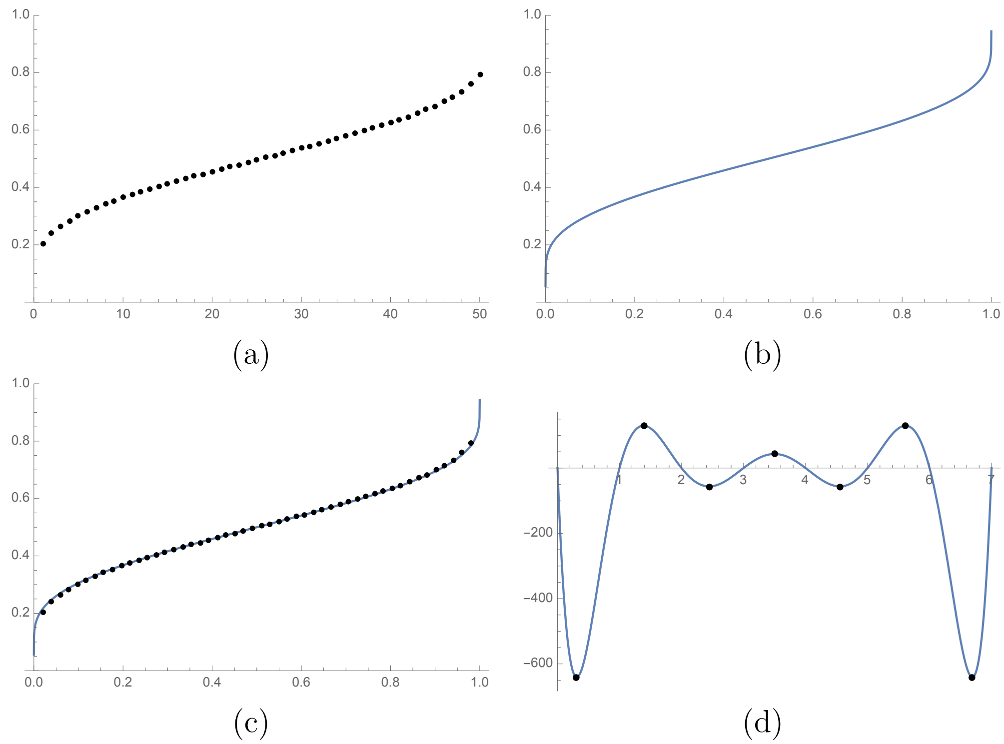

Figure 1: (a) A plot of the fractional parts of the roots of $Q_n(x)$ for $n = 50$. (b) The limiting function $R(t)$. (c) The two previous plots combined. (d) The polynomial $P_7(x)$. Note that the roots of $Q_7(x)$ correspond to the local minima and maxima of $P_7(x)$, which are highlighted.

integral equals $\Gamma(\alpha)$, except that it is divided by $\Gamma(\alpha)$ so that it becomes a probability density function. See Figure 2 for an illustration.

Show that for each $\alpha, \beta > 0$ we have

$$\gamma_\alpha * \gamma_\beta = \gamma_{\alpha+\beta}.$$

That is, the family of density functions $(\gamma_\alpha)_{\alpha>0}$ is closed under the convolution operation. This fact is one of the reasons why the family of gamma densities plays a very important role in probability theory and appears in many real-life applications.

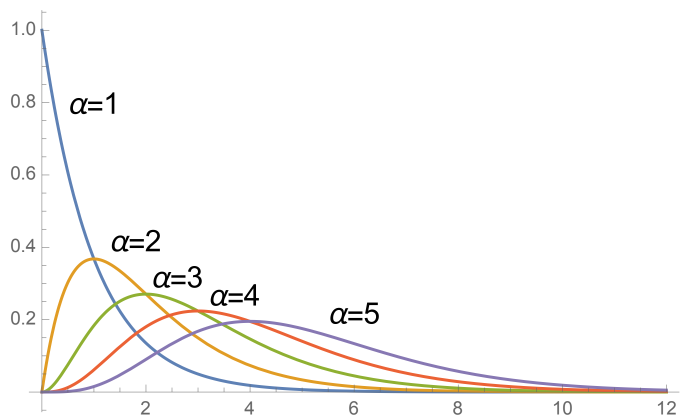

Figure 2: The gamma densities $\gamma_\alpha(x)$ for $\alpha = 1, 2, 3, 4, 5$.