Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #7

February 28, 2018

1. (a) Show that the Laurent expansion of $\Gamma(s)$ around $s = 0$ is of the form

$$\Gamma(s) = \frac{1}{s} - \gamma + O(s)$$

(where γ is the Euler-Mascheroni constant). If you're feeling especially energetic, derive the more detailed expansion

$$\Gamma(s) = \frac{1}{s} - \gamma + \left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right) s + O(s^2)$$

and proceed to derive (by hand, or if you prefer using a software package such as `Mathematica`) as many additional terms in the expansion as you have the patience to do.

(b) Show that the Laurent expansion of $\zeta(s)$ around $s = 1$ is of the form

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

2. Show that the symmetric version of the functional equation for the zeta function

$$\zeta^*(1-s) = \zeta^*(s),$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, can be rewritten in the equivalent form

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

3. Show that the Taylor expansion of the digamma function $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ around $s = 1$ is given by

$$\psi(s) = -\gamma - \sum_{n=1}^{\infty} (-1)^{n-1} \zeta(n+1)(s-1)^n \qquad (|s-1| < 1).$$

4. Define a function $D(s)$ of a complex variable s by

$$D(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots.$$

(a) Prove that the series defining $D(s)$ converges uniformly on any half-plane of the form $\mathrm{Re}(s) \geq \alpha$ where $\alpha > 0$, and conclude that $D(s)$ is defined and holomorphic in the half-plane $\mathrm{Re}(s) > 0$.

(b) Show that $D(s)$ is related to the Riemann zeta function by the formula

$$D(s) = (1 - 2^{1-s})\zeta(s) \qquad (\mathrm{Re}(s) > 1).$$

(c) Using this relation, deduce a new proof that the zeta function can be analytically continued to a meromorphic function on $\mathrm{Re}(s) > 0$ that has a simple pole at $s = 1$ with residue 1 and is holomorphic everywhere else in the region.

5. Let $\psi(x) = \sum_{p^k \leq x} \log p$ denote von Mangoldt's weighted prime counting function. Show that $\psi(n) = \log \mathrm{lcm}(1, 2, \ldots, n)$, where for integers $a_1, \ldots, a_k$, $\mathrm{lcm}(a_1, \ldots, a_k)$ denotes the least common multiple of $a_1, \ldots, a_k$.

Note that this implies that an equivalent formulation of the prime number theorem is the interesting statement that

$$\mathrm{lcm}(1, \ldots, n) = e^{(1+o(1))n} \qquad \text{as } n \to \infty.$$

6. (a) Prove that for all $x \geq 1$,

$$\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} \geq \log x$$

(where the product is over all prime numbers p that are $\leq x$).

(b) Pass to the logarithm and deduce that for some constant $K > 0$ we have the bound

$$\sum_{p \leq x} \frac{1}{p} \geq \log \log x - K \qquad (x \geq 1).$$

That is, the *harmonic series of primes* $\sum_p \frac{1}{p}$ diverges as $\log \log x$, in contrast to the usual harmonic series which diverges as $\log x$.

7. Recall that we defined the Jacobi theta function by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \qquad (t > 0),$$

and showed that it satisfies the functional equation

$$\vartheta\left(\frac{1}{t}\right) = \sqrt{t}\vartheta(t).$$

(a) Use the residue theorem to evaluate the contour integral

$$\oint_{\gamma_N} \frac{e^{-\pi z^2 t}}{e^{2\pi i z} - 1}\, dz,$$

where γ_N is the rectangle with vertices $\pm(N + 1/2) \pm i$ (with N a positive integer), then take the limit as $N \to \infty$ to derive the integral representation

$$\vartheta(t) = \int_{-\infty-i}^{\infty-i} \frac{e^{-\pi z^2 t}}{e^{2\pi i z} - 1}\, dz - \int_{-\infty+i}^{\infty+i} \frac{e^{-\pi z^2 t}}{e^{2\pi i z} - 1}\, dz$$

for the function $\vartheta(t)$.

(b) In this representation, expand the factor $(e^{2\pi i z} - 1)^{-1}$ as a geometric series in $e^{-2\pi i z}$ (for the first integral) and as a geometric series in $e^{2\pi i z}$ (for the second integral). Evaluate the resulting infinite series, rigorously justifying all steps, to obtain an alternative proof of the functional equation for $\vartheta(t)$.