Math 205A: Complex Analysis, Winter 2018

Homework Problem Set #8

March 7, 2018

1. (a) Reprove the "toy Riemann hypothesis" — the theorem that the Riemann zeta function has no zeros on the line $\mathrm{Re}(s) = 1$ by considering the behavior of

$$Y = \mathrm{Re}\left[-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} - \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \right]$$

for $t \in \mathbb{R} \setminus \{0\}$ fixed and $\sigma \searrow 1$, instead of the quantity

$$X = \log |\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)|.$$

Use the series expansion

$$-\frac{\zeta(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s},$$

where $\Lambda(n)$ is von Mangoldt's function (equal to $\log p$ if $n = p^k$ is a prime power, and 0 otherwise).

(b) Try to reprove the same theorem in yet a third way by considering

$$Z = \log |\zeta(\sigma)^{10} \zeta(\sigma+it)^{15} \zeta(\sigma+2it)^6 \zeta(\sigma+3it)|,$$

and attempting to repeat the argument involving expanding the logarithm in a power series and deducing that $Z \geq 0$. Does this give a proof of the theorem? If not, what goes wrong?

Hint. $(a+b)^6 = a^6 + 6a^5b + 10a^4b^2 + 15a^3b^3 + 10a^2b^4 + 6ab^5 + b^6$.

2. Define arithmetic functions taking an integer argument n, as follows:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise,} \end{cases}$$
(the Möbius μ-function),

$$d(n) = \sum_{d|n} 1, \qquad \text{(the number of divisors function),}$$

$$\sigma(n) = \sum_{d|n} d, \qquad \text{(the sum of divisors function),}$$

$$\phi(n) = \#\{1 \le k \le n-1 \,:\, \gcd(k,n) = 1\}, \quad \text{(the Euler totient function),}$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k,\ p \text{ prime,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(the von Mangoldt } \Lambda\text{-function).}$$

We saw that the zeta function and its logarithmic derivative have the Dirichlet series representations

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$
$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Use the Euler product formula for the zeta function or other elementary manipulations to prove the following identities (valid for $\mathrm{Re}(s) > 1$):

$$\zeta'(s) = -\sum_{n=1}^{\infty} \log n \cdot n^{-s},$$
$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s},$$
$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} |\mu(n)| n^{-s}.$$

Other famous Dirichlet series representations you may want to think about or look up are

$$\zeta(s)^2 = \sum_{n=1}^{\infty} d(n) n^{-s},$$
$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \phi(n) n^{-s},$$
$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \sigma(n) n^{-s}.$$

3. Solve exercises 15–16 on Chapter 6, pages 178–179 of [Stein-Shakarchi].

4. As you know, the final exam is approaching. This week is a good time to start systematically reviewing the course material (including past homework problem sets) and looking for and resolving any gaps in your understanding. Consider this one of your homework assignments for the week.