Complex Analysis Lecture Notes — Additional Material

Dan Romik

March 8, 2020

Solution to problem 24

24. **Characterizing some important families of holomorphic functions on $\mathbb{C}$ and $\widehat{\mathbb{C}}$.** Denote

$$\begin{aligned}
\widehat{\mathbb{C}} &= \mathbb{C} \cup \{\infty\} = \text{the Riemann sphere}, \\
\mathcal{K} &= \text{the set of constant functions } z \mapsto c \in \mathbb{C}, \\
\mathcal{L} &= \text{the set of linear functions } z \mapsto az + b, \quad a, b \in \mathbb{C}, \\
\mathcal{P} &= \text{the set of complex polynomials } z \mapsto \sum_{k=0}^{n} a_k z^k, \\
\mathcal{R} &= \text{the set of rational functions } z \mapsto \frac{p(z)}{q(z)}, \quad p, q \in \mathcal{P}, \\
\mathcal{M} &= \text{the set of Möbius transformations } z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}.
\end{aligned}$$

Note the containment relations $\mathcal{K} \subset \mathcal{L} \subset \mathcal{P} \subset \mathcal{R} \supset \mathcal{M} \supset \mathcal{L} \supset \mathcal{K}$.

(a) **Claim.** An entire function has a removable singularity at ∞ if and only if it is constant.

Proof. If $f(z) = c$ (a constant function) then $f(1/z) = c$ for $z \neq 0$, so $f(1/z)$ has a removable singularity at $z = 0$, therefore by definition $f(z)$ has a removable singularity at ∞.

Conversely, if f is entire and has a removable singularity at ∞, then $g(z) = f(1/z)$ has a removable singularity at $z = 0$. In particular, $g(z)$ is bounded near $z = 0$, i.e., satisfies $|g(z)| < M$ for some constant $M > 0$, for all z in some punctured neighborhood $\{|z| < r\} \setminus \{0\}$ of 0. That implies that $|f(z)| = |g(1/z)| < M$ for z satisfying $|z| > 1/r$. On the other hand, f is also bounded in the closed disc $|z| \leq 1/r$ since it is continuous there. It follows that f is bounded on all of $\mathbb{C}$. Since it is a bounded entire function, it is constant by Liouville's theorem. $\square$

(b) **Claim.** The set of entire functions $f : \mathbb{C} \to \mathbb{C}$ that have a nonessential singularity at ∞ is $\mathcal{P}$, the polynomials.

Proof. If $f(z) = a_n z^n + \ldots + a_1 z + a_0$ is a polynomial of degree n, then we can write $f(1/z) = g(z)/z^n$, where

$$g(z) = z^n f(1/z) = a_0 z^n + a_1 z^{n-1} \ldots + a_n.$$

Since $g(z)$ is analytic at 0 and nonzero ($a_n \neq 0$ since we assumed f is a polynomial of degree n), by definition, $f(1/z) = g(z)/z^n$ has a pole of order n at 0, and therefore, again by definition, $f(z)$ has a pole of order n at ∞. In the case $n = 0$, this is a removable singularity. In general, we showed that polynomials are entire functions that have a nonessential singularity at ∞.

For the converse, assume that f is entire and has a pole of order n at ∞. We will show that f is a polynomial of degree n. Working from the definitions again, this means that $f(1/z)$ has a pole of order n at 0, and so can be expressed in some punctured neighborhood $\mathcal{N}_0 \setminus \{0\}$ of 0, with $\mathcal{N}_0 = \{|z| < r_0\}$, in the form

$$f(1/z) = \frac{g(z)}{z^n},$$

where $g(z)$ is analytic in $\mathcal{N}_0$ and satisfies $g(0) \neq 0$. On the other hand, inspired by this relationship between $f(1/z)$ and $g(z)$, we can define a function $h(z)$ by

$$h(z) = z^n f(1/z),$$

which is of course identical to $g(z)$ on the punctured neighborhood $\mathcal{N}_0 \setminus \{0\}$, but is in fact analytic on the larger region $\mathbb{C} \setminus \{0\}$. It also has a removable singularity at $z = 0$, since $g(z)$ is analytic at 0. So really after defining the value of $h(z)$ at 0 to be $g(0)$ we can say that we have found an entire function $h(z)$ that satisfies

$$f(z) = z^n h(1/z)$$

for all $z \in \mathbb{C} \setminus \{0\}$. In particular, since $|h(z)|$ is bounded by some constant $A > 0$ on the closed unit disc $\{|z| \leq 1\}$, it follows that $f(z)$ satisfies

$$|f(z)| \leq A|z|^n$$

for $|z| \geq 1$. Also, since f is entire, $|f|$ is bounded on the closed unit disc $\{|z| \leq 1\}$, that is, we have

$$|f(z)| \leq B \qquad (|z| \leq 1)$$

for some constant $B > 0$. Combining the above two inequalities, we see that $f(z)$ satisfies

$$|f(z)| \leq A|z|^n + B$$

for all $z \in \mathbb{C}$. We now appeal to a previous homework problem in which we proved that an entire function satisfying this inequality must be a polynomial of degree at most n. This finishes the proof. □

(c) **Claim.** The set of meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ that have a nonessential singularity at ∞ is $\mathcal{R}$, the rational functions.

Proof. If $f(z)$ is a rational function, write it in the form $f(z) = cp(z)/q(z)$, where $c \neq 0$ and $q(z)$ are monic polynomials with respective degrees $m = \deg p$, $n = \deg q$. Then the possibilities for the behavior of $f(z)$ at ∞ are:

1. If $m = n$ then it is easy to see that $\lim_{z\to\infty} f(z) = c$. In particular, $f(1/z)$ is bounded in a neighborhood of 0 and hence has a removable singularity by the Riemann removable singularity theorem. So $f(z)$ has a removable singularity at ∞. (Note: one can also show this directly through a short calculation without appealing to the Riemann removable singularity theorem.)

2. If $m > n$ then similarly one can show that $\lim_{z\to\infty} f(z)/z^{m-n} = c$, and therefore that $f(z)$ has a pole of order $m - n$ at ∞ by similar reasoning as above.

3. If $m < n$ then similarly one can show that $\lim_{z\to\infty} z^{n-m} f(z) = c$, and therefore that $f(z)$ has a zero of order $n - m$ at ∞ by similar reasoning.

Thus, in each of the three cases, $f(z)$ is a meromorphic function with a nonessential singularity at ∞.

Now let us prove the converse statement: assume that $f(z)$ is a meromorphic function with a nonessential singularity at ∞. The goal is to show that it is a rational function.

We start by showing that $f(z)$ has only finitely many poles in $\mathbb{C}$. Let $n \geq 0$ be the order of the pole f has at ∞. By the usual logic (I skip some of the argumentation involving the coordinate change $z \mapsto 1/z$, which is becoming obvious at this point) the function $f(z)/z^n$ has a removable singularity, and hence is bounded in modulus, in a neighborhood $\{|z| > R_0\}$ of the point at infinity. Therefore all poles of $f(z)$ in $\mathbb{C}$ must be in the disc $\{|z| \leq R_0\}$. But poles are isolated points, so in any compact set there can be only finitely many of them.

Having shown there are only finitely many poles, denote the poles of $f(z)$ by $z_1, \ldots, z_m$ and their respective degrees by $d_1, \ldots, d_m$. The function

$$g(z) = f(z) \cdot \prod_{k=1}^{m} (z - z_k)^{d_k}$$

is meromorphic, has removable singularities at $z_1, \ldots, z_m$ and is analytic everywhere else, and by an easy calculation is seen to still have a nonessential singularity at ∞ (a pole of order $n + \sum_{k=1}^{m} d_k$, to be precise). By the claim of part (b), $g(z)$ (after redefining its values at $z_1, \ldots, z_m$ to remove the singularities, turning it into an entire function) is a

polynomial. Therefore
$$f(z) = \frac{g(z)}{\prod_{k=1}^m (z - z_k)^{d_k}}$$
is a rational function, as claimed. □

(d) **Claim.** The set of meromorphic, one-to-one and onto functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is $\mathcal{M} \setminus \mathcal{K}$, the set of nonconstant Möbius transformations.

Proof. If $f(z) = \frac{az+b}{cz+d}$ is a nonconstant Möbius transformation (so $ad - bc \neq 0$), then clearly $f(z)$ is meromorphic, and it is one-to-one and onto since the function $g(z) = \frac{dz-b}{-cz+a}$ satisfies
$$f(g(z)) = z, \qquad g(f(z)) = z$$
(check this by direct computation for generic z, and check it separately if z is one of the special values $0, \infty, f(0) = \frac{b}{d}, f(\infty) = \frac{a}{c}, g(0) = -\frac{b}{a}, g(\infty) = -\frac{d}{c}$).

We now turn to the converse claim. Let $f(z)$ be a meromorphic, one-to-one and onto function on $\widehat{C}$. Clearly f is nonconstant. By the assumption, $f(z)$ has a nonessential singularity at ∞, so by the result proved above, $f(z)$ is a rational function, i.e., we can write it as before as $f(z) = cp(z)/q(z)$, where $c \neq 0$ and $p(z)$ $q(z)$ are monic polynomials with respective degrees $m = \deg p$, $n = \deg q$.

The goal is now to show that only the case when $\max(m, n) = 1$ — that is, the case of a Möbius transformation — is compatible with the assumption that f is one-to-one and onto. To show this, assume by contradiction that at least one of m, n is greater than 1. The injectiveness requires that for any $w_0 \in \mathbb{C}$ the equation
$$f(z) = w_0$$
has precisely one solution in z. But now, imagine that w_0 is of the form $w_0 = f(z_0)$ for some complex number z_0 (left unspecified momentarily — we will say below what properties we require it to have in order for this approach to work). We rewrite the equation $f(z) = w_0$ as
$$g_{z_0}(z) := cp(z) - w_0 q(z) = cp(z) - c\frac{p(z_0)}{q(z_0)} q(z) = 0 \qquad \Longleftrightarrow \qquad p(z) - \frac{p(z_0)}{q(z_0)} q(z) = 0.$$
But now observe that for this equation to have only one solution seems like a difficult thing to achieve under our assumption that $\max(m, n) > 1$. The reason for this is because, we now claim, for a "generic" value of z_0, this will be a polynomial equation in z of degree $K := \max(m, n) > 1$ with simple roots, which therefore must have K solutions. To make precise the meaning of "generic," first, require that $q(z_0) \neq 0$ to avoid division by zero in the above equation — that's not a problem; we can easily avoid the bad situation where $q(z_0) = 0$ by excluding a finite number of possible values for z_0. Second, note that the degree of $g_{z_0}(z)$ as a polynomial in z is indeed $K = \max(m, n)$ for all values of w_0,

except possibly one value (which is a situation that arises only in the case $m = n$, where one can cause cancellation of the leading coefficients of p and q, lowering the degree in the definition of g_{z_0}), and that one value of w_0 is of the form $f(z_0)$ for a unique number z_0, by the injectivity assumption. So again, excluding that value of z_0 still leaves an uncountably infinite number of allowed values for choosing z_0.

Third, for a given choice of z_0 as described above, the equation $g_{z_0}(z) = 0$ will have $K > 1$ solutions in z, *counting multiplicities*, with one of the solutions being $z = z_0$. If $z = z_0$ were actually a simple zero of $g_{z_0}(z)$, then we would reach a contradiction, since that would mean there has to be at least one other (simple or otherwise) zero, which would contradict the injectiveness assumption. And this situation of a simple zero would happen precisely if

$$g'_{z_0}(z_0) \neq 0 \qquad \Longleftrightarrow \qquad cp'(z_0) - \frac{cp(z_0)}{q(z_0)} q'(z_0) \neq 0,$$

or, equivalently, if

$$p'(z_0)q(z_0) - p(z_0)q'(z_0) \neq 0.$$

This is again a polynomial inequality in z_0: it is satisfied for all but finitely many values of z_0, so in particular for *some* value of z_0 (which is all we care about).

To summarize this somewhat strange chain of logical reasoning, we showed that the equation $f(z) = w_0 = f(z_0)$ will have more than one solution (as an equation in z) for any choice of z_0 for which the following conditions are satisfied:

a. $\deg g_{z_0} = K$;
b. $q(z_0) \neq 0$;
c. $p'(z_0)q(z_0) - p(z_0)q'(z_0) \neq 0$.

And we showed that there exists at least one value z_0 for which these conditions are satisfied — in fact, all but finitely many values of z_0 have the required properties. This contradicts the injectivity assumption, and therefore finishes the proof that f is a Möbius transformation. □

Remark. An alternative proof of the fact that f is a Möbius transformation if it is an injective, onto rational function uses injectivity to infer first of all that $f(z)$ must be of the form

$$f(z) = c\frac{(z - z_0)^m}{(z - z_1)^n}$$

for some integers $m, n \geq 0$: indeed, this follows from the fact that, after ensuring that the fraction $p(z)/q(z)$ is in reduced form, that is, $p(z)$ and $q(z)$ don't share any roots, $p(z)$ must have at most one root (*not* counting multiplicities in this case), since $f(z) = 0$ has at most one solution, and similarly $q(z)$ must have at most one root, not counting multiplicities, since $f(z) = \infty$ must have at most one solution.

Once the fact that $f(z)$ is of the concrete form $c(z-z_0)^m/(z-z_1)^n$ has been established, the problem becomes conceptually a bit easier. It is then a bit tedious, but not particularly difficult, to prove that $\max(m,n)=1$ as before (for example, by considering each of the subcases $m=n$, $m<n$ and $m>n$ separately).

(e) **Claim.** The set of entire functions $f:\mathbb{C}\to\mathbb{C}$ that are one-to-one and onto is precisely $\mathcal{L}\setminus\mathcal{K}$, the set of nonconstant linear functions.

Proof. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire, one-to-one and onto function.[1] If f has a nonessential singularity at ∞, by part (b) it is a polynomial, and since it is injective it must be a nonconstant linear function (this is a special case of the subclaim proved in (d) above for rational functions, but it is also easy to prove directly that all polynomials of degree ≥ 2 are not injective without going through that claim), and we are done.

It remains to consider the situation when f has an essential singularity at ∞. Let $\mathbb{D}$ denote the open unit disc. Denote $w_0=f(0)$. Since f is an open mapping by the open mapping theorem, the image $f(\mathbb{D})$ of $\mathbb{D}$ under f contains an open neighborhood E of w_0. But by the Casorati-Weierstrass theorem, the image $f(\mathbb{C}\setminus\overline{D_R(0)})$ of the complement of any closed disc around 0 (i.e., any neighborhood of ∞) is dense in $\mathbb{C}$, and therefore has a nonempty intersection with E. This intersection means the existence of points $z_1\in\mathbb{D}$ and $z_2\in\mathbb{C}\setminus D_R(0)$ for which

$$f(z_1)=f(z_2).$$

Of course, if $R>1$ then $z_1\neq z_2$, in which case we get a contradiction to the assumption that f was injective. Thus, the situation when f is injective but has an essential singularity at ∞ is impossible, and the proof is finished. □

Remarks. Given a region $\Omega\subset\mathbb{C}$, or more generally a Riemann surface Σ, complex analysts are interested in understanding the structure of its set of holomorphic functions ($\mathbb{C}$-valued holomorphic functions on Σ); its set of meromorphic functions ($\widehat{\mathbb{C}}$-valued holomorphic functions on Σ); and its set of holomorphic automorphisms (holomorphic, one-to-one and onto mappings from Σ to itself). Although we won't get into the general theory of Riemann surfaces, once one defines these concepts it easy to see that the above exercises essentially prove the following conceptually important results:

(i) The constant functions are the only holomorphic functions on $\widehat{\mathbb{C}}$.

(ii) The rational functions are the meromorphic functions on $\widehat{\mathbb{C}}$.

(iii) The nonconstant linear functions are the holomorphic automorphisms of $\mathbb{C}$.

(iv) The nonconstant Möbius transformations are the holomorphic automorphisms of $\widehat{\mathbb{C}}$.

Another related result that is not very difficult to prove is:

[1] Actually this proof does not use the assumption that f is onto: the assumption that f is entire and one-to-one is sufficient to imply the conclusion.

(v) The holomorphic automorphisms of the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ are the Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. (Try to prove that any such map is indeed an automorphism of $\mathbb{H}$; the reverse implication that all automorphisms of $\mathbb{H}$ are of this form is a bit more difficult and requires a result known as the Schwarz lemma, which will likely be covered at some point in MAT205A/B.)

Note that the set of holomorphic functions on $\mathbb{C}$ (a.k.a. entire functions) and the set of meromorphic functions on $\mathbb{C}$ are much larger families of functions that do not have such a simple description as the functions in the relatively small families $\mathcal{L}, \mathcal{P}, \mathcal{R}, \mathcal{M}$. This is related to the fact that $\mathbb{C}$ is a non-compact Riemann surface.