MAT 21B — Solutions to Midterm 2

Dan Romik

Question 1

A mechanical engineer is evaluating two springs made by competing manufacturers for possible use in a sliding tray mechanism for a DVD player her company is designing.

The first spring obeys Hooke's law for the spring force,

$$F = kx$$

where x is the compression of the spring relative to its natural length, and the spring constant is $k = 0.02\ \frac{\text{N}}{\text{cm}}$. (N=Newtons; cm=centimeters).

The second spring is a "nonlinear spring" whose restoring force equation is

$$F = cx^{3/2}$$

(this equation is known as the *Hertz elastic contact law*), with the constant c equal to $c = 0.015\ \frac{\text{N}}{\text{cm}^{3/2}}$.

Calculate the work needed to compress each of the springs by 4 centimeters from their natural lengths.

Solution. The work needed to compress the first spring is

$$W_1 = \int_0^4 F(x)\,dx = \int_0^4 kx\,dx = \frac{1}{2}kx^2\Big|_0^4 = \frac{1}{2} \times 0.02 \times 4^2 = 0.16.$$

(The units are Newton-centimeter = one hundredths of Joules, so in Joules W_1 is equal to 0.0016.) For the second spring, the work is

$$W_2 = \int_0^4 F(x)\,dx = \int_0^4 cx^{3/2}\,dx = \frac{2}{5}cx^{5/2}\Big|_0^4 = \frac{2}{5} \times 0.015 \times 4^{5/2} = \frac{64}{5} \times 0.015 = 0.192,$$

again in units of Newton-centimeter.

Question 2

Use integration to find the volume and surface area of the solid body which is the solid of revolution (whose technical name is a *conical frustum*) formed by revolving the straight line connecting the points $(0, 2)$ and $(5, 1)$ in the plane around the x-axis.

Solution. The equation for the straight line is $y = 2 - x/5$. The volume is computed as

$$V = \int_0^5 \pi y(x)^2 \, dx = \pi \int_0^5 \left(4 - \frac{4}{5}x + \frac{1}{25}x^2\right) dx$$
$$= \pi \left(4 \times 5 - \frac{4}{5} \times \frac{1}{2} \times 5^2 + \frac{1}{25} \times \frac{1}{3} \times 5^3\right) = \frac{35\pi}{3}.$$

The arc length element is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx = \frac{1}{5}\sqrt{26}\, dx.$$

The surface area element is

$$dS = 2\pi y \, ds = 2\pi(2 - x/5) \times \frac{1}{5}\sqrt{26}\, dx.$$

The surface area is therefore

$$S = \int_0^5 \frac{2}{5}\pi\sqrt{26}(2 - x/5)\, dx = \frac{2}{5}\pi\sqrt{26}\left(2 \cdot 5 - \frac{1}{2} \cdot \frac{1}{5} \cdot 5^2\right) = 3\pi\sqrt{26}.$$

Question 3

Find the center of mass of a thin plate of material shaped like the top half of the ellipse

$$x^2 + 4y^2 = 1$$

(that is, the part of the ellipse that lies above the x-axis) if the material has uniform density $\delta = 1$.

Solution. The semi-ellipse is bounded between the x-axis and the graph of the function

$$y(x) = \frac{1}{2}\sqrt{1-x^2} \qquad (-1 \le x \le 1).$$

The area of the semi-ellipse is $\pi/4$, which is the same as the total mass when the density is 1:

$$M = \frac{\pi}{4}.$$

Next, we compute the moment M_y and from it the x-component $\bar{x}$ of the center of mass. We use the method of vertical strips, and observe that $\tilde{x} = x$, which means that M_y is given by the integral

$$M_y = \int_{-1}^{1} \tilde{x} \cdot \frac{1}{2}\sqrt{1-x^2}\,dx = \int_{-1}^{1} x \cdot \frac{1}{2}\sqrt{1-x^2}\,dx = 0,$$

since the function being integrated is an odd function of x. The x-coordinate of the center of mass is therefore

$$\bar{x} = \frac{M_y}{M} = \frac{2}{\pi} \cdot 0 = 0,$$

a fact that is also obvious from the symmetry of the half-ellipse relative to the y-axis.

Now we compute the numbers M_x and $\bar{y}$. The average y-value $\tilde{y}$ on each vertical strip at coordinate x is half the height of the semi-ellipse at that strip:

$$\tilde{y} = \frac{1}{4}\sqrt{1-x^2}.$$

Therefore M_x can be calculated as

$$M_x = \int_{-1}^{1} \tilde{y} \cdot \frac{1}{2}\sqrt{1-x^2}\,dx = \int_{-1}^{1} \frac{1}{8}(1-x^2)\,dx = \frac{1}{8} \times 2\int_{0}^{1} (1-x^2)\,dx = \frac{1}{4} \times \left(1 - \frac{1}{3}\right) = \frac{1}{6}.$$

The y-coordinate of the center of mass is therefore

$$\bar{y} = \frac{M_x}{M} = \frac{2}{3\pi}.$$

To summarize, the center of mass of the semi-ellipse is

$$(\bar{x}, \bar{y}) = \left(0, \frac{2}{3\pi}\right).$$

Question 4

The density of air in the Earth's atmosphere as a function of altitude follows an exponential decay law of the form

$$d(y) = \delta_0 e^{-ky},$$

where

y	is the altitude above sea level, measured in meters,
$d(y)$	is the density of air at altitude y, measured in kilograms per cubic meter,
k	is a constant, with units of meters^{-1},
$\delta_0 = 1.225 \ \frac{\text{kg}}{\text{m}^3}$	is the density of air at sea level.

It is known that the density at an altitude of 1000 meters is 88% of its value at sea level.

(a) Find the value of k.

Solution. We have the equation

$$0.88 = \frac{y(1000)}{y(0)} = \frac{\delta_0 e^{-1000k}}{\delta_0 e^0} = e^{-1000k},$$

which implies that

$$k = -\frac{1}{1000}\ln(0.88) = 0.000127833 \quad \text{(approximately)},$$

in units of m^{-1}.

(b) The *Karman line* is an altitude above the Earth's surface above which "outer space" officially begins. It is known that at the Karman line, the density of air is

$$\delta_{\text{Karman}} = 3.43877 \times 10^{-6} \ \frac{\text{kg}}{\text{m}^3}.$$

Find the altitude y_{Karman} where outer space begins according to this definition, in meters.

Solution. The equation for y_{Karman} is

$$d(y_{\text{Karman}}) = \delta_0 e^{-k\, y_{\text{Karman}}} = \delta_{\text{Karman}}.$$

Solving for y_{Karman} gives

$$y_{\text{Karman}} = -\frac{1}{k}\ln\left(\frac{\delta_{\text{Karman}}}{\delta_0}\right) = -\frac{1}{0.000127833}\ln\left(\frac{3.43877 \times 10^{-6}}{1.225}\right) = 100000 \ \text{(meters)}.$$

Thus, according to the Karman line definition, outer space begins at an altitude of 100 kilometers above sea level, or around 62 miles.