1 Write the sum $1 + 3 + 5 + 7 + \ldots + 97 + 99$ in $\sum$ notation, and evaluate it. You may use the formula $\sum_{m=1}^{N} m = N(N + 1)/2$.

**Solution.** The sum in question has 50 terms, where the $k$th term is equal to $2k - 1$, so, expressing the sum in $\sum$ notation and then manipulating it using the sum rule and constant multiple rule, and finally making use of the formula mentioned in the question, we see that

$$\sum_{k=1}^{50} (2k - 1) = 2 \sum_{k=1}^{50} k - \sum_{k=1}^{50} 1 = 2 \times 50 \times 51/2 - 50 \times 1 = 50^2 = 2500.$$

2 (a) Evaluate the definite integral $A = \int_{1}^{10} \frac{1}{x} \, dx$. (An answer expressed in terms of values of standard mathematical functions is acceptable.)

**Solution.** $A = \int_{1}^{10} \frac{1}{x} \, dx = \ln|x| \bigg|_{1}^{10} = \ln(10) - \ln(1) = \ln(10).$

(b) Define $B = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10}$. Without using a calculator, determine which of $A$ and $B$ is larger, and explain how you can tell.

**Solution.** $B$ is an upper Riemann sum associated with an approximation to the integral defining $A$, as shown in Figure 1 below. This tells us that $B > A$.

(c) Define $C = B - 1 + \frac{1}{10}$. Without using a calculator, determine which of $A$ and $C$ is larger, and explain how you can tell.

**Solution.** $C$ is a lower Riemann sum associated with an approximation to the integral defining $A$, as shown in the figure. That implies that $C < A$. 
(a) $A$ is the area bounded between the graph of $y = \frac{1}{x}$ and the $x$-axis above the interval $[1, 10]$.

(b) $B$ is the upper sum associated with an approximation to the integral defining $A$, corresponding to the sum of the areas of the shaded rectangles.

(c) $C$ is the lower sum associated with an approximation to the integral defining $A$, corresponding to the sum of the areas of the shaded rectangles.

Figure 1: Illustration of the numbers $A$, $B$ and $C$ from question 2.
Compute the indefinite integral \( \int 10 \cos(x) \sin^4(x) \, dx \).

**Solution.** Make the substitution \( u = \sin(x) \). We then have that \( du = \cos(x) \, dx \), so that

\[
\int 10 \cos(x) \sin^4(x) \, dx = \int 10u^4 \, du = 2u^5 + C = 2 \sin(x)^5 + C.
\]

As usual, the answer can be verified by differentiating \( 2 \sin(x)^5 + C \).

(b) Find an antiderivative for \( f(x) = \frac{1}{(x-2)^3} \).

**Solution.** With the substitution \( u = x - 2 \) this reduces to the problem of finding an antiderivative for \( \frac{1}{u^3} \). One correct answer is \( -\frac{1}{3u^3} = -\frac{1}{3(x-2)^3} \).

(c) Find all antiderivatives \( F(x) \) for \( f(x) = 2 \sin x \cos x \) that satisfy \( F(0) = 3 \).

**Solution.** Apply the substitution method with \( u = \sin x \), then \( du = \cos x \, dx \), and the general form for an antiderivative of \( f(x) \) is

\[
F(x) = \int 2 \sin x \cos x \, dx = \int 2u \, du = u^2 + C = \sin^2 x + C,
\]

with \( C \) an integration constant. The condition \( F(0) = 3 \) gives the equation \( C = 3 \), which determines the value of the constant. Thus we have shown that \( F(x) = \sin^2 x + 3 \) is the unique antiderivative of \( f(x) \) satisfying the required condition.

Use the midpoint rule and a subdivision of the interval \([-1, 2]\) into 3 subintervals of equal lengths to compute a numerical approximation to the definite integral \( \int_{-1}^{2} |x| \, dx \).

**Solution.** Denoting \( f(x) = |x| \), the numerical approximation associated with the midpoint rule and the specified interval subdivision is

\[
1 \cdot f(-1/2) + 1 \cdot f(1/2) + 1 \cdot f(3/2) = |-1/2| + |1/2| + |3/2| = 1/2 + 1/2 + 3/2 = 5/2.
\]

**Remark.** Incidentally, the number \( 5/2 \) is actually the correct (precise, not approximate) value for \( \int_{-1}^{2} |x| \, dx \) (exercise: check this). There is a geometric explanation for why the midpoint rule gives the correct answer in this particular case — try to draw a picture and understand this phenomenon.

The temperature in Davis on the first week of January 2018 was measured to be (in Fahrenheit degrees) approximately

\[ f(t) = 53 + 10 \sin(2\pi t), \quad 1 \leq t \leq 8, \]

where we model the week as the interval \([1, 8]\), with the subinterval \([1, 2]\) corresponding to Monday, \([2, 3]\) corresponding to Tuesday, etc. What was the average temperature in Davis that week?
Solution. The average temperature is

$$\frac{1}{8 - 1} \int_1^8 f(t) \, dt = \frac{1}{7} \int_1^8 \left( 53 + 10 \sin(2\pi t) \right) \, dt$$

$$= \frac{1}{7} \int_1^8 53 \, dt + \frac{10}{7} \int_1^8 \sin(2\pi t) \, dt$$

$$= \frac{1}{7} \times 7 \cdot 53 - \frac{10}{7} \times \frac{1}{2\pi} \cos(2\pi t) \bigg|_1^8$$

$$= 53 - \frac{5}{7\pi} (\cos(16\pi) - \cos(2\pi)) = 53.$$

Note that the answer makes intuitive sense: the temperature function is the sum of a constant (53) and a constant multiple of a periodic wave function \(\sin(2\pi t)\) with a period of one day. When averaging the function over a period of several days, the periodic component averages out to zero, so we are left with the constant value 53 as the average temperature.

6. According to data published on the website zeroto60times.com, a 2017 Tesla Model S 60 car accelerates from 0 to 60 mph in 5 seconds. (Note that a speed of 60 mph is equivalent to 88 feet per second.) Assume that the acceleration is uniform over that 5-second interval. How many feet has the car traveled at the moment it hits 60 mph?

Solution. Let \(u(t)\) denote the distance the car has traveled after \(t\) seconds, let \(v(t)\) denote the velocity, and let \(a(t)\) denote the acceleration. Then we have the relations

\[ v(t) = u'(t) \quad \text{(velocity is the rate of change of the position)}, \]
\[ a(t) = v'(t) \quad \text{(acceleration is the rate of change of the velocity)}. \]

Said differently, \(v(t)\) is the antiderivative of \(a(t)\), and \(u(t)\) is the antiderivative of \(v(t)\). We also have

\[ a(t) = \frac{88}{5} = 17.6, \]

by the assumption that the car accelerates uniformly by 88 feet per second over a time interval of 5 seconds. This implies that

\[ v(t) = \frac{88}{5} t + C = 17.6t + C, \]

where \(C\) is an integration constant, but since \(v(0) = 0\) (the car starts accelerating from a speed of 0 mph), the constant is equal to 0, so \(v(t) = 17.6t\). Now, integrating again to find the antiderivative \(u(t)\), we get

\[ u(t) = \frac{44}{5} t^2 + D = 8.8t^2 + D, \]

where \(D\) is another integration constant. Since \(u(0) = 0\), again the constant \(D\) is equal to 0, so

\[ u(t) = \frac{44}{5} t^2. \]
Finally, setting \( t = 5 \) gives
\[
u(5) = \frac{44}{5} \times 5^2 = 5 \times 44 = 220.
\]
That is, the car has traveled 220 feet at the moment its velocity hits 60 mph.

7 If \( f(x) \) is integrable and \( \int_{0}^{4} f(x) \, dx = 10, \int_{2}^{4} f(x) \, dx = 6, \) what is \( \int_{0}^{1} f(2x) \, dx? \)

**Solution.** Make the substitution \( u = 2x \) in the integral \( \int_{0}^{1} f(2x) \, dx. \) This gives \( du = 2 \, dx, \) and so, by the substitution rule,
\[
\int_{0}^{1} f(2x) \, dx = \int_{0}^{2} f(u) \frac{1}{2} du = \frac{1}{2} \int_{0}^{2} f(u) \, du = \frac{1}{2} \left( \int_{0}^{4} f(x) \, dx - \int_{2}^{4} f(x) \, dx \right) = \frac{1}{2} (10 - 6) = 2.
\]

8 (a) Write a Riemann sum \( S_n \) with \( n \) summands for \( f(x) = x - 1 \) in the interval \([2, 6]\).

**Solution.**
\[
S_n = \sum_{k=1}^{n} f\left(2 + (6 - 2) \frac{k}{n}\right) \times \frac{6 - 2}{n} = \sum_{k=1}^{n} f\left(2 + \frac{4k}{n}\right) \frac{4}{n} = \sum_{k=1}^{n} \left(1 + \frac{4k}{n}\right) \frac{4}{n}
\]
\[
= \sum_{k=1}^{n} \left(\frac{4}{n} + \frac{16k}{n^2}\right)
\]

(b) Calculate the limit of \( S_n \) as \( n \to \infty \) to obtain the value of \( \int_{2}^{6} f(x) \, dx. \) (You may find the formula \( \sum_{m=1}^{N} m = N(N + 1)/2 \) helpful for this purpose.)

**Solution.**
\[
S_n = \sum_{k=1}^{n} \frac{4}{n} + \sum_{k=1}^{n} \frac{16k}{n^2} = n \times \frac{4}{n} + \frac{16}{n^2} \sum_{k=1}^{n} k = 4 + \frac{16}{n^2} \times \frac{n(n + 1)}{2}
\]
\[
= 4 + \frac{8n(n + 1)}{n^2} = 4 + \frac{8n^2}{n^2} + \frac{8n}{n^2} = 12 + \frac{8}{n} \xrightarrow{n \to \infty} 12.
\]
So the value of the integral is 12.
9 (a) Define a function \( f(n) \) of an integer \( n \geq 1 \) by \( f(n) = \sum_{k=1}^{n} (-1)^{k+1} k^2 \). Compute the values \( f(1), f(2), f(3), f(4), f(5), f(6) \).

Solution.

\[
\begin{align*}
  f(1) &= (-1)^2 1^2 = 1, \\
  f(2) &= (-1)^2 1^2 + (-1)^3 2^2 = 1 - 4 = -3, \\
  f(3) &= (-1)^2 1^2 + (-1)^3 2^2 + (-1)^4 3^2 = 1 - 4 + 9 = 6, \\
  f(4) &= (-1)^2 1^2 + (-1)^3 2^2 + (-1)^4 3^2 + (-1)^5 4^2 = 1 - 4 + 9 - 16 = -10, \\
  f(5) &= (-1)^2 1^2 + (-1)^3 2^2 + \ldots + (-1)^6 5^2 = 1 - 4 + 9 - 16 + 25 = 15, \\
  f(6) &= (-1)^2 1^2 + (-1)^3 2^2 + \ldots + (-1)^7 6^2 = 1 - 4 + 9 - 16 + 25 - 36 = -21. 
\end{align*}
\]

(b) Similarly, define two functions \( g(n) \) and \( h(n) \) of an integer \( n \geq 1 \) by

\[
g(n) = \sum_{k=1}^{n} k^2 \quad \text{and} \quad h(n) = \frac{1}{6} n(n+1)(2n+1). \]

Show that

\[
g(n) - g(n - 1) = n^2 \quad \text{and} \quad h(n) - h(n - 1) = n^2.
\]

Solution.

\[
\begin{align*}
g(n) &= 1^2 + 2^2 + 3^2 + \ldots + n^2, \\
  g(n) - g(n - 1) &= (1^2 + 2^2 + \ldots + (n - 1)^2 + n^2) - (1^2 + 2^2 + \ldots + (n - 1)^2) = n^2, \\
  h(n) - h(n - 1) &= \frac{1}{6} n(n+1)(2n+1) - \frac{1}{6} (n-1)(n-1+1)(2(n-1)+1) \\
  &= \frac{1}{6} n \left( (n+1)(2n+1) - (n-1)(2n-1) \right) \\
  &= \frac{1}{6} n \left( 2n^2 + 3n + 1 - 2n^2 - 3n + 1 \right) = \frac{1}{6} n(3n + 3n) = n^2.
\end{align*}
\]

10 Evaluate the following definite integrals:

(a) \( \int_{-1}^{3} \sqrt{1 + x} \, dx \)

Solution.

\[
\int_{-1}^{3} \sqrt{1 + x} \, dx = \frac{2}{3} (1 + x)^{3/2} \bigg|_{-1}^{3} = \frac{2}{3} (4^{3/2} - 0^{3/2}) = \frac{16}{3}.
\]
(b) \[ \int_{0}^{2} e^{x^3} x^2 \, dx \] (a non-numerical answer in terms of \( e \) or other standard mathematical constants or functions is acceptable)

Solution.

\[ \int_{0}^{2} e^{x^3} x^2 \, dx = \frac{1}{3} e^{x^3} \bigg|_{0}^{2} = \frac{1}{3} (e^8 - 1). \]