Solutions to practice problem set for Midterm Exam 1

MAT 21B (UC Davis, Winter 2018)

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1 Write the sum $1+3+5+7+\ldots+97+99$ in $\sum$ notation, and evaluate it. You may use the formula $\sum_{m=1}^{N} m = N(N+1)/2$.

Solution. The sum in question has 50 terms, where the kth term is equal to $2k-1$, so, expressing the sum in $\sum$ notation and then manipulating it using the sum rule and constant multiple rule, and finally making use of the formula mentioned in the question, we see that

$$\begin{aligned} 1+3+5+7+\ldots+97+99 &= \sum_{k=1}^{50}(2k-1) = 2\sum_{k=1}^{50} k - \sum_{k=1}^{50} 1 \\ &= 2 \times 50 \times 51/2 - 50 \times 1 = 50^2 = 2500. \end{aligned}$$

2 (a) Evaluate the definite integral $A = \int_1^{10} \frac{1}{x}\,dx$. (An answer expressed in terms of values of standard mathematical functions is acceptable.)

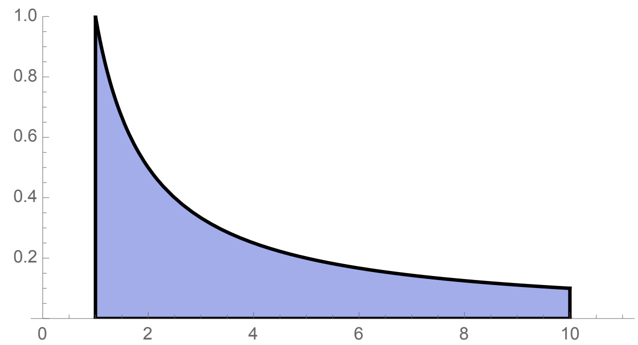
Solution. $A = \int_1^{10} \frac{1}{x}\,dx = \ln|x|\Big|_1^{10} = \ln(10) - \ln(1) = \ln(10).$

(b) Define $B = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{9}$. Without using a calculator, determine which of A and B is larger, and explain how you can tell.

Solution. B is an upper Riemann sum associated with an approximation to the integral defining A, as shown in Figure 1 below. This tells us that $B > A$.

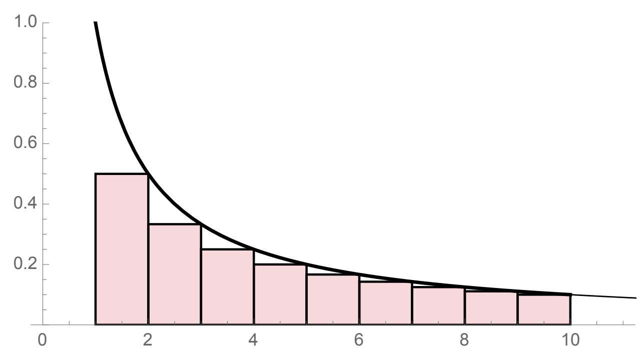
(c) Define $C = B - 1 + \frac{1}{10}$. Without using a calculator, determine which of A and C is larger, and explain how you can tell.

Solution. C is a lower Riemann sum associated with an approximation to the integral defining A, as shown in the figure. That implies that $C < A$.



(a) A is the area bounded between the graph of $y = \frac{1}{x}$ and the x-axis above the interval $[1, 10]$.

(b) B is the upper sum associated with an approximation to the integral defining A, corresponding to the sum of the areas of the shaded rectangles.



(c) C is the lower sum associated with an approximation to the integral defining A, corresponding to the sum of the areas of the shaded rectangles.

Figure 1: Illustration of the numbers A, B and C from question 2.

3 (a) Compute the indefinite integral $\int 10\cos(x)\sin^4(x)\,dx$.

Solution. Make the substitution $u = \sin(x)$. We then have that $du = \cos(x)\,dx$, so that

$$\int 10\cos(x)\sin^4(x)\,dx = \int 10u^4\,du = 2u^5 + C = 2\sin(x)^5 + C.$$

As usual, the answer can be verified by differentiating $2\sin(x)^5 + C$.

(b) Find an antiderivative for $f(x) = \frac{1}{(x-2)^4}$.

Solution. With the substitution $u = x - 2$ this reduces to the problem of finding an antiderivative for $\dfrac{1}{u^4}$. One correct answer is $-\dfrac{1}{3u^3} = -\dfrac{1}{3(x-2)^3}$.

(c) Find *all* antiderivatives $F(x)$ for $f(x) = 2\sin x\cos x$ that satisfy $F(0) = 3$.

Solution. Apply the substitution method with $u = \sin x$, then $du = \cos x\,dx$, and the general form for an antiderivative of $f(x)$ is

$$F(x) = \int 2\sin x\cos x\,dx = \int 2u\,du = u^2 + C = \sin^2 x + C,$$

with C an integration constant. The condition $F(0) = 3$ gives the equation $C = 3$, which determines the value of the constant. Thus we have shown that $F(x) = \sin^2 x + 3$ is the *unique* antiderivative of $f(x)$ satisfying the required condition.

4 Use the midpoint rule and a subdivision of the interval $[-1, 2]$ into 3 subintervals of equal lengths to compute a numerical approximation to the definite integral $\displaystyle\int_{-1}^{2} |x|\,dx$.

Solution. Denoting $f(x) = |x|$, the numerical approximation associated with the midpoint rule and the specified interval subdivision is

$$1 \cdot f(-1/2) + 1 \cdot f(1/2) + 1 \cdot f(3/2) = |-1/2| + |1/2| + |3/2| = 1/2 + 1/2 + 3/2 = 5/2.$$

Remark. Incidentally, the number $5/2$ is actually the correct (precise, not approximate) value for $\int_{-1}^{2} |x|\,dx$ (exercise: check this). There is a geometric explanation for why the midpoint rule gives the correct answer in this particular case — try to draw a picture and understand this phenomenon.

5 The temperature in Davis on the first week of January 2018 was measured to be (in Fahrenheit degrees) approximately

$$f(t) = 53 + 10\sin(2\pi t), \qquad 1 \le t \le 8,$$

where we model the week as the interval $[1, 8]$, with the subinterval $[1, 2]$ corresponding to Monday, $[2, 3]$ corresponding to Tuesday, etc. What was the average temperature in Davis that week?

Solution. The average temperature is

$$\begin{aligned}
\frac{1}{8-1}\int_1^8 f(t)\,dt &= \frac{1}{7}\int_1^8 \Big(53 + 10\sin(2\pi t)\Big)\,dt \\
&= \frac{1}{7}\int_1^8 53\,dt + \frac{10}{7}\int_1^8 \sin(2\pi t)\,dt \\
&= \frac{1}{7}\times 7\cdot 53 - \frac{10}{7}\times\frac{1}{2\pi}\cos(2\pi t)\Big|_1^8 \\
&= 53 - \frac{5}{7\pi}\Big(\cos(16\pi) - \cos(2\pi)\Big) = 53.
\end{aligned}$$

Note that the answer makes intuitive sense: the temperature function is the sum of a constant (53) and a constant multiple of a periodic wave function $\sin(2\pi t)$ with a period of one day. When averaging the function over a period of several days, the periodic component averages out to zero, so we are left with the constant value 53 as the average temperature.

6 According to data published on the website zeroto60times.com, a 2017 Tesla Model S 60 car accelerates from 0 to 60 mph in 5 seconds. (Note that a speed of 60 mph is equivalent to 88 feet per second.) Assume that the acceleration is uniform over that 5-second interval. How many feet has the car traveled at the moment it hits 60 mph?

Solution. Let $u(t)$ denote the distance the car has traveled after t seconds, let $v(t)$ denote the velocity, and let $a(t)$ denote the acceleration. Then we have the relations

$$\begin{aligned}
v(t) &= u'(t) &&\text{(velocity is the rate of change of the position)},\\
a(t) &= v'(t) &&\text{(acceleration is the rate of change of the velocity)}.
\end{aligned}$$

Said differently, $v(t)$ is the antiderivative of $a(t)$, and $u(t)$ is the antiderivative of $v(t)$. We also have

$$a(t) = \frac{88}{5} = 17.6,$$

by the assumption that the car accelerates uniformly by 88 feet per second over a time interval of 5 seconds. This implies that

$$v(t) = \frac{88}{5}t + C = 17.6t + C,$$

where C is an integration constant, but since $v(0) = 0$ (the car starts accelerating from a speed of 0 mph), the constant is equal to 0, so $v(t) = 17.6t$. Now, integrating again to find the antiderivative $u(t)$, we get

$$u(t) = \frac{44}{5}t^2 + D = 8.8t^2 + D,$$

where D is another integration constant. Since $u(0) = 0$, again the constant D is equal to 0, so

$$u(t) = \frac{44}{5}t^2.$$

Finally, setting $t = 5$ gives

$$u(5) = \frac{44}{5} \times 5^2 = 5 \times 44 = 220.$$

That is, the car has traveled 220 feet at the moment its velocity hits 60 mph.

7 If $f(x)$ is integrable and $\int_0^4 f(x)\,dx = 10$, $\int_2^4 f(x)\,dx = 6$, what is $\int_0^1 f(2x)\,dx$?

Solution. Make the substitution $u = 2x$ in the integral $\int_0^1 f(2x)\,dx$. This gives $du = 2\,dx$, and so, by the substitution rule,

$$\begin{aligned}\int_0^1 f(2x)\,dx &= \int_0^2 f(u)\,\tfrac{1}{2}du = \tfrac{1}{2}\int_0^2 f(u)\,du \\ &= \tfrac{1}{2}\left(\int_0^4 f(x)\,dx - \int_2^4 f(x)\,dx\right) = \tfrac{1}{2}(10-6) = 2.\end{aligned}$$

8 (a) Write a Riemann sum S_n with n summands for $f(x) = x - 1$ in the interval $[2, 6]$.

Solution.

$$\begin{aligned}S_n &= \sum_{k=1}^{n} f\left(2 + (6-2)\frac{k}{n}\right) \times \frac{6-2}{n} = \sum_{k=1}^{n} f\left(2 + \frac{4k}{n}\right)\frac{4}{n} = \sum_{k=1}^{n}\left(1 + \frac{4k}{n}\right)\frac{4}{n} \\ &= \sum_{k=1}^{n}\left(\frac{4}{n} + \frac{16k}{n^2}\right)\end{aligned}$$

(b) Calculate the limit of S_n as $n \to \infty$ to obtain the value of $\int_2^6 f(x)\,dx$. (You may find the formula $\sum_{m=1}^{N} m = N(N+1)/2$ helpful for this purpose.)

Solution.

$$\begin{aligned}S_n &= \sum_{k=1}^{n}\frac{4}{n} + \sum_{k=1}^{n}\frac{16k}{n^2} = n \times \frac{4}{n} + \frac{16}{n^2}\sum_{k=1}^{n} k = 4 + \frac{16}{n^2} \times \frac{n(n+1)}{2} \\ &= 4 + \frac{8n(n+1)}{n^2} = 4 + \frac{8n^2}{n^2} + \frac{8n}{n^2} = 12 + \frac{8}{n} \xrightarrow[n\to\infty]{} 12.\end{aligned}$$

So the value of the integral is 12.

9 (a) Define a function $f(n)$ of an integer $n \geq 1$ by $f(n) = \sum_{k=1}^{n} (-1)^{k+1} k^2$. Compute the values $f(1), f(2), f(3), f(4), f(5), f(6)$.

Solution.

$$
\begin{aligned}
f(1) &= (-1)^2 1^2 = 1, \\
f(2) &= (-1)^2 1^2 + (-1)^3 2^2 = 1 - 4 = -3, \\
f(3) &= (-1)^2 1^2 + (-1)^3 2^2 + (-1)^4 3^2 = 1 - 4 + 9 = 6, \\
f(4) &= (-1)^2 1^2 + (-1)^3 2^2 + (-1)^4 3^2 + (-1)^5 4^2 = 1 - 4 + 9 - 16 = -10, \\
f(5) &= (-1)^2 1^2 + (-1)^3 2^2 + \ldots + (-1)^6 5^2 = 1 - 4 + 9 - 16 + 25 = 15, \\
f(6) &= (-1)^2 1^2 + (-1)^3 2^2 + \ldots + (-1)^7 6^2 = 1 - 4 + 9 - 16 + 25 - 36 = -21.
\end{aligned}
$$

(b) Similarly, define two functions $g(n)$ and $h(n)$ of an integer $n \geq 1$ by $g(n) = \sum_{k=1}^{n} k^2$ and $h(n) = \frac{1}{6} n(n+1)(2n+1)$. Show that

$$
g(n) - g(n-1) = n^2 \qquad \text{and} \qquad h(n) - h(n-1) = n^2.
$$

Solution.

$$
\begin{aligned}
g(n) &= 1^2 + 2^2 + 3^2 + \ldots + n^2, \\
g(n) - g(n-1) &= \big(1^2 + 2^2 + \ldots + (n-1)^2 + n^2\big) - \big(1^2 + 2^2 + \ldots + (n-1)^2\big) = n^2, \\
h(n) - h(n-1) &= \frac{1}{6} n(n+1)(2n+1) - \frac{1}{6}(n-1)(n-1+1)(2(n-1)+1) \\
&= \frac{1}{6} n\Big((n+1)(2n+1) - (n-1)(2n-1)\Big) \\
&= \frac{1}{6} n\Big((2n^2 + 3n + 1) - (2n^2 - 3n + 1)\Big) = \frac{1}{6} n(3n + 3n) = n^2.
\end{aligned}
$$

10 Evaluate the following definite integrals:

(a) $\displaystyle\int_{-1}^{3} \sqrt{1+x}\, dx$

Solution.

$$
\int_{-1}^{3} \sqrt{1+x}\, dx = \frac{2}{3}(1+x)^{3/2}\Big|_{-1}^{3} = \frac{2}{3}(4^{3/2} - 0^{3/2}) = \frac{16}{3}.
$$

(b) $\int_0^2 e^{x^3} x^2\,dx$ (a non-numerical answer in terms of e or other standard mathematical constants or functions is acceptable)

Solution.

$$\int_0^2 e^{x^3} x^2\,dx = \frac{1}{3}e^{x^3}\Big|_0^2 = \frac{1}{3}(e^8 - 1).$$