Practice problem set for Midterm Exam 2 — Solutions

MAT 21B (UC Davis, Winter 2018)

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1. The main span of the Golden Gate Bridge is 1280 meters long, and is suspended by a thick steel cable whose shape is described to a good level of approximation as the *catenary* curve

$$y(x) = -1234 + 652\left(e^{x/1304} + e^{-x/1304}\right) \qquad (-640 \le x \le 640)$$

(see the figure below). Compute the length of the steel cable.

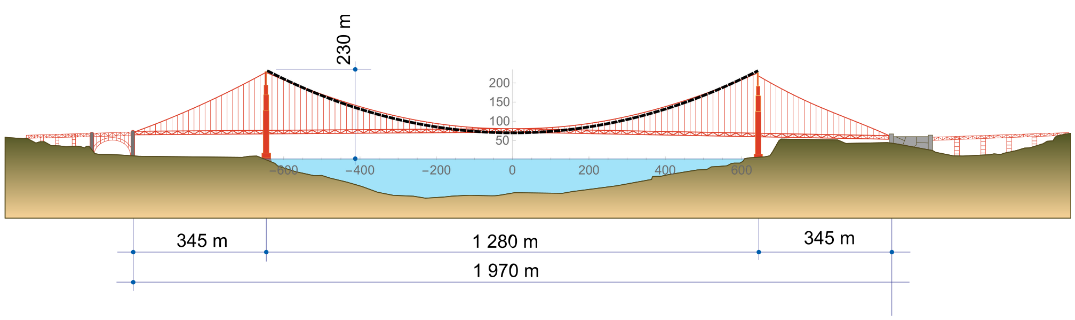


Figure 1: The catenary curve $y(t)$ superposed on top of an image (from Wikipedia) of the Golden Gate Bridge

Solution. In calculus terms, the question is about the arc length of the curve $y(x)$. Remembering the formula $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$, we express the arc length as the integral

$$L = \int_{-640}^{640} \sqrt{1 + y'(x)^2}\,dx.$$

Before doing the calculation it is useful to simplify the expression in the square root:

$$\begin{aligned} 1 + y'(x)^2 &= 1 + 652^2\left(\frac{1}{1304}e^{x/1304} - \frac{1}{1304}e^{-x/1304}\right)^2 \\ &= 1 + \left(\frac{1}{2}e^{x/1304} - \frac{1}{2}e^{-x/1304}\right)^2 = \left(\frac{1}{2}e^{x/1304} + \frac{1}{2}e^{-x/1304}\right)^2. \end{aligned}$$

So the integral becomes

$$L = \int_{-640}^{640} \frac{1}{2}\left(e^{x/1304} + e^{-x/1304}\right)\,dx = \ldots \approx 1332 \text{ meters}$$

after a short calculation.

2. A soap bubble that forms between two circular pieces of wire assumes the shape of a type of surface known as a *catenoid*, which is the surface of revolution of the catenary curve

$$y(x) = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right) = c \cdot \cosh(x/c),$$

for some appropriate value of the parameter c that is determined by the geometry of the problem.

If the circular wires are held 20 centimeters apart (so that they are located at positions $x = -10$ and $x = 10$ along the x-axis in our coordinate system), and each one has a radius of 25 centimeters, it can be found that the value of c is given approximately by the numerical value $c = 22.7685$ (this is the value for which the equation $y(10) = 25$ holds). Given this value, find the surface area of the bubble, in square centimeters.

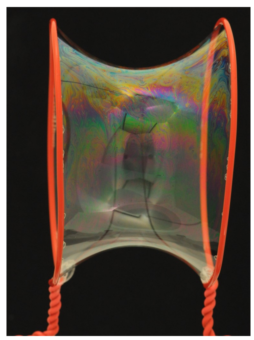


Figure 2: A catenoid-shaped soap bubble between two circular wires. (Image source: http://soapbubble.dk/english/science/the-geometry-of-soap-films-and-soap-bubbles/)

Solution. As we saw in the previous problem, the catenary has the nice property that the arc length element $ds = \sqrt{1 + y'(x)^2}\, dx$ simplifies in a nice way, namely:

$$\sqrt{1 + y'(x)^2} = \sqrt{1 + \left(\frac{e^{x/c} - e^{-x/c}}{2}\right)^2} = \sqrt{\left(\frac{e^{x/c} + e^{-x/c}}{2}\right)^2} = \frac{1}{2}\left(e^{x/c} + e^{-x/c}\right).$$

This can be exploited in a surface area calculation using the formula

$$S = \int 2\pi y \sqrt{1 + y'^2}\, dx$$

for the surface area. In our case we get

$$S = \int_{-10}^{10} 2\pi y(x)\sqrt{1 + y'(x)^2}\, dx = 2\pi \frac{c}{2} \times 2 \int_0^{10} \frac{1}{2}\left(e^{x/c} + e^{-x/c}\right)\left(e^{x/c} + e^{-x/c}\right) dx$$

$$= \pi c \int_0^{10} \left(e^{2x/c} + 2 + e^{-2x/c}\right) dx = \pi c \left(20 + \frac{c}{2}e^{20/c} - \frac{c}{2}e^{-20/c} - \frac{c}{2}e^0 + \frac{c}{2}e^0\right)$$

$$= 3052.38.$$

3. (a) Use the method we learned in class to compute the centroid (center of mass) of the triangle with vertices at $(0,0)$, $(1,0)$ and $(A,1)$ (where A is a parameter; you can assume for simplicity that A is positive, or repeat the calculation also in the case where A is negative if you feel like it).

   Note: the answer will be a pair of numbers $(\bar{x},\bar{y})$, expressed as a function of the parameter A.

   Solution. The question is about centroids, which means that we treat the shape as a thin plate with uniform planar mass density $\delta = 1$. The total mass is equal to the area, which is $1/2$ (one half the base times the height) independently of the value of A:

$$M = 1/2.$$

   Now, compute the moments M_y and M_x using the method of vertical strips. The way to set up the integral depends on the value of the parameter A, with two separate cases that need to be dealt with: $0 < A < 1$ and $A > 1$ (the case $A = 1$ can be thought of as a limiting case of either so it is not necessary to do a separate calculation for it). I will do the calculation for $0 < A < 1$. In this case the upper boundary of the triangle is given by the graph of the function

$$y(x) = \begin{cases} \frac{x}{A} & \text{if } 0 \le x \le A, \\ -\frac{1}{1-A}x + \frac{1}{1-A} & \text{if } A \le x \le 1, \end{cases}$$

   and we also have $\tilde{x} = x$, $\tilde{y} = y(x)/2$, so that

$$\begin{aligned} M_y &= \int_0^1 \tilde{x}\, y(x)\, dx = \int_0^A x\left(\frac{x}{A}\right)\, dx + \int_A^1 x\left(-\frac{1}{1-A}x + \frac{1}{1-A}\right)\, dx \\ &= \ldots \text{[some tedious calculations]} \ldots = \frac{A+1}{6}, \\ M_x &= \int_0^1 \tilde{y}\, y(x)\, dx = \int_0^A \tilde{y}\left(\frac{x}{A}\right)\, dx + \int_A^1 \tilde{y}\left(-\frac{1}{1-A}x + \frac{1}{1-A}\right)\, dx \\ &= \int_0^A \left(\frac{x}{2A}\right)\left(\frac{x}{A}\right)\, dx + \int_A^1 \frac{1}{2}\left(-\frac{1}{1-A}x + \frac{1}{1-A}\right)\left(-\frac{1}{1-A}x + \frac{1}{1-A}\right)\, dx \\ &= \ldots \text{[some tedious calculations]} \ldots = \frac{1}{6}. \end{aligned}$$

   Finally, the centroid is obtained by dividing (M_y, M_x) by M:

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{A+1}{3}, \frac{1}{3}\right).$$

   As a sanity check, trying the special value $A = 1/2$ we get $\bar{x} = 1/2$ (could be guessed by symmetry), $\bar{y} = 1/3$ (seems reasonable...).

(b) Check the results of your calculation by comparing it to the point $(\hat{x}, \hat{y})$ obtained as the common intersection point of the three lines connecting each vertex of the triangle with the midpoint of the triangle edge opposing it. If your calculations are correct, you will find that $(\bar{x}, \bar{y}) = (\hat{x}, \hat{y})$. This is in fact true for any triangle by a theorem in geometry; see the figure below, and the web page `http://mathworld.wolfram.com/TriangleCentroid.html` for a more detailed explanation.

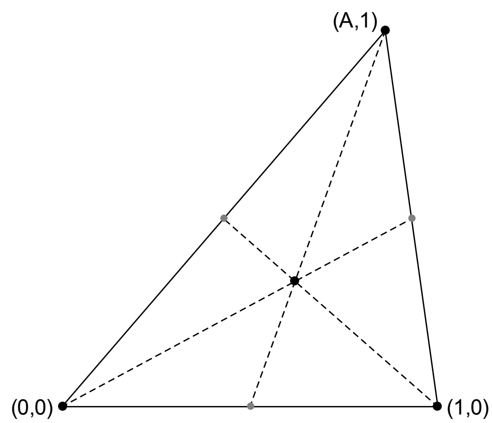


Figure 3: Finding the centroid of a triangle by intersecting the median lines

Solution. The line that connects $(0,0)$ to the midpoint of the opposite edge of the triangle, whose coordinates are $((A+1)/2, 1/2)$, has the equation $y = \frac{x}{1+A}$. Similarly, the line that connects $(1,0)$ to $(A/2, 1/2)$ has the equation $y = \frac{-x+1}{2-A}$. It's easy to find the intersection point of these two lines and check that it is equal to the centroid $(\bar{x}, \bar{y})$ found in part (a) above. It can also be checked that the third line connecting $(A,1)$ to $(1/2, 0)$ intersects these two lines at the same point.

4. A hydroelectric battery is a type of energy-storage facility which stores energy in a water reservoir by using electric power to pump the water uphill at times during the day when a surplus of electric power is available in the electric grid, and harvesting the energy later, turning it back into electricity by letting the water flow back downhill when demand for electric energy is high and electric power is more scarce.

   Assume that the water reservoir is shaped like a cylinder with a radius of 10 meters and a height of 20 meters, and stands upright on a hill 30 meters above ground level. The density of water is 1000 kilograms per cubic meter, and Earth's surface gravitational acceleration is 9.8 meters per second squared. Calculate the amount of energy (in Joules) that can be stored in the reservoir, defined as the work that the water will perform by flowing out of the cylinder and down to ground level.

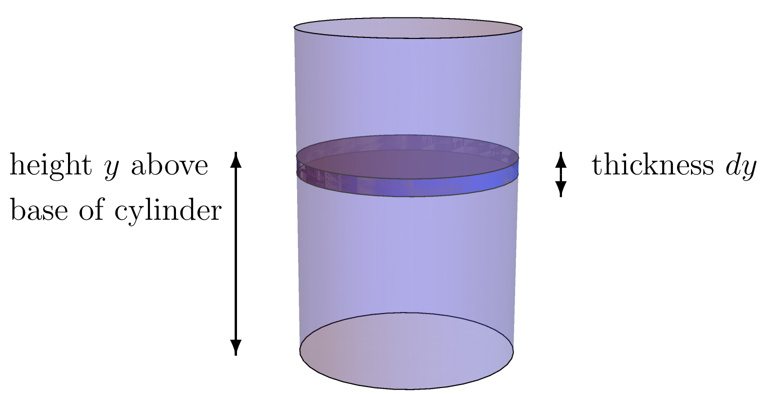
   (Note: in this idealized calculation we are neglecting various sources of waste, such as the impracticality of converting mechanical motion of fluid into electricity with 100% efficiency.)

   Solution. We calculate the work by summing up (i.e., integrating) the contributions from the work performed by many thin cylindrical layers of different heights that stratify the cylindrical reservoir, as illustrated in the figure below. For a cylindrical layer at height y above the cylinder base that has thickness dy, the volume of the layer is $\pi \times 10^2 \times dy$ (cubic meters), so the mass of the water in that layer is $\pi \times 10^2 \times dy \times 1000$ (kilograms), so its weight (the downwards force it exerts on a turbine while flowing downhill) is $\pi \times 10^2 \times dy \times 1000 \times 9.8$ (Newtons). By the formula "work equals force times distance," the work this water will perform while flowing to ground level from its current height $30 + y$ meters above the ground is

$$\pi \times 10^2 \times dy \times 1000 \times 9.8 \times (30 + y) \quad \text{Joules.}$$

   Thus, the total work performed by the water in the reservoir is

$$\begin{aligned} W &= \int_0^{20} \pi \times 10^2 \times 1000 \times 9.8 \times (30 + y)\, dy \\ &= 9.8\pi \times 10^5 \left(30y + \frac{1}{2}y^2\right)\Big|_0^{20} = 9.8\pi \times 10^5 (600 + 200) = 2.46 \times 10^9 \quad \text{Joules.} \end{aligned}$$



5. A flu epidemic in a big city spreads in its first few months in such a way that the number $y(t)$ of infected people satisfies the exponential growth equation

$$y'(t) = ky(t),$$

where time t is measured in days, and k is a positive constant. Assuming the number of infected people doubles every 11 days and that initially at time $t = 0$ there were 1000 infected people, determine:

(a) how many people will be infected after 30 days?

(b) after how many days will the number of infected people reach $50,000$?

Solution. The general solution of the exponential growth equation $y' = ky$ is $y(t) = Ae^{kt}$ for an arbitrary constant A. We are given the information that $y(0) = 1000$ (whereas on the other hand $y(0) = Ae^0 = A$), so we find that

$$A = 1000.$$

We are also told in words that $y(11)/y(0) = 2$, but $y(11)/y(0) = Ae^{k\cdot 11}/Ae^0$, so this gives us an equation $e^{11k} = 2$, which we can solve for the constant k, to get

$$k = \frac{\ln 2}{11} \approx 0.063.$$

The answer to (a) is therefore $y(30) = Ae^{30k} \approx 6622$. The answer to (b) is the value t_1 for which $y(t_1) = Ae^{kt_1} = 50000$, or

$$t_1 = \frac{1}{k} \ln\left(\frac{50000}{A}\right) \approx 62 \text{ days}.$$

6. (a) Show that $\ln(1/10) = -\ln(10)$, using $\ln(x) = \int_1^x \frac{1}{x}\,dx$ as the definition of the natural logarithm function and without relying on any known special identities satisfied by this function.

Solution. By the definition, $\ln(1/10)$ is the integral

$$\ln(1/10) = \int_1^{1/10} \frac{1}{x}\,dx.$$

Making the substitution $u = \frac{1}{x}$, we have $du = -\frac{1}{x^2}\,dx = -u^2 dx$, so that

$$\ln(1/10) = \int_{1/1}^{1/(1/10)} u(-u^{-2})\,du = \int_{1/1}^{1/(1/10)} -\frac{1}{u}\,du = -\int_1^{10} \frac{1}{u}\,du = -\ln(10),$$

as was to be shown.

(b) The mathematical constant e is defined as the unique positive number for which $\int_1^e \frac{1}{x}\,dx = 1$. Show that $\int_1^{e^2} \frac{1}{x}\,dx = 2$, using only standard manipulations on integrals and without relying on any known special identities satisfied by the number e or the exponential or logarithm functions.

Solution. Split up the interval $[1, e^2]$ into two parts, $[1, e]$ and $[e, e^2]$. The integral then becomes the sum of two integrals:

$$\int_1^{e^2} \frac{1}{x}\,dx = \int_1^e \frac{1}{x}\,dx + \int_e^{e^2} \frac{1}{x}\,dx.$$

The first of the two integrals is equal to 1 by the definition of e. The second integral $\int_e^{e^2} \frac{1}{x}\,dx$ can be evaluated by making the substitution $t = x/e$, which using the usual rules for substitutions brings it to the form

$$\int_e^{e^2} \frac{1}{x}\,dx = \int_{e/e}^{e^2/e} \frac{1}{et}(e\,dt) = \int_1^e \frac{1}{t}\,dt = 1.$$

Thus the sum of the two integrals is $1 + 1 = 2$.

(c) What is the slope at $x = 1$ of the following functions:

i. $y = 10^x$ ii. $y = x^e$ iii. $y = x^x$

Solution.

$$\frac{d}{dx}\bigg|_{x=1}(10^x) = \frac{d}{dx}_{|x=1}\left(e^{x\ln(10)}\right) = \left[\text{value at } x = 1 \text{ of } \ln(10)e^{x\ln(10)}\right]$$

$$= \ln(10)e^{\ln(10)} = 10\ln(10) \approx 23.06.$$

$$\frac{d}{dx}\bigg|_{x=1}(x^e) = \left[\text{value at } x = 1 \text{ of } ex^{e-1}\right] = e \approx 2.718.$$

$$\frac{d}{dx}\bigg|_{x=1}(x^x) = \frac{d}{dx}\bigg|_{x=1}\left(e^{x\ln(x)}\right) = \left[\text{value at } x = 1 \text{ of } (\ln(x)+1)e^{x\ln(x)}\right] = 1.$$