Practice problem set for Midterm Exam 2

MAT 21B (UC Davis, Winter 2018)

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Notes — please read carefully:

- The second midterm exam will be held on Wednesday 2/28 in the regular lecture hall (Giedt 1001) at the regular class meeting time (W 5:10-6)
- The midterm will be 50 minutes long.
- No books or notes will be allowed in the exam.
- Mobile phones, tablets, Kindles, or any other wi-fi/network/programmable devices will also not be allowed.
- Non-programmable, non-graphing hardware calculators will be allowed.
- The exam will cover Sections 5.6, 6.1, 6.3–6.6, 7.1, and 7.2 in the book Thomas' Calculus (13th Ed.), except for the following subtopics:
  - fluid pressure (subsection of 6.5 — won't be on the final either)
  - Fluid forces and centroids (subsection of 6.6 — won't be on the final either)
  - Pappus' theorems (subsection of 6.6 — won't be on the final either)
  - Separable differential equations (subsection of 7.2 — will be on the final but not on the midterm)
- The problems below are offered as a supplementary aid to help you challenge yourself and test your understanding of the material. I have designed them to be at a somewhat higher level of difficulty than the exam questions and to take a bit more time to solve, so try not to panic if they seem difficult; nonetheless, if you find that you are completely lost or unable to solve them, this should be a signal to you that your mastery of the material is lacking and that you need to review the relevant concepts some more.
- I have not attempted to cover all the topics in this problem set, so don't neglect the topics that are not covered in your review.
- Solutions will be posted online in the next few days.

1. The main span of the Golden Gate Bridge is 1280 meters long, and is suspended by a thick steel cable whose shape is described to a good level of approximation as the *catenary* curve

$$y(x) = -1234 + 652\left(e^{x/1304} + e^{-x/1304}\right) \qquad (-640 \leq x \leq 640)$$

(see the figure below). Compute the length of the steel cable.

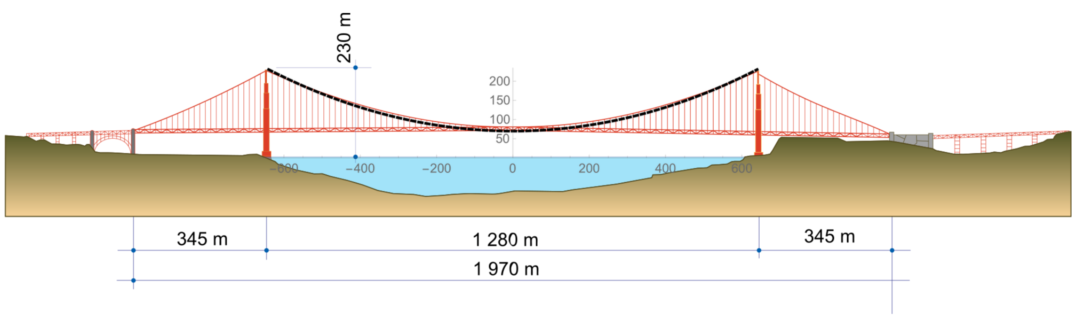

Figure 1: The catenary curve $y(t)$ superposed on top of an image (from Wikipedia) of the Golden Gate Bridge

2. A soap bubble that forms between two circular pieces of wire assumes the shape of a type of surface known as a *catenoid*, which is the surface of revolution of the catenary curve

$$y(x) = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right) = c \cdot \cosh(x/c),$$

for some appropriate value of the parameter c that is determined by the geometry of the problem.

If the circular wires are held 20 centimeters apart (so that they are located at positions $x = -10$ and $x = 10$ along the x-axis in our coordinate system), and each one has a radius of 25 centimeters, it can be found that the value of c is given approximately by the numerical value $c = 22.7685$ (this is the value for which the equation $y(10) = 25$ holds). Given this value, find the surface area of the bubble, in square centimeters.

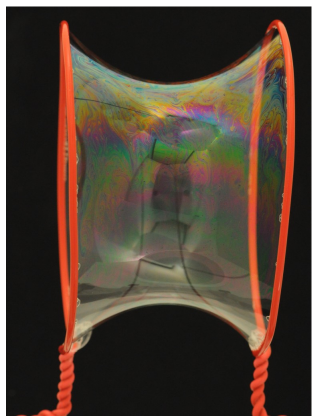

Figure 2: A catenoid-shaped soap bubble between two circular wires. (Image source: `http://soapbubble.dk/english/science/the-geometry-of-soap-films-and-soap-bubbles/`)

3. (a) Use the method we learned in class to compute the centroid (center of mass) of the triangle with vertices at $(0,0)$, $(1,0)$ and $(A,1)$ (where A is a parameter; you can assume for simplicity that A is positive, or repeat the calculation also in the case where A is negative if you feel like it).

   Note: the answer will be a pair of numbers $(\bar{x}, \bar{y})$, expressed as a function of the parameter A.

   (b) Check the results of your calculation by comparing it to the point $(\hat{x}, \hat{y})$ obtained as the common intersection point of the three lines connecting each vertex of the triangle with the midpoint of the triangle edge opposing it. If your calculations are correct, you will find that $(\bar{x}, \bar{y}) = (\hat{x}, \hat{y})$. This is in fact true for any triangle by a theorem in geometry; see the figure below, and the web page `http://mathworld.wolfram.com/TriangleCentroid.html` for a more detailed explanation.

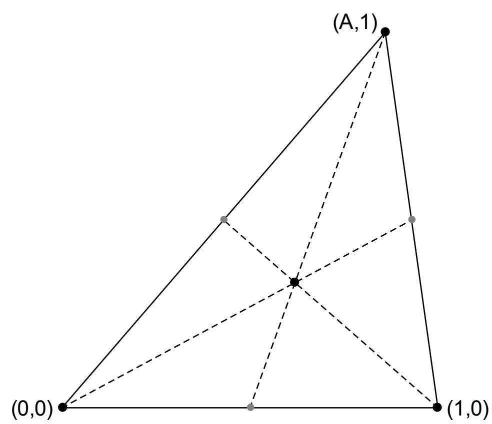

Figure 3: Finding the centroid of a triangle by intersecting the median lines

4. A hydroelectric battery is a type of energy-storage facility which stores energy in a water reservoir by using electric power to pump the water uphill at times during the day when a surplus of electric power is available in the electric grid, and harvesting the energy later, turning it back into electricity by letting the water flow back downhill when demand for electric energy is high and electric power is more scarce.

   Assume that the water reservoir is shaped like a cylinder with a radius of 10 meters and a height of 20 meters, and stands upright on a hill 30 meters above ground level. The density of water is 1000 kilograms per cubic meter, and Earth's surface gravitational acceleration is 9.8 meters per second squared. Calculate the amount of energy (in Joules) that can be stored in the reservoir, defined as the work that the water will perform by flowing out of the cylinder and down to ground level.

   (Note: in this idealized calculation we are neglecting various sources of waste, such as the impracticality of converting mechanical motion of fluid into electricity with 100% efficiency.)

5. A flu epidemic in a big city spreads in its first few months in such a way that the number $y(t)$ of infected people satisfies the exponential growth equation

$$y'(t) = ky(t),$$

where time t is measured in days, and k is a positive constant. Assuming the number of infected people doubles every 11 days and that initially at time $t = 0$ there were 1000 infected people, determine:

(a) how many people will be infected after 30 days?

(b) after how many days will the number of infected people reach $50,000$?

6. (a) Show that $\ln(1/10) = -\ln(10)$, using $\ln(x) = \int_1^x \frac{1}{x}\, dx$ as the definition of the natural logarithm function and without relying on any known special identities satisfied by this function.

(b) The mathematical constant e is defined as the unique positive number for which $\int_1^e \frac{1}{x}\, dx = 1$. Show that $\int_1^{e^2} \frac{1}{x}\, dx = 2$, using only standard manipulations on integrals and without relying on any known special identities satisfied by the number e or the exponential or logarithm functions.

(c) What is the slope at $x = 1$ of the following functions:

i. $y = 10^x$ ii. $y = x^e$ iii. $y = x^x$