Exponential growth and differential equations: a worked example

A social network has $y(0) = 100$ (millions of user) at a given moment in time $t = 0$, and its rate of growth is described by the differential equation

$$\frac{dy}{dt} = \frac{1000 - y}{500} \cdot y,$$

where t is time, measured in months, and $y(t)$ is the number of millions of users.

When will the number of users hit 800 million?

Note. The equation above is a separable differential equation meant to model growth that is exponential in the initial growth phase but then is followed by a rapid decrease in the rate of growth as the number of users approaches one billion. Thus the equation is analogous to the usual exponential growth equation $\frac{dy}{dt} = ky$, except that now the growth constant k itself varies as a function of the number of users, being equal to $9/5$ when $y = 100$, $1/5$ when $y = 900$, $1/50$ when $y = 990$, etc. — as $y(t)$ approaches 1000 the ratio between $\frac{dy}{dt}$ and y converges to 0.

Solution. We solve the equation in the usual way, by moving all terms associated with t and dt to one side of the equation, and all terms associated with y and dy to the other side:

$$2\,dt = \frac{1000}{y(1000 - y)} dy = \left(\frac{1}{y} - \frac{1}{y - 1000}\right) dy.$$

Integrating both sides of the equation gives

$$2t + C = \ln|y| - \ln|y - 1000| = \ln(y) - \ln(1000 - y) = \ln\left(\frac{y}{1000 - y}\right)$$

where C is an arbitrary integration constant, and where I've replaced $|y|$ with y (since y measures the number of users, which is always positive) and $|y - 1000|$ with $1000 - y$ (since I know y will never exceed 1000 based on my understanding of the meaning of the differential equation) inside the logarithms. Now exponentiating both sides of the equation results in

$$\frac{y}{1000 - y} = Ae^{2t}$$

where we replaced e^C with $A = e^C$, also an arbitrary constant. By algebra one can now solve this for y, giving after a short calculation

$$y = 1000 \left(1 - \frac{1}{Ae^{2t} + 1}\right).$$

Thus, we have expressed y as a function of t, except for the value of the constant A. This can be determined using the data that $y(0) = 1000 \left(1 - \frac{1}{A+1}\right)$ is equal to 100, giving the equation

$$1 - \frac{1}{A+1} = \frac{1}{10},$$

which is easily solved to give the value $A = 1/9$. Summarizing, we found that

$$y(t) = 1000 \left(1 - \frac{1}{\frac{1}{9}e^{2t} + 1}\right).$$

(See the graph of $y(t)$ shown below.) Finally, we were asked to find the value t_1 for which $y(t) = 800$. This gives the equation

$$1000 \left(1 - \frac{1}{\frac{1}{9}e^{2t_1} + 1}\right) = 800.$$

After a bit of algebra and the extraction of a logarithm we find that

$$t_1 = \ln(6) \approx 1.79.$$

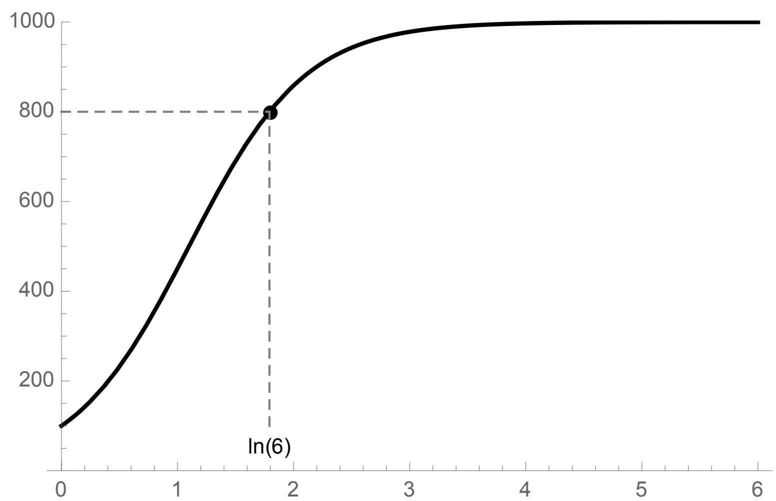

Figure 1: A plot of $y(t)$