Solutions to Midterm 1

Question 1

A cannonball is shot vertically upwards from the ground with an initial velocity of \( v_0 = 320 \) feet per second. Neglecting the effects of air resistance, find its height \( y = h(t) \) above the ground as a function of time at \( t = 5, \ t = 10, \) and \( t = 20. \)

( Reminder: the force of gravity causes falling objects a downwards acceleration of 32 feet per second per second. )

Solution:

\[
y = h(t) = \text{height of the cannonball above the ground at time } t, \\
\frac{dy}{dt} = h'(t) = v(t) = \text{velocity of the cannonball in the y direction}, \\
\frac{d^2y}{dt^2} = h''(t) = v'(t) = a(t) = \text{acceleration of the cannonball} = -32 \text{ (gravity)}, \\
h(0) = 0, \\
v(0) = v_0 = 320.
\]

It follows that \( v(t) \) can be recovered from \( a(t) \) by integration, and \( h(t) \) can be recovered from \( v(t) \) by integration:

\[
v(t) = v(0) + \int_0^t v'(s) \, ds = 320 + \int_0^t (-32) \, ds = 320 - 32s \bigg|_0^t = 320 - 32t,
\]

\[
h(t) = h(0) + \int_0^t h'(s) \, ds = \int_0^t v(s) \, ds = \int_0^t (320 - 32s) \, ds = (320s - 16s^2) \bigg|_0^t = 320t - 16t^2.
\]

Finally, plug in the values \( t = 5, 10, 20 \) to the formula for \( h(t) \):

\[
h(5) = 320 \times 5 - 16 \times 25 = 1600 - 400 = 1200, \\
h(10) = 320 \times 10 - 16 \times 100 = 1600, \\
h(20) = 320 \times 20 - 16 \times 400 = 6400 - 6400 = 0.
\]

(All three values are in units of feet.)
Question 2

Write the sum \(S = (0 \times 2) + (1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + (9 \times 11)\) in \(\sum\) notation, and evaluate it.

These formulas may prove useful for your solution:

\[
\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}, \quad (a - 1)(a + 1) = a^2 - 1
\]

Solution:

\[
S = (0 \times 2) + (1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + (9 \times 11)
\]

\[
= \sum_{k=1}^{10} (k - 1)(k + 1) = \sum_{k=1}^{10} (k^2 - 1) = \sum_{k=1}^{10} k^2 - \sum_{k=1}^{10} 1
\]

\[
= \frac{10 \times 11 \times 21}{6} - 10 = \frac{2 \times 5 \times 11 \times 3 \times 7}{2 \times 3} - 10 = 5 \times 7 \times 11 - 10 = 375.
\]

An alternative solution would be to use a different indexing scheme where we regard the summation index as ranging between the values 0 and 9, and write the sum (using a different letter, \(j\), for the summation index) as

\[
S = (0 \times 2) + (1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots + (9 \times 11)
\]

\[
= \sum_{j=0}^{9} j(j + 2) = \sum_{j=1}^{9} (j^2 + 2j) \quad \text{(the term with } j = 0 \text{ is 0 so can be omitted)}
\]

\[
= \sum_{j=1}^{9} j^2 + 2 \sum_{j=1}^{9} j = \frac{9 \times 10 \times 19}{6} + 2 \times \frac{9 \times 10}{2}.
\]

This still comes out to 375.
Question 3

(a) Evaluate the definite integral \( \int_1^4 \sqrt{x} \, dx \).

**Solution:**

\[
\int_1^4 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \bigg|_1^4 = \frac{2}{3} (4^{3/2} - 1^{3/2}) = \frac{2}{3} \times (8 - 1) = \frac{14}{3}.
\]

(b) Evaluate the indefinite integral \( \int \cos(x) \sqrt{2 + \sin(x)} \, dx \).

**Solution:** Make the substitution \( u = 2 + \sin(x) \), so that \( du = \cos(x) \, dx \). Then

\[
\int \cos(x) \sqrt{2 + \sin(x)} \, dx = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (2 + \sin(x))^{3/2} + C,
\]

where (of course) \( C \) is an integration constant.

(c) Evaluate the definite integral \( \int_1^3 f(x)f'(x) \, dx \) if \( f(x) \) is a differentiable function that satisfies: \( f(0) = 0, \ f(1) = 4, \ f(2) = -10, \ f(3) = 5 \).

**Solution:** Note that \( f(x)f'(x) = \frac{d}{dx} \left( \frac{1}{2} f(x)^2 \right) \), that is, \( \frac{1}{2} f(x)^2 \) is an antiderivative of \( f(x)f'(x) \) — this can be verified using the chain rule, and can be found by making the substitution \( u = f(x) \). This allows us to compute the integral as

\[
\int_1^3 f(x)f'(x) \, dx = \frac{1}{2} f(x)^2 \bigg|_1^3 = \frac{1}{2} f(3)^2 - \frac{1}{2} f(1)^2 = \frac{1}{2} \times (5^2 - 4^2) = \frac{1}{2} \times (25 - 16) = \frac{9}{2}.
\]
Question 4

Determine which of the following sums is a Riemann sum for the integral $\int_0^1 (1 - x)^2 \, dx$.

For each of the sums that is a Riemann sum, explain whether it is a lower sum, an upper sum, a sum associated with the midpoint rule, or something else. For a sum that is not a Riemann sum, give a brief explanation why it isn’t.

(a) $A = \left(1 - \frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1 - \frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1 - \frac{3}{4}\right)^2 \times \frac{1}{4}

Answer: $A$ is a Riemann sum for the integral. Specifically, it is an upper sum, as shown in the figure below illustrating the rectangles the sum of whose areas $A$ is calculating.

![Figure 1](image)

(b) $B = \left(1 - \frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1 - \frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1 - \frac{3}{4}\right)^2 \times \frac{1}{4}

Answer: $B$ is a Riemann sum for the integral. It is a lower sum, see the figure below.

Note: some students wrote that $B$ is not a Riemann sum because there are only 3 terms in the sum but the intervals are of length $1/4$ so the fourth term is missing. Partial credit was given for that answer.
(c) \[ C = \left( \frac{3}{4} \right)^2 \times \frac{1}{4} + \left( \frac{1}{2} \right)^2 \times \frac{1}{4} + \left( \frac{5}{8} \right)^2 \times \frac{1}{2} \]

**Answer:** \( C \) is a Riemann sum for the integral, associated with the partition \( \{x_0, x_1, x_2, x_3\} = \{0, \frac{1}{4}, \frac{1}{2}, 1\} \) of the interval \([0, 1]\) (which partitions \([0, 1]\) into subintervals of unequal length, but that is permitted in the definition of Riemann sums), and with the intermediate points \( c_1 = \frac{3}{4}, c_2 = \frac{1}{2}, c_3 = \frac{5}{8} \). Because \( c_3 \) is neither a minimum point or a maximum point for \((1-x)^2\) in the subinterval \([\frac{1}{2}, 1]\), nor the midpoint of the subinterval, the sum is not a lower sum or an upper sum or a sum associated with the midpoint rule. Again, it’s good to keep in mind that the concept of Riemann sums is more general than those particular kinds of sums.
(d) \[ D = \sum_{k=1}^{n} \left( 1 - \frac{k - 1/2}{n} \right)^2 \times \frac{1}{n} \]

**Answer:** \( D \) is a Riemann sum for the integral, computed according to the midpoint rule with a partition of \([0, 1]\) into \(n\) subintervals of equal length. One can see that \( \frac{k - 1/2}{n} \), the point where the function gets evaluated in the \(k\)th summand, is the midpoint between the two points \( x_{k-1} = \frac{k-1}{n} \) and \( x_k = \frac{k}{n} \) which are the endpoints of the \(k\)th partition subinterval.
**Question 5**

Evaluate the limit

\[
\lim_{n \to \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n} \right)
\]

by relating it to a definite integral. (An answer expressed in terms of standard mathematical constants and functions is acceptable.)

**Solution:** the key is to observe that the second way of writing the sum inside the limit expresses it as a Riemann sum. Take \( f(x) = \frac{1}{1+x} \), then the sum is of the form

\[
\frac{1}{1+1/n} \cdot \frac{1}{n} + \frac{1}{1+2/n} \cdot \frac{1}{n} + \frac{1}{1+3/n} \cdot \frac{1}{n} + \ldots + \frac{1}{1+n/n} \cdot \frac{1}{n}
\]

which is a Riemann sum (associated with a partition of the interval \([0, 1]\) into \(n\) subintervals of equal length) for the definite integral

\[
\int_0^1 f(x) \, dx = \int_0^1 \frac{1}{1+x} \, dx = \ln |1 + x| \bigg|_0^1 = \ln |2| - \ln |1| = \ln(2).
\]

Thus, the limit is equal to \(\ln(2)\).