Solutions to Midterm 1

Question 1

A cannonball is shot vertically upwards from the ground with an initial velocity of $v_0 = 320$ feet per second. Neglecting the effects of air resistance, find its height $y = h(t)$ above the ground as a function of time at $t = 5$, $t = 10$, and $t = 20$.

(Reminder: the force of gravity causes falling objects a downwards acceleration of 32 feet per second per second.)

Solution:

$$\begin{aligned} y &= h(t) = \text{height of the cannonball above the ground at time } t, \\ \frac{dy}{dt} &= h'(t) = v(t) = \text{velocity of the cannonball in the } y \text{ direction}, \\ \frac{d^2y}{dt^2} &= h''(t) = v'(t) = a(t) = \text{acceleration of the cannonball} = -32 \quad \text{(gravity)}, \\ h(0) &= 0, \\ v(0) &= v_0 = 320. \end{aligned}$$

It follows that $v(t)$ can be recovered from $a(t)$ by integration, and $h(t)$ can be recovered from $v(t)$ by integration:

$$v(t) = v(0) + \int_0^t v'(s)\,ds = 320 + \int_0^t (-32)\,ds = 320 - 32s\Big|_0^t = 320 - 32t,$$

$$h(t) = h(0) + \int_0^t h'(s)\,ds = \int_0^t v(s)\,ds = \int_0^t (320 - 32s)\,ds = (320s - 16s^2)\Big|_0^t = 320t - 16t^2.$$

Finally, plug in the values $t = 5, 10, 20$ to the formula for $h(t)$:

$$\begin{aligned} h(5) &= 320 \times 5 - 16 \times 25 = 1600 - 400 = 1200, \\ h(10) &= 320 \times 10 - 16 \times 100 = 1600, \\ h(20) &= 320 \times 20 - 16 \times 400 = 6400 - 6400 = 0. \end{aligned}$$

(All three values are in units of feet.)

Question 2

Write the sum $S = (0 \times 2) + (1 \times 3) + (2 \times 4) + (3 \times 5) + \ldots (9 \times 11)$ in $\sum$ notation, and evaluate it.

These formulas may prove useful for your solution:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad (a-1)(a+1) = a^2 - 1$$

Solution:

$$\begin{aligned} S &= \underbrace{(0 \times 2)}_{k=1} + \underbrace{(1 \times 3)}_{k=2} + \underbrace{(2 \times 4)}_{k=3} + \underbrace{(3 \times 5)}_{k=4} + \ldots + \underbrace{(9 \times 11)}_{k=10} \\ &= \sum_{k=1}^{10} (k-1)(k+1) = \sum_{k=1}^{10} (k^2 - 1) = \sum_{k=1}^{10} k^2 - \sum_{k=1}^{10} 1 \\ &= \frac{10 \times 11 \times 21}{6} - 10 = \frac{2 \times 5 \times 11 \times 3 \times 7}{2 \times 3} - 10 = 5 \times 7 \times 11 - 10 = 375. \end{aligned}$$

An alternative solution would be to use a different indexing scheme where we regard the summation index as ranging between the values 0 and 9, and write the sum (using a different letter, j, for the summation index) as

$$\begin{aligned} S &= \underbrace{(0 \times 2)}_{j=0} + \underbrace{(1 \times 3)}_{j=1} + \underbrace{(2 \times 4)}_{j=2} + \underbrace{(3 \times 5)}_{j=3} + \ldots + \underbrace{(9 \times 11)}_{j=9} \\ &= \sum_{j=0}^{9} j(j+2) = \sum_{j=1}^{9} (j^2 + 2j) \quad \text{(the term with } j = 0 \text{ is 0 so can be omitted)} \\ &= \sum_{j=1}^{9} j^2 + 2\sum_{j=1}^{9} j = \frac{9 \times 10 \times 19}{6} + 2 \times \frac{9 \times 10}{2}. \end{aligned}$$

This still comes out to 375.

Question 3

(a) Evaluate the definite integral $\int_1^4 \sqrt{x}\,dx$.

Solution:

$$\int_1^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3} \times (8-1) = \frac{14}{3}.$$

(b) Evaluate the indefinite integral $\int \cos(x)\sqrt{2+\sin(x)}\,dx$.

Solution: Make the substitution $u = 2 + \sin(x)$, so that $du = \cos(x)\,dx$. Then

$$\int \cos(x)\sqrt{2+\sin(x)}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(2+\sin(x)\right)^{3/2} + C,$$

where (of course) C is an integration constant.

(c) Evaluate the definite integral $\int_1^3 f(x)f'(x)\,dx$ if $f(x)$ is a differentiable function that satisfies: $f(0) = 0,\ f(1) = 4,\ f(2) = -10,\ f(3) = 5.$

Solution: Note that $f(x)f'(x) = \frac{d}{dx}\left(\frac{1}{2}f(x)^2\right)$, that is, $\frac{1}{2}f(x)^2$ is an antiderivative of $f(x)f'(x)$ — this can be verified using the chain rule, and can be found by making the substitution $u = f(x)$. This allows us to compute the integral as

$$\int_1^3 f(x)f'(x)\,dx = \frac{1}{2}f(x)^2\Big|_1^3 = \frac{1}{2}f(3)^2 - \frac{1}{2}f(1)^2 = \frac{1}{2} \times \left(5^2 - 4^2\right) = \frac{1}{2} \times (25-16) = \frac{9}{2}.$$

Question 4

Determine which of the following sums is a Riemann sum for the integral $\int_0^1 (1-x)^2\,dx$. For each of the sums that is a Riemann sum, explain whether it is a lower sum, an upper sum, a sum associated with the midpoint rule, or something else. For a sum that is *not* a Riemann sum, give a brief explanation why it isn't.

(a) $A = \left(1-0\right)^2 \times \frac{1}{4} + \left(1-\frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1-\frac{1}{2}\right)^2 \times \frac{1}{4} + \left(1-\frac{3}{4}\right)^2 \times \frac{1}{4}$

Answer: A **is** a Riemann sum for the integral. Specifically, it is an upper sum, as shown in the figure below illustrating the rectangles the sum of whose areas A is calculating.

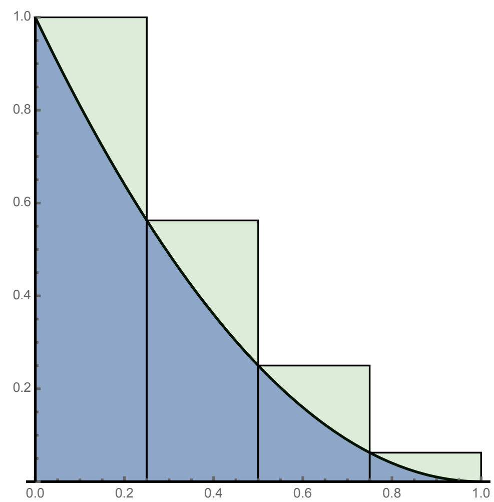

Figure 1: The graph of $f(x) = (1-x)^2$ and the Riemann sum A

(b) $B = \left(1-\frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1-\frac{1}{2}\right)^2 \times \frac{1}{4} + \left(1-\frac{3}{4}\right)^2 \times \frac{1}{4}$

Answer: B **is** a Riemann sum for the integral. It is a lower sum, see the figure below.

Note: some students wrote that B is not a Riemann sum because there are only 3 terms in the sum but the intervals are of length 1/4 so the fourth term is missing. Partial credit was given for that answer.

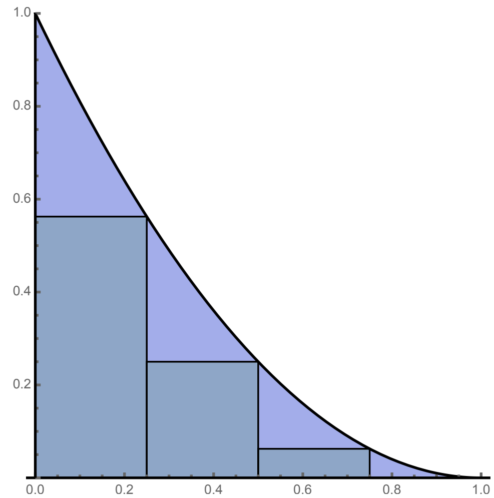

Figure 2: Illustration of the Riemann sum B

(c) $C = \left(1 - \frac{1}{4}\right)^2 \times \frac{1}{4} + \left(1 - \frac{1}{2}\right)^2 \times \frac{1}{4} + \left(1 - \frac{5}{8}\right)^2 \times \frac{1}{2}$

Answer: C **is** a Riemann sum for the integral, associated with the partition $\{x_0, x_1, x_2, x_3\} = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ of the interval $[0, 1]$ (which partitions $[0, 1]$ into subintervals of *unequal* length, but that is permitted in the definition of Riemann sums), and with the intermediate points $c_1 = \frac{1}{4}, c_2 = 12, c_3 = \frac{5}{8}$. Because c_3 is neither a minimum point or a maximum point for $(1 - x)^2$ in the subinterval $[\frac{1}{2}, 1]$, nor the midpoint of the subinterval, the sum is not a lower sum or an upper sum or a sum associated with the midpoint rule. Again, it's good to keep in mind that the concept of Riemann sums is more general than those particular kinds of sums.

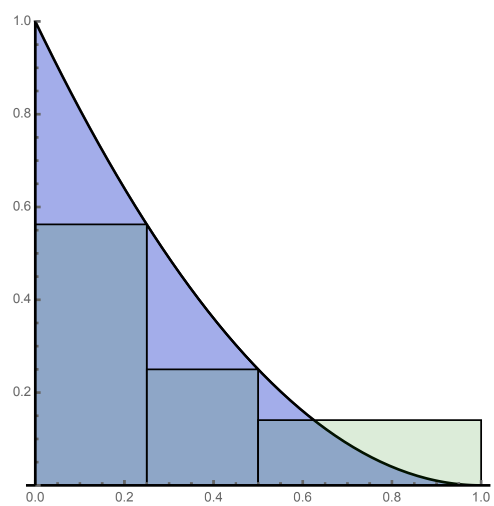

Figure 3: Illustration of the Riemann sum C

(d) $D = \sum_{k=1}^{n} \left(1 - \frac{k-1/2}{n}\right)^2 \times \frac{1}{n}$

Answer: D **is** a Riemann sum for the integral, computed according to the midpoint rule with a partition of $[0,1]$ into n subintervals of equal length. One can see that $\frac{k-1/2}{n}$, the point where the function gets evaluated in the kth summand, is the midpoint between the two points $x_{k-1} = \frac{k-1}{n}$ and $x_k = \frac{k}{n}$ which are the endpoints of the kth partition subinterval.

Question 5

Evaluate the limit

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{n+n}\right)$$
$$=\lim_{n\to\infty}\left(\frac{1}{1+1/n}\cdot\frac{1}{n}+\frac{1}{1+2/n}\cdot\frac{1}{n}+\frac{1}{1+3/n}\cdot\frac{1}{n}+\ldots+\frac{1}{1+n/n}\cdot\frac{1}{n}\right)$$

by relating it to a definite integral. (An answer expressed in terms of standard mathematical constants and functions is acceptable.)

Solution: the key is to observe that the second way of writing the sum inside the limit expresses it as a Riemann sum. Take $f(x)=\frac{1}{1+x}$, then the sum is of the form

$$\frac{1}{1+1/n}\cdot\frac{1}{n}+\frac{1}{1+2/n}\cdot\frac{1}{n}+\frac{1}{1+3/n}\cdot\frac{1}{n}+\ldots+\frac{1}{1+n/n}=\sum_{k=1}^{n}\frac{1}{1+k/n}\cdot\frac{1}{n}$$
$$=\sum_{k=1}^{n}f(k/n)\cdot\frac{1}{n},$$

which is a Riemann sum (associated with a partition of the interval $[0,1]$ into n subintervals of equal length) for the definite integral

$$\int_0^1 f(x)\,dx=\int_0^1\frac{1}{1+x}\,dx=\ln|1+x|\Big|_0^1=\ln|2|-\ln|1|=\ln(2).$$

Thus, the limit is equal to $\ln(2)$.