Problem 1

You are given a system of 4 linear equations in 5 unknowns:

$$\begin{cases} 2x_1 - 4x_2 + 2x_3 + 6x_5 = 6 \\ x_3 - x_4 = -1 \\ 2x_1 - 4x_2 - x_3 + x_4 + 8x_5 = 7 \\ x_3 + x_4 - 2x_5 = 1 \end{cases}$$

(a) Represent the system as an augmented matrix.

Solution.

$$\left(\begin{array}{ccccc|c} 2 & -4 & 2 & 0 & 6 & 6 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 2 & -4 & -1 & 1 & 8 & 7 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

(b) Use the Gaussian elimination method to bring the augmented matrix to Reduced Row-Echelon Form (RREF).

Solution.

$$\left(\begin{array}{ccccc|c} 2 & -4 & 2 & 0 & 6 & 6 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 2 & -4 & -1 & 1 & 8 & 7 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow \frac{1}{2} R_1} \left(\begin{array}{ccccc|c} 1 & -2 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & -3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{\substack{R_1 \leftarrow R_1 - R_2 \\ R_3 \leftarrow R_3 + 3R_2 \\ R_4 \leftarrow R_4 - R_2}} \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 2 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{\substack{R_3 \leftarrow -\frac{1}{2} R_3 \\ R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 + R_3 \\ R_4 \leftarrow R_4 - 2R_3}} \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 4 & 3 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \text{RREF}$$

(c) Use the RREF obtained in part (b) above to write the general form of the solution to the original system.

Solution. The free variables in the RREF are x_2 and x_5. Denoting $x_2 = s, x_5 = t$ where s, t are arbitrary real numbers, we get that the general form of the solution is

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2s - 4t + 3 \\ s \\ t \\ t + 1 \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(d) Use the general form of the solution obtained in part (c) to write a specific solution to the system. That is, write specific numbers x_1, x_2, x_3, x_4, x_5 that solve the system. Substitute the numbers into the system to verify that they actually satisfy the equations.

Solution. Setting $s = t = 0$ gives that the vector $\begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a solution.

Problem 2

(a) Find a basis for $\mathbb{R}^2$ of eigenvectors of the matrix $\begin{pmatrix} 8 & -3 \\ -3 & 0 \end{pmatrix}$.

Solution. Denote the matrix by A. The characteristic polynomial of A is

$$p_A(x) = \det \begin{pmatrix} x-8 & 3 \\ 3 & x \end{pmatrix} = (x-8)x - 9 = x^2 - 8x - 9 = (x+1)(x-9),$$

so the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 9$. For $\lambda_1 = -1$, the eigenvectors are solutions of

$$(A+I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which leads to the eigenvector $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Similarly, for the eigenvalue $\lambda_2 = 9$ we get the equation

$$(A-9I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ -3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives the second eigenvector $v_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. Together v_1 and v_2 form a basis.

(b) Given real numbers a, b, c, find a formula for the eigenvalues λ_1, λ_2 of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Show that the eigenvalues are always real numbers, and characterize for what values of the parameters a, b, c is it true that $\lambda_1 = \lambda_2$.

Solution. The characteristic polynomial of this matrix is

$$p(x) = \det \begin{pmatrix} x-a & -b \\ -b & x-c \end{pmatrix} = (x-a)(x-c) - b^2 = x^2 - (a+c)x + ac - b^2.$$

The eigenvalues are the solutions of the equation $p(x) = 0$, which gives

$$\lambda_{1,2} = \frac{a + c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2}$$

Note that $(a+c)^2 - 4ac = a^2 + c^2 - 2ac = (a-c)^2$, so this can be rewritten as

$$\lambda_{1,2} = \frac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

which shows that the solutions are always real numbers (since the expression in the square root is non-negative). The two eigenvalues λ_1, λ_2 are equal exactly when $(a-c)^2 = b^2 = 0$, or in other words when $b = 0$ and $a = c$. Note that in this case the matrix is simply

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

i.e., it is a scalar multiple of the identity matrix.

(c) Define what it means for a matrix to be diagonalizable.

Solution. A matrix is diagonalizable if there is a basis of the vector space on which the matrix acts consisting of eigenvectors of the matrix.

(d) Prove that the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ (where a, b, c are real numbers as above) is always diagonalizable.

Hint: divide into two cases according to whether $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$.

First solution. If $\lambda_1 = \lambda_2$ then as noted in the solution to part (b) above the matrix is a scalar multiple of the identity matrix, which we know is diagonalizable (any basis is a basis of eigenvectors, since any non-zero vector is an eigenvector). On the other hand if $\lambda_1 \neq \lambda_2$ then the two eigenvectors v_1, v_2 are linearly independent (there is a theorem that says that eigenvectors associated with distinct eigenvalues are always linearly independent; for two vectors this is also very easy to see directly, since two vectors are linearly dependent only if one is a scalar multiple of the other).

Second solution.[1] The matrix is a real symmetric matrix, which is a special case of a self-adjoint or Hermitian matrix. The spectral theorem discussed in the last lecture contains as part of its statement that such matrices are always diagonalizable.

[1] This solution uses material from Chapter 11 that you are not required to know for the final exam.

Problem 3

Let A be the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix}.$$

(a) Compute A^{-1}.

Solution. $A^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. It can be computed either using Gaussian elimination (starting with the augmented matrix $(A \,|\, I)$ and bringing the matrix on the left to RREF, which gives $\big(I \,|\, A^{-1}\big)$), or using the formula $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.

(b) Multiply the matrix you obtained in part (a) above by A to check that it is indeed the inverse matrix of A.

(c) Find all the eigenvalues of A.

Hint: you should be able to answer this without any computations (or with a very short computation that will make the answer obvious).

Solution. The characteristic polynomial of A is

$$p_A(x) = \det(xI - A) = \det \begin{pmatrix} x+1 & 0 & 0 \\ 2 & x-1 & 0 \\ 2 & 1 & x-2 \end{pmatrix} = (x-(-1))(x-1)(x-2),$$

(since $xI - A$ is a lower triangular matrix, so the determinant is the product of the diagonal entries), so the eigenvalues are $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$. In general, the eigenvalues of a (lower or upper) triangular matrix are exactly the diagonal entries.

(d) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

Solution. By writing the three matrices $A - I, A + I$ and $A - 2I$ and solving the corresponding systems of equations, after a short computation one finds that the vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$. $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$, and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ corresponds to the eigenvalue $\lambda_3 = 2$.

Problem 4

(a) Compute the following determinants:

i. $$\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 5 & 0 & 0 & 0 \end{pmatrix} = (-1)\det\begin{pmatrix} 0 & -1 & 0 \\ 2 & 1 & 2 \\ 5 & 0 & 0 \end{pmatrix} = (-1)\left[-(-1)\det\begin{pmatrix} 2 & 2 \\ 5 & 0 \end{pmatrix}\right]$$
$$= -(2\cdot 0 - 5\cdot 2) = 10 \quad \text{(using first row expansion)}$$

ii. $$\det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1\cdot 2\cdot 3\cdot 2\cdot 1 = 12$$

(the determinant of a triangular matrix is equal to the product of the diagonal entries)

iii. $$\det\overbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}}^{n \text{ columns}} = 0 \text{ if } n \geq 2 \text{ since the matrix has identical rows and is}$$

therefore not invertible. For $n = 1$ the determinant is 1.

(b) Let A, B be square matrices of order n. Assume that B is invertible. Prove that

$$\det(BAB^{-1}) = \det(A).$$

Solution. We use the known facts that the determinant of a product of matrices is equal to the product of the determinants, and the determinant of the inverse matrix of B (assuming it exists) is the reciprocal of the determinant of B. Hence we have

$$\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \det(B)\det(B)^{-1}\det(A) = \det(A).$$

(c) A square matrix A of order n is called *anti-symmetric* if it satisfies the condition

$$A^\top = -A$$

For example, the matrix

$$\begin{pmatrix} 0 & 5 & -1 \\ -5 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}$$

is anti-symmetric. Prove that if n is an odd number and A is an anti-symmetric matrix of order n then A is not invertible. (**Hint:** use determinants.)

Solution. Recall that the transpose operation does not change the determinant, and for a matrix A of order n and a scalar c we have the relation $\det(cA) = c^n A$. If n is odd and A is anti-symmetric then

$$\det(A) = \det(-A^\top) = (-1)^n \det(A^\top) = (-1)^n \det(A) = -\det(A),$$

so $\det(A) = 0$ (the only number equal to the negative of itself is 0), which means that A is not invertible.

Problem 5

(a) If a linear operator $T : V \to V$ has two eigenvectors v_1, v_2. Assume that the associated eigenvalues λ_1 and λ_2 are distinct. Are v_1, v_2 necessarily linearly independent? Prove that they are, or give an example that shows they don't have to be.

Solution. Yes, they are linearly independent. Two vectors are linearly dependent if and only if one of them is a scalar multiple of the other, but if that were the case then both of them would be eigenvectors associated with the same eigenvalue.

(b) Let V be a vector space with $\dim(V) = 3$, and let $T : V \to V$ be a linear operator on V. Assume that v_1, v_2, v_3 are eigenvectors of T, with associated eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 1.$$

Assume also that v_1 and v_2 are linearly independent. Prove that $\{v_1, v_2, v_3\}$ is a basis of V.

Hint: If v_1, v_2, v_3 are linearly dependent, show that you can find linearly dependent eigenvectors u, w such that $T(u) = \lambda_1 u$ and $T(w) = \lambda_3 w$, and explain why this leads to a contradiction.

Solution. We prove that v_1, v_2, v_3 are linearly independent, and since $\dim(V) = 3$ that also implies that they form a basis. Assume that we have a representation of the 0 vector as a linear combination of the form

$$av_1 + bv_2 + cv_3 = 0.$$

This can also be written as $u + cv_3 = 0$, where $u = av_1 + bv_2$. Since u is a linear combination of v_1, v_2 (which are both eigenvectors associated with the eigenvalue λ_1), it also satisfies $T(u) = \lambda_1 u$. If $u = 0$ that means that $av_1 + bv_2 = 0$, but since we assumed that v_1, v_2 are linearly independent then $a = b = 0$, which also implies $c = 0$ (which implies that the linear combination $av_1 + bv_2 + cv_3 = 0$ is the trivial combination with all coefficients equal to 0, which is exactly what we need to show).

On the other hand, if $u \neq 0$, then u is an eigenvector associated with the eigenvalue λ_1. v_3 is an eigenvector associated with the eigenvalue λ_3, which is different than λ_1, so by part (a) above, u and v_3 are linearly independent. This implies that $c = 0$, which forces u to be 0, a contradiction.