HW Assignment #1

1.

(a) $\begin{cases} x + y = 22 \\ x - 2y = 7 \end{cases} \implies x = 17, y = 5$

(b) $\begin{cases} -4x + 2y = 10 \\ 2x - y = 10 \end{cases} \implies -5 = 2x - y = 10 \implies$ no solutions

(c) $\begin{cases} x + y = 10 \\ 5x + 5y = 50 \end{cases} \implies x + y = 10 \implies$ infinitely many solutions, the general solution is of the form $(x, 10 - x), x \in \mathbb{R}$

(d) $\begin{cases} x + 2y + 3z = 11 \\ 2x + 6z = 14 \\ x + y + z = 5 \end{cases} \implies x = 1, y = 2, z = 2$

2. ... Assume that the crown was made of a mixture of gold and silver, weighed 1 pound (equal to about 454 grams) and had a volume of 30 cubic centimeters. The density of gold is known to be 19.3 grams per cubic centimeter, and the density of silver is 10.5 grams per cubic centimeter. What percentage of the crown's weight is gold?

   Solution. Denote by x the weight of the gold and by y the weight of the silver, both measured in grams. The question leads to the linear system of equations

$$\begin{cases} x + y = 454 \\ \frac{1}{19.3}x + \frac{1}{10.5}y = 30 \end{cases}$$

   which can be solved to give

$$x = 304.852, \quad y = 149.148$$

   The percentage of gold is therefore

$$\frac{x}{x+y} = \frac{304.852}{454} = 0.67148,$$

   i.e., approximately 67%.

3. Solve the following exercises in the textbook:

   (a) Chapter 1: "Proof-writing exercise" number 1.

   Solution. Assume that $ad - bc \neq 0$ and that the numbers x_1, x_2 solve the equations

$$\begin{cases} ax_1 + bx_2 = 0 \\ cx_1 + dx_2 = 0 \end{cases}$$

One such solution is $x_1 = x_2 = 0$, but we are asked to show that there are no other solutions. Multiply the first equation by c and the second equation by $(-a)$ and add the two; this gives the new equation

$$(bc - ad)x_2 = (ac - ac)x_1 + (bc - ad)x_2 = 0$$

Since we assumed that $ad - bc \neq 0$, we can divide this last equation by $bc - ad$ to get that $x_2 = 0$. A modified version of this manipulation (multiplying the first equation by d and the second by $-b$, and adding the two) also shows that $x_1 = 0$. So, we showed that necessarily $x_1 = x_2 = 0$, which means that no other pair of numbers can satisfy the equations.

(b) Chapter 2: "Calculational exercises" number 1, 2, 3, 4(a), 4(b), 5(c), 5(d).

Solution to 2. (a)

$$\frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{\overline{z}}{|z|^2} \cdot \frac{\overline{z}}{|z|^2} = \frac{1}{|z|^4}(x - iy)^2 = \frac{1}{|z|^4}\left((x^2 - y^2) - 2xyi\right)$$
$$= \frac{x^2 - y^2}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} i,$$

so the real part of $1/z^2$ is $\frac{x^2-y^2}{(x^2+y^2)^2}$ and the imaginary part is $\frac{-2xy}{(x^2+y^2)^2}$.

(b)

$$\frac{1}{3z + 2} = \frac{3\overline{z} + 2}{|3z + 2|^2} = \frac{(3x + 2) - 3yi}{(3x + 2)^2 + y^2}$$
$$\implies \operatorname{Re}\left(\frac{1}{3z + 2}\right) = \frac{3x + 2}{(3x + 2)^2 + y^2}, \quad \operatorname{Im}\left(\frac{1}{3z + 2}\right) = \frac{-3y}{(3x + 2)^2 + y^2}.$$

(d)

$$z^3 = (x^3 - 3xy^2) + (3x^2y - y^3)i \implies \operatorname{Re}(z^3) = x^3 - 3xy^2, \quad \operatorname{Im}(z) = 3x^2y - y^3.$$

Solution to 3. $r = 1$, $\theta = 7\pi/4$.

Solution to 4(a).

$$z_1 = \sqrt[5]{2} \approx 1.1487$$
$$z_2 = \sqrt[5]{2}(\cos(2\pi/5) + i\sin(2\pi/5)) \approx 0.355 + i \cdot 1.0925$$
$$z_3 = \sqrt[5]{2}(\cos(4\pi/5) + i\sin(4\pi/5)) \approx -0.9293 + i \cdot 0.6752$$
$$z_4 = \sqrt[5]{2}(\cos(6\pi/5) + i\sin(6\pi/5)) \approx -0.9293 - i \cdot 0.6752$$
$$z_5 = \sqrt[5]{2}(\cos(8\pi/5) + i\sin(8\pi/5)) \approx 0.355 - i \cdot 1.0925$$

Solution to 4(b).

$$\begin{aligned}
z_1 &= \cos(3\pi/8) + i\sin(3\pi/8)) \approx 0.3827 + i\cdot 0.9239 \\
z_2 &= \cos(3\pi/8 + 2\pi/4) + i\sin(3\pi/8 + 2\pi/4)) \approx -0.9239 + i\cdot 0.3827 \\
z_3 &= \cos(3\pi/8 + 4\pi/4) + i\sin(3\pi/8 + 4\pi/4)) \approx -0.3287 - i\cdot 0.9239 \\
z_5 &= \cos(3\pi/8 + 6\pi/4) + i\sin(3\pi/8 + 6\pi/4)) \approx 0.9239 - i\cdot 0.3827
\end{aligned}$$

Solution to 5(c) The modulus of a product is the product of the moduli, so

$$\left|\frac{i(2+3i)(5-2i)}{-2-i}\right| = \frac{|i|\cdot|2+3i|\cdot|5-2i|}{|-2-i|} = \frac{1\cdot\sqrt{2^2+3^2}\cdot\sqrt{5^2+2^2}}{\sqrt{2^2+1^2}} = \sqrt{377/5}$$

(c) Chapter 2: "Proof-writing exercises" number 2, 3.

Solution to 2. Denote $z = x + iy$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. Proof of the "if" claim: if $y = \mathrm{Im}(z) = 0$ then $z = x + iy = x + i\cdot 0 = x = \mathrm{Re}(z)$. Proof of the "only if" claim: if $z = \mathrm{Re}(z) = x$ then $0 = z - x = iy = i\,\mathrm{Im}(z)$, so $\mathrm{Im}(z) = 0$.

Solution to 3.

$$\begin{aligned}
|z-w|^2 + |z+w|^2 &= (z-w)(\overline{z-w}) + (z+w)(\overline{z+w}) \\
&= (z-w)(\overline{z}-\overline{w}) + (z+w)(\overline{z}+\overline{w}) \\
&= (z\overline{z} + w\overline{w} - z\overline{w} - w\overline{z}) + (z\overline{z} + w\overline{w} + z\overline{w} + w\overline{z}) \\
&= 2z\overline{z} + 2w\overline{w} = 2(|z|^2 + |w|^2)
\end{aligned}$$

HW Assignment #2

1. (a) We rewrite the second and third equations in the system as

$$\begin{aligned} (5-z)x &+ 2y &= 0 \\ 2x &+ (8-z)y &= 0 \end{aligned}$$

We are looking for a solution other than the zero solution $x = y = 0$. By a previous exercise mentioned in the problem, such a solution exists if and only if $(5-z)(8-z) - 2 \cdot 2 = 0$. This gives the quadratic equation $z^2 - 13z + 36 = 0$, which has the two solutions $z = 4$ and $z = 9$.

(b) When $z = 4$, the solution of the original system of three equations is $x = 2, y = -1$. This doesn't make sense since in the original problem x and y referred to the relative importances of two websites, and therefore had to be positive numbers (or 0).

When $z = 9$, the solution is $x = 1/3, y = 2/3$, which makes sense and is the same solution given in the lecture notes.

2. (a) $f(x) = x^3 - x^2 + 2, \quad g(x) = x - 2 \quad \Longrightarrow q(x) = x^2 + x + 2, r(x) = 6$

(b) $f(x) = x^3 - x^2 + 2, \quad g(x) = x + 1 \quad \Longrightarrow q(x) = x^2 - 2x + 2, r(x) = 0$

(c) $f(x) = x^4 + x, \quad g(x) = x^2 + 1 \quad \Longrightarrow q(x) = x^2 - 1, r(x) = x + 1$

3. If $p(x) = x^3 - 4x^2 + 2x + 3$ and $p(3) = 0$, then $p(x)$ is divisible (without remainder) by the factor $x - 3$. By performing polynomial long division we find that

$$p(x) = (x-3)(x^2 - x - 1)$$

It follows that the roots of $p(x)$ are $x_1 = 3$ and the two roots of $x^2 - x - 1$, which are

$$x_2 = \frac{1-\sqrt{5}}{2}, \quad x_3 = \frac{1+\sqrt{5}}{2}$$

4.

$$x^n - 1 = (x-1)(1 + x + x^2 + \ldots + x^{n-1})$$

5. Solve the following problems in the textbook:

(a) Calculational exercise 1 in Chapter 3.

Solution to both parts (a) and (b). The polynomial $p(x) = x$ satisfies the condition (it is of degree 1 which is ≤ 2).

(b) Proof-writing exercise 2(a) in Chapter 3.

Solution to part (a). First, assume that $p(z)$ is just a single monomial of the form $a_k z^k$ for some power k. In this case, the equation $\overline{p(z)} = \overline{p}(\overline{z})$ which we need to prove becomes

$$\overline{(a_k z^k)} = \overline{a_k}(\overline{z})^k.$$

This is true, and follows from the property $\overline{wz} = \overline{w} \cdot \overline{z}$ ("the conjugate element of a product is the product of the conjugate elements") which holds for any two complex numbers.

Next, if the claim is true separately for each of two polynomaials $p_1(z)$ and $p_2(z)$, then it is true for their sum $p(z) = p_1(z) + p_2(z)$, since

$$\overline{p(z)} = \overline{p_1(z) + p_2(z)} = \overline{p_1(z)} + \overline{p_2(z)} = \overline{p_1}(\overline{z}) + \overline{p_2}(\overline{z}) = \overline{p}(\overline{z})$$

(here, we use the fact that the conjugate element of a sum of numbers is the sum of the conjugate elements).

By combining the two observations above (1. the claim is true for a monomial; 2. if it is true for two polynomials then it is true for their sum), we deduce that the claim is true also for a sum of monomials, so it is true for a general polynomial.

(c) Calculational exercise 1 in Chapter 4.

Solution.

(a) It is a vector space.

(b) It is a vector space.

(c) It is not a vector space. For example, the vectors $(0, 1)$ and $(1, 1)$ are both in the set but their sum $(0, 2)$ is not in the set. Thus, this subset of the vector space $\mathbb{R}^2$ is not closed under addition.

(d) It is not a vector space. For example, the vector $(1, 0)$ is in the set but multiplying it by the scalar -1 gives $(-1, 0)$ which is not in the set. So, the set is not closed under scalar multiplication.

(e) It is not a vector space, for similar reasons as the previous two examples.

(f) It is a vector space.

(g) It is not a vector space since it is not closed under addition (the "1" in the corner makes all the difference...)

(d) Proof-writing exercise 1 in Chapter 4.

Solution. Assume that $a \in \mathbb{F}$, $v \in V$ satisfy $av = 0$. We claim that either $a = 0$ or $v = 0$. If $a = 0$, we are done. Otherwise, we show that v must be the zero vector. Since $a \neq 0$, we can multiply the equation $av = 0$ by the reciprocal scalar a^{-1}. The left-hand side becomes

$$a^{-1}(av) = (a^{-1}a)v = 1 \cdot v = v$$

(the first equality follows from one of the "associativity" properties of a vector space; the second one follows from the "multiplicative identity" property). The right-hand side becomes $a^{-1}0$, which is also equal to the vector 0 (see Proposition 4.2.4 in the textbook). The fact that the two expressions are equal gives the desired claim that $v = 0$.

HW Assignment #3

1. Solve the following problems in the textbook:

   (a) Calculational exercise 5 in Chapter 4.

   Solution. It is not hard to check that

$$\begin{aligned} W &= \{p(z) \in \mathbb{F}[z] \,:\, p''(0) = p'''''(0) = 0\} \\ &= \{p(z) = c_0 + c_1 z + c_3 z^3 + c^4 z^4 + c^6 z^6 + \ldots + c_n z^n \,: \\ &\qquad\qquad c_0, c_1, c_3, c_4, c_6, \ldots, c_n \in \mathbb{F}\} \end{aligned}$$

   satisfies the requirement.

   (b) Proof-writing exercises 3, 4 in Chapter 4.

   Solution. Both claims are false. To see this, consider the example in which

$$V = \mathbb{R}^2, \qquad W_1 = \text{span}(1,0), \qquad W_2 = \text{span}(1,1), \qquad W_3 = \text{span}(0,1).$$

   In this case we have the direct sums $V = W_1 \oplus W_3$ and $V = W_2 \oplus W_3$ (check that there is a non-direct sum, and that $W_1 \cap W_3 = W_2 \cap W_3 = \{0\}$), but $W_1 \neq W_2$.

2. Show two matrices A, B that have the property that $AB \neq BA$ (where AB, BA refer to matrix multiplication, described in the appendix). Thus matrix multiplication is not commutative, unlike the multiplication of real and complex numbers.

   Solution. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are an example, since

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

3. Show two matrices A, B that have the property that $AB = 0$ but both A and B are different from the zero matrix (that is, the zero matrix of the same respective width and height as A, B). This shows another way that matrix multiplication differs from number multiplication.

   Solution. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are an example.

4. Define a 3×3 matrix M that depends on a parameter α by $M = \begin{pmatrix} \alpha & 0 & 3 \\ 3 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$.

   (a) For which value of α is the matrix $A = \begin{pmatrix} -2 & 0 & -3 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix}$ equal to the inverse matrix M^{-1} of M? (The inverse matrix is defined on page 140 in the textbook.)

Solution. The multiplication of AM gives

$$AM = \begin{pmatrix} -2 & 0 & -3 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 3 \\ 3 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 3-2\alpha & 0 & 0 \\ \alpha-1 & 1 & 0 \\ \alpha-1 & 0 & 1 \end{pmatrix}.$$

This becomes the identity matrix $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ when $\alpha = 1$. One can easily check that for that value of α we also have $MA = I$, which means that when $\alpha = 1$ the matrix A is indeed the inverse of M. (Note: although it is good to check that $AM = I$ and $MA = I$ separately, we will later see that if for two square matrices A, B we have $AB = I$ then necessarily the equation $BA = I$ must hold as well, so that A and B are inverse to each other and only one of the two equations needs to be checked.)

(b) Use the answer to (a) above to solve the linear system

$$\begin{cases} -2x \phantom{{}+y} - 3z = 5 \\ x + y + 4z = 0 \\ x \phantom{{}+y} + z = -1 \end{cases}$$

Solution. In matrix notation, this system can be expressed as

$$\begin{pmatrix} -2 & 0 & -3 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

or

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

where A is the matrix discussed in part (a). Multiplying both sides of this equation by $M = A^{-1}$ (with the parameter value $\alpha = 1$) and using the associativity of matrix multiplication, we see that

$$MA \begin{pmatrix} x \\ y \\ z \end{pmatrix} = MM^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = I \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

must be equal to

$$M \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -3 \end{pmatrix}.$$

That is, the system has the unique solution $x = 2, y = 10, z = -3$.

Assignment #4

1. Solve Calculational exercises 2, 3 in Chapter 5.

   Solution to calculational exercise 2. (a) Denote $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, the standard basis vectors. Note that

   $$e_1 = (-i)v_1, \qquad e_2 = v_2 - v_1, \qquad e_3 = -v_3 + v_1 + ie_2 = -v_3 + v_1 + i(v_2 - v_1),$$

   i.e., $e_1, e_2, e_3 \in \operatorname{span}(v_1, v_2, v_3)$, and therefore also $V = \mathbb{C}^3$, which is the span of e_1, e_2, e_3, is contained in $\operatorname{span}(v_1, v_2, v_3)$ (since the span is closed under linear combinations). We also have the opposite containment relation $\operatorname{span}(v_1, v_2, v_3) \subset V$, so combining the two we get the claim that $V = \operatorname{span}(v_1, v_2, v_3)$.

   (b) We know from (a) above that (v_1, v_2, v_3) is a spanning set of vectors, and since it has exactly 3 vectors (the dimension of V), it must be linearly independent, since otherwise (by the "basis reduction theorem", theorem 5.3.4 on page 55 of the textbook) we could discard some of the vectors and get a basis with less than 3 elements.

   Solution to calculational exercise 3.

   (a) $\dim = 3$, since $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ is a basis for the space.

   (b) $\dim = 3$, since $\{(1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 0)\}$ is a basis.

   (c) $\dim = 2$, since $\{(1, 0, 1, 1), (0, 1, -1, 1)\}$ is a basis.

   (d) $\dim = 2$, the same as (c) above since the extra equation is just the sum of the two previous equations.

   (e) $\dim = 1$, $\{(1, 1, 1, 1)\}$ is a basis.

2. In this problem you are asked to solve each of the homogeneous linear systems (a)–(f) below in 3 steps: (i) write the coefficient matrix; (ii) use the Gaussian elimination technique to bring the system to Reduced Row-Echelon Form (RREF) by applying elementary row operations to the coefficient matrix; (iii) using the reduced row-echelon form, write the general form of a solution to the system.

3. (a) $$\begin{cases} x \qquad + 5z = 0 \\ \qquad y \qquad = 0 \end{cases}$$

   Solution. The coefficient matrix is $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \end{pmatrix}$ which is already in RREF. The solution set is $(-5z, 0, z) \ : \ z \in \mathbb{R}\} = \operatorname{span}\{(-5, 0, 1)\}$.

   (b) $$\begin{cases} x + 3y + 0z - w = 0 \\ -x - 2y - z + w = 0 \\ \qquad y \qquad\quad + w = 0 \end{cases}$$

   Solution. The coefficient matrix is $\begin{pmatrix} 1 & 3 & 0 & -1 \\ -1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. We apply Gaussian elimination to bring this matrix to reduced row-echelon form using elementary row operations:

$$\begin{pmatrix} 1 & 3 & 0 & -1 \\ -1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + R_3]{R_1 \leftarrow R_1 - 3R_3} \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

This new RREF matrix represents the equivalent system of equations

$$\begin{cases} x & & & - & 4w & = & 0 \\ & y & & + & w & = & 0 \\ & & z & + & w & = & 0 \end{cases}$$

The solution set can be described in terms of the "free" (non-pivot) variable w, as

$$\{(4w, -w, -w, w) \ : \ w \in \mathbb{R}\} = \{w(4, -1, -1, 1) \ : \ w \in \mathbb{R}\} = \text{span}\{(4, -1, -1, 1)\}.$$

(c) $\begin{cases} x & + & 3y & = & 0 \\ -x & - & 2y & = & 0 \end{cases}$

Solution.

$$\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which represents the equations $x = y = 0$, so the solution set is $\{0\}$.

(d) $\begin{cases} x & + & 3y & = & 0 \\ 3x & + & 9y & = & 0 \end{cases}$

Solution. The coefficient matrix $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$ leads to the RREF $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, which corresponds to the equation $x + 3y = 0$. So the solution set is

$$\{(-3y, y) \ : \ y \in \mathbb{R}\} = \text{span}\{(-3, 1)\}.$$

(e) $\begin{cases} x & + & y & + & 5z & = & 0 \\ & & y & - & 10z & = & 0 \\ & & & & 2z & = & 0 \end{cases}$

Solution. The coefficient matrix $\begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -10 \\ 0 & 0 & 2 \end{pmatrix}$ leads after a short computation to the RREF $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is the *identity matrix* of order 3, i.e., it corresponds to the trivial equations $x = y = z = 0$. By the way, it is easy to see that this will always happen if we start from an *upper triangular matrix* — a matrix whose entries below the main diagonal are all 0 — if the entries on the diagonal are all non-zero, and can therefore act as pivots. So, the solution set is $\{0\}$.

(f) $\left\{\begin{array}{rcrcrcl} x & + & y & + & 5z & = & 0 \\ 2x & + & 3y & - & 10z & = & 0 \end{array}\right.$

Solution.

$$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & -10 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -20 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 25 \\ 0 & 1 & -20 \end{pmatrix}$$

So the solution set is

$$\{z(-25, 20, 1) \ : \ z \in \mathbb{R}\} = \text{span}\{(-25, 20, 1)\}.$$

4. (a) Solve each of the following inhomogeneous linear systems.

$$\text{(i)} \qquad \left\{\begin{array}{rcrcrcl} x & & & + & 4z & = & 2 \\ & & y & & & = & -1 \end{array}\right.$$

The solution set is $\{(2-4z, -1, z) \ : \ z \in \mathbb{R}\} = \{(2, -1, 0) + z(-4, 0, 1) \ : \ z \in \mathbb{R}\}$

$$\text{(ii)} \qquad \left\{\begin{array}{rcrcrcrcl} 3x & - & 3y & + & 15z & & & = & -6 \\ x & + & 2y & - & z & - & w & = & -3 \\ x & & & + & 3z & + & w & = & -1 \end{array}\right.$$

The solution set is $\{(-2-3z, 2z, z, 1) \ : \ z \in \mathbb{R}\} = \{(-2, 0, 0, 1) + z(-3, 2, 1, 0) \ : \ z \in \mathbb{R}\}$

$$\text{(iii)} \qquad \left\{\begin{array}{rcrcl} x & + & 3y & = & 1 \\ -x & - & 2y & = & 3 \end{array}\right. \implies \text{unique solution } (x, y) = (-11, 4)$$

$$\text{(iv)} \qquad \left\{\begin{array}{rcrcl} x & + & 3y & = & 0 \\ 3x & + & 9y & = & 1 \end{array}\right. \implies \text{no solutions}$$

$$\text{(v)} \qquad \left\{\begin{array}{rcrcrcl} x & + & y & + & 5z & = & 13 \\ & & y & - & 10z & = & 0 \\ & & & & 2z & = & 4 \end{array}\right. \implies \text{unique solution } (x, y, z) = (-17, 20, 2)$$

$$\text{(vi)} \qquad \left\{\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & + & 4x_4 & + & 4x_5 & = & 5 \\ 2x_1 & + & 4x_2 & - & x_3 & + & 5x_4 & + & 8x_5 & = & -5 \\ x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & 4x_5 & = & -10 \\ x_1 & + & 2x_2 & + & 0x_3 & + & 6x_4 & + & 8x_5 & = & 0 \end{array}\right.$$

The general form of the solution is $(x, y, z, s, t) = (-2y-3s-4t, y, 5-s, s, t), \ \ y, s, t \in \mathbb{R}$.

Alternatively, the solution set can be written as

$\{(0, 0, 5, 0, 0) + y(-2, 1, 0, 0, 0) + s(-3, 0, -1, 1, 0) + t(-4, 0, 0, 0, 1) \ : \ y, s, t \in \mathbb{R}\}$